Overview

Use analysis, geometry, etc. to understand representation theory, number theory, etc.

Along the way, discover the truth behind weird phenomena in representation theory, combinatorics, etc.

\[ \text{e.g., Langlands correspondence} \]

\[ \{ \text{n-dim Galois} \rightarrow \text{irreducible, cuspidal, automorphic reps of } \text{GL}_n(\mathbb{A}) \} \]

appearance and construction of "dual" groups
(weird exchanging laws for generalizations)
of above.

Setting
\[ F - \text{non-Archimedean local field (complete)} \]
\[ \text{e.g., } \mathbb{Q}_p, \mathbb{R}((t)) \text{ for } k = F_p \]
\[ \mathcal{O} - \text{ring of integers, local in ring sense} \]
\[ \text{e.g., } \mathbb{Z}_p, \mathbb{Z}[[t]] \]
\[ \mathfrak{m} \subset \mathcal{O} \text{ maximal ideal} \]
\[ \text{e.g., } (p), (t) \]
\[ k = \mathcal{O}/\mathfrak{m} \text{ some } \mathbb{F}_q \]
\[ \text{e.g., } \mathbb{F}_p, k = \mathbb{F}_q \]
$G$-split, connected reductive group over $F$

$T \leq B$, torus / Borel pair

$W = N_G(T)/T$ Weyl group

e.g. $SL_2 = \{ A \in \text{Mat}_2 \mid \det A = 1 \}$

$T = \{(a, a^{-1}) \}$

$B = \{(a, b, a^{-1}) \}$

Let $X^*(T) = \text{Hom}(T, F^*)$, characters

$X_*(T) = \text{Hom}(F^*, T)$ cocharacters

Then composition $X^*(T) \times X_*(T) \to \text{Aut}(F^*) \cong \mathbb{Z}$

gives a natural pairing

Let $\Phi \subset X^*(T)$ roots, $\check{\Phi} \subset X_*(T)$ coroots

e.g. for $SL_2$, $\Phi = \{ (a, a^{-1}) \mapsto a, (a, a^{-1}) \mapsto a^{-2} \}$

$\check{\Phi} = \{ a \mapsto (a, a^{-1}), a \mapsto (a^{-1}, a) \}$

and $X^*(T) \cong \check{X}^*_*(T) \cong \mathbb{Z}$

Pictorially

$X^*(T)$

$\Phi$

$X_*(T)$

Pairing multiplication

Fact: $X^*(T)$ controls structure of $G$, $\Phi$

via $\check{\Phi}$. $X^*(T)$ is something dual $\Phi$
The spherical Hecke algebra $\mathcal{H}$ comes with a topology given by a metric. It is locally compact.

- $\mathcal{O} \subset F$ is the closed unit disk and is compact.
- Hence $k = G(\mathcal{O}) \subset G(F)$ is compact.
- $G(F)$ locally compact.

Let's do some integration.

$$\mathcal{H}(G, K) = \{ f: G \to \mathbb{C} \mid f \text{ locally constant and } \text{compactly supported} \}$$

Algebra under

$$(f \ast g)(x) = \int_G f(x) g(x^{-1} z) \, dx$$

$dx$ normalized Haar measure

$(dx(K) = 1)$

Fact: $G(F) = \prod_{\ell = 1}^{r} G(O_{\ell}(F))$, $G(O_{\ell})$

$\Gamma$ uniformizer (e.g. $\ell \in \mathbb{Z}_{\ell}$, $\ell \in \mathcal{O}$)

Thus: $\mathcal{H}(G, K) = \bigoplus_{\chi \in \hat{\mathcal{O}}} \mathcal{H}(\chi, K)$, $\chi \in \mathcal{O}(\mathbb{C})$

This is unexpected—why cocharacters?

Fact: $\mathcal{H}(G, K)$ is commutative

W.c. $(G, K)$ Gelfand pair (anti-involutive)

exists, fixing $T$

By a quick calculation on # cosets, $\mathcal{H}(T, T(0)) \cong \mathbb{Z}[X, T]/\mathbb{Z}[X]$.

By $c_1 \hookrightarrow T$. 
The Satake Transform

In the Haar measure on \( N \cong \mathbb{C}^* \times \mathbb{R} \),

let \( \sigma \) be with \( d\sigma(N(O)) = 1 \),

and \( S : \mathcal{B} \to \mathbb{R}^* \) given by \( d\sigma(bn^{-1}) = \varphi(b) d\sigma(n) \).

the modular function, trivial on \( N \).

(\( \varphi \) is a character of \( T \)).

Define \( \Phi : \mathcal{H}(G,K) \to \mathcal{H}(T, T(O)) \otimes \mathbb{Z}[\varphi^{\pm \frac{1}{2}}] \)

by

\[
\Phi(f) = \varphi(\varphi) \sum_{n} f(tn) d\sigma(n) \;
\]

(\( \varphi^{\pm \frac{1}{2}} \) comes from \( \varphi^{\frac{1}{2}} \)).

\( S \) is an injection. Furthermore,

**Theorem (Satake iso).** The image of \( \Phi \) lies in \( \mathcal{H}(T, T(O)) \otimes \mathbb{Z}[\varphi^{\pm \frac{1}{2}}] \).

Recalling that \( \mathbb{C}[X,T] \otimes \mathbb{Z}[\varphi^{\pm \frac{1}{2}}] \)

is an iso.

\[
\Phi : \mathcal{H}(G,K) \cong \mathcal{R}(G) \otimes \mathbb{Z}[\varphi^{\pm \frac{1}{2}}] \]

where \( \mathcal{R}(G) \) is the representation ring of the dual group (that with dual root data).

\[
\mathcal{H}(SL_2(F), SL_2(O)) \cong \mathcal{R}(PGL_2(C)) \otimes \mathbb{Z}[\varphi^{\pm \frac{1}{2}}] \]
So, the spherical Hecke algebra encodes info on the representations of the dual.

Common dual pairs

\[
\begin{array}{c|c|c|c|c}
\text{GL}_n & \text{SL}_n & \text{SO}_{2n+1} & \text{SO}_{2n} & \text{E}_8 \\
\text{GL}_n & \text{PGL}_n & \text{Sp}_{2n} & \text{SO}_{2n} & \text{E}_8 \\
\end{array}
\]

"swaps root lengths"

Proof idea

If \( T, \mu \in P^+ \), we can find \( S(c_\mu)(t) \) for \( t = \mu(\tau) \). Each \( x_i = t(x_i) n(x_i) + TN = \beta_i \).

Then \( S(c_\mu)(t) = S^{\frac{1}{2}}(t) \sum_{\mathfrak{n}} c_\mu(\tau n) \).

\[
= \sum_{\mathfrak{n} \in \mathfrak{N}, x_i \in \mathfrak{k}} \prod_{i=1}^{n} \mathfrak{n}(x_i) \quad \text{property of} \ S
\]

\[
= \sum_{\mathfrak{n} \in \mathfrak{N}, x_i \in \mathfrak{k}} \prod_{i=1}^{n} \mathfrak{n}(x_i) \quad \text{if} \ t(x_i) = \mu(\tau) \mod T\text{(Alg)}
\]

\( \implies S(c_\mu) \) counts diagonal entries with valuation = to that of \( \mu(\tau) \).
Sample calc: \( \mathcal{L}(\alpha_x) = \frac{1}{a^2} \mathcal{L}(\alpha) \cdot \mathcal{L}(\pi) \cdot \mathcal{L}(\chi) \)

since \( \mathcal{L}(\alpha_x) \cdot \mathcal{L}(\mu) \neq 0 \quad \Rightarrow \quad \mu \in \mathcal{L} \),

\[ \mathcal{L}(\alpha_x) = \frac{1}{a^2} \mathcal{L}(\alpha) \cdot \mathcal{L}(\pi) \cdot \sum_{\mu \in \mathcal{L}} \alpha(\mu) \mathcal{L}(\chi) \]

**Applications**

- Representation theory of \( R(G, \mathbb{C}) \)
- Influences that of \( R(G) \)

- Useful in \( L \)-functions

Characters of \( R(G) \) \( \otimes \) \( \mathbb{C} \) \( \rightarrow \) \( \mathbb{C} \)

\( \leftrightarrow \) simple conjugacy classes in \( G(C) \).

Then \( \omega(x_\alpha) = x_\alpha \cdot (s) = \exp(sL_v) \).

Then \( \mathcal{L} : H(G, \mathbb{C}) \otimes \mathbb{C} \rightarrow R(G) \)

gives a parametrization