1. Why crystalline cohomology
   2. What is...  
   3. Katz's conjecture  
   4. Number of rational points.

Ⅲ. Why.

Def: A Weil cohomology theory is a contravariant functor.
   \( H^* : \) Smooth projective varieties over \( k \) → graded \( k \)-algebra \( k \), char
   \( H^*(X) \).

S.t. Satisfies axioms:
   - Finiteness  
   - Vanishing property  
   - Poincaré duality  
   - Künneth isomorphism.

- Cycle maps  
- Weak and Strong Lefschetz.

E.g.: 1. Singular coho  
       2. Algebraic de Rham coho  
       3. Étale coho  
       4. Crystalline coho (p-adic).

- Zeta function of a variety over a finite field:
   \( X \)-proper smooth over \( k = \mathbb{F}_q \), \( \text{deg} X = d \).

\[
Z(T ; X_0 / k) = \exp \left( \sum_{i=1}^{\infty} \frac{N_i}{i} T^i \right)
\]

where \( l \neq \text{char} \), \( X = \mathbb{F}_q \), \( F_r \) is the relative Frobenius.

- Weil conjecture: \( \text{Pic}(T) \in Z[T] \), the eigenvalues of \( F_r \) have complex absolute value \( |\lambda| = q^{\frac{d}{2}} \).
- Take log det: \( N_r = \sum_{0 \leq i \leq 2d} (-1)^i \frac{1}{(2i)!} \lambda_i^{2i} \).

Cor: \( N_r = q^r + 1 + O(\text{char}^{d-r+1}) \).

- l-adic value: \( x \) is a.e. of \( H^i \), then \( q^d \) is a.e. of \( H^{2d-i} \) (Poincaré duality).
- alg. int \( \Rightarrow \) \( x \), l-adic unit.

- P-adic value? (p-adic cohomology theory).

Crystalline coho: \( H^*_c(X, \mathbb{F}) := \lim H^*_c(X, \mathbb{F}_l) \).

Étale: \( l \)-adic (\( \ll \lim \) l.c. e.c. sheaves).

\( l \)-adic, p-adic topo are not compatible, exacting taking (local systems).

(Zariski topology is too coarse, so we need étale site).

For local systems...
Ag de Rham: \( H^1(X, \Omega^k_{X/k}) \) (tangent bundle, Zariski open is enough).

Crystalline: Zariski, all thinking all orders.

Let \( k \): perfect field of char \( k = p \). \( W = W(k) \) Witt vector, \( W_n = W(k)/p^nW(k) \).
\( W_n \): \( n \)-th Witt vector of \( k \) (e.g. \( k = \mathbb{F}_p \), \( W(k) = \mathbb{Z}_p \), \( W_n(\mathbb{Z}_p) \)). For a scheme \( X \) over \( k \), define a site \( \text{Cris}(X/W_n) \) (\( X \) not over \( W_n \)).

Obj: \((U, T) \in \text{X} \). \( U \subseteq X \). \( T \): \( W_n \)-scheme. \( U \hookrightarrow T \) with defining ideal of \( U \).

Nilpotent and has a PD-structure compatible with PD-structure on \( W_n \).

Mor: \[ U \to T \]
\[ U' \to T' \]

Covering: \( \{(U_i, T_i)\} \) is a covering of \((U, T) \) if \( T_i \to T \) open immersion and \( T = U T_i \).

PD-Structure (Deformed Power Structure): \( A \) is a comm ring. Ideal in \( \mathfrak{a} \). \( \mathfrak{d} \) on \( I \) are a collection of maps \( \xi_i : I \to \mathfrak{a} \) s.t. (axioms).

\[ \xi_i(x+y) = \xi_i(xy) = \xi_i(y) + \xi_i(x) \] abstract from: if \( A \) is a \( \mathbb{Q} \)-algebra, let \( \xi(x) = \frac{x^n}{n!} \).

Why PD-struct? (de Rham of formal lifting).
\[ O \to \mathbb{C}(t) \to \mathbb{C}(t)dt \to 0 \]
\[ O \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^3dt \to 0 \]
\[ T^d dt \]
\[ T^d dt \]

Extra PD-struct allows us to do diff and integral. PD-structure in \( W_n \).

Site \( \text{crys}(X/W_n) \to \text{Topos}(\text{X}/W_n) \text{cris} \). Let \( F \in (X/W_n) \text{cris} \) (e.g. \( O_{X/W_n} \), \( O_{X/W_n}(\text{O}(T)) \)). \( \Gamma(F) \): = \( \Gamma(e, F) := \text{Mor}(e, F) \). \( e \) is the final object in \( (X/W_n) \text{cris} \). (not representable). Enough objects on sheaves of abgp. \( H^i \text{cris}(X/W_n, \mathbb{F}) := i \)-th derived functor of \( \Gamma(F) \).

If \( X \) sm. proper / \( k \). \( H^i \text{cris}(X/W) := H^i \text{cris}(X/W, O_{X/W}) = \lim H^i \text{cris}(X/W_n, O_{X/W}) \).

Theorem: If \( Y/W \) is a smooth lifting of \( X/k \), then \( \exists \)

\[ H^i \text{cris}(X/W) \cong H^i \text{cris}(Y/W) \] (independent to lifting).

Theorem: \( H^*(X/W) \otimes_{W} [\text{Quot}(W(k))] \) is a Weil coho theory.

\[ \Phi(T) = \det(1 - TF^*|_{H^i_{et}}) = \det(1 - TF^*|_{H^i \text{cris}}) \] for \( \Phi \).
Katz's conjecture:

\[ \phi = F^* : H^{cris}(X/W) \to H^{cris}(X/W) \] is a \( G \)-linear map \( (\phi(ax) = a^p \phi(x)) \).

is an isogeny, i.e., \( \phi \otimes k \) is bijective.

\[ H^m = H^{cris}(X/W) / (Torsion) \] is free of finite rank over \( W \).

Theorem (Dieudonné-Manin). Fix any \( m \).

\[ H^m \otimes W(k) \cong \bigoplus_{i=1}^t W(k) / \mathfrak{m}_i^{n_i} \cdot L^{\cdot n_i} . \]

where \( \mathfrak{m}_i \), \( n_i \in \mathbb{Z} \), \( (n_i) = 1 \), the rational number \( \frac{n_i}{m_i} \) are called slopes with multiplicity \( m_i \).

e.g.: If \( k = \mathbb{F}_p \), \( \phi : H^m \to H^m \) has e.g. \( \{d, m \} \) with \( ord_p d, m \) precisely the slopes with mult.

Newton polygon of \( X/W \) at \( dim = m \): Ntw

Hodge polygon of \( X/W \) at \( dim = m \): Let \( h^i = h^{i-1} = \dim p^i H^m(X, \Omega^i) \)

Theorem (Katz's conjecture):

If \( X/k \) is smooth and proper, then the Newton's polygon lies on or above the Hodge polygon of \( X(k) \). Moreover, assume \( H^{cris}(X/W) \) is torsion-free and Hodge to de Rham spectral \( E_1 \) degenerates at \( E_1 \), then \( h^i \) is the multiplicity of the elementary divisor \( p^i \) of the \( W \)-linear map \( \phi : o H^m(X/W) \to H^m(X/W) \) defined by \( \phi \).

Also, Ntw and Hdg \( m \) have the same endpoint \( (b_m, C_m) \), \( b_m = \text{rank} H^{cris}(X/W) \), \( C_m = \text{length} H^{cris}(X/W) / \text{Im} \phi \).

e.g.: \( X/\mathbb{F}_p \): a curve of genus 3.

\[ h^{0,1} = 3 \ h^0 = 3. \]
cor. If \( c = \min \{ i \in \mathbb{Z}_{\geq 0} \mid h^{i-1} \neq 0 \} \), then we have \( \text{ord } \alpha_{m,i} \geq c \)

cor. Assume \( \kappa = \mathbb{F}_q \). Let \( X/k \) be smooth complete intersection of \( d \) hypersurfaces with multidegree \( d \cdot a_1, \ldots, a_d \) with \( \text{deg } a_i = d \cdot a_i \) in \( \mathbb{P}^{d+1} \) over \( k \). Then,

\[
\mathbb{Z}(T, X/k) / \mathbb{Z}(P^d(T), P^d/k) \subseteq \mathbb{Z}[q^{c+1}] \quad \text{(weak Lefschetz)}
\]

or equivalently:

\[
\deg(X/k) = \deg(P^d(k)) \mod q^{c+1}
\]

( Deligne: SGA7. Gives a formula of \( c \) depend on \( d \).)