Iwahori–Hecke Algebras in Multiple Contexts

1) Hecke algebras for a reductive group
2) Presentation of spherical/finite/affine Hecke algebras
3) Quantum Schur–Weyl Duality

1) Reductive Groups

\( G \): reductive gp. / \( F \): nonarch local field

\( \mathcal{O} \): ring of integers of \( F \)

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\( \mathcal{P} \): maximal ideal of \( \mathcal{O} \)

\( \mathcal{B} = \) Borel subgp.

\( \mathcal{K}^0 = \) maximal compact subgp.

\( \mathcal{I} = \) Iwahori subgp.

Favorite example:

\( G = \text{GL}_n \left( \mathbb{Q}_p \right) \)

\( \mathcal{O} = \mathbb{Z}_p \)

\( \mathcal{P} = \langle \mathfrak{p} \rangle \)

\( \mathcal{B} = \begin{pmatrix} \mathcal{O} & \ldots & \mathcal{O} \\ \vdots & \ddots & \vdots \\ \mathcal{O} & \ldots & \mathcal{O} \end{pmatrix} \)

\( \mathcal{K}^0 = \begin{pmatrix} \mathcal{O} & \ldots & \mathcal{O} \\ \cdots & \ddots & \cdots \\ \mathcal{O} & \ldots & \mathcal{O} \end{pmatrix} \)

\( \mathcal{I} = \begin{pmatrix} \mathcal{O} & \ldots & \mathcal{O} \\ \vdots & \ddots & \vdots \\ \mathcal{O} & \ldots & \mathcal{O} \end{pmatrix} \)
Let $k$ be a compact open subgp. of $G$. The Hecke algebra of $G$ relative to $k$ is the set of smooth, compactly supported $k$-biinvariant functions on $G$:

$$H_k := \{ \phi: G \to \mathbb{C}, \text{ smooth, cpt. supp} \mid \phi(kgk') = \phi(g) \ \forall k, k' \in K, g \in G \},$$

w/ mult. defined by convolution.

1) Reductive gps. are hard
2) Hecke algebras are relatively simple: often finite(-ish) dim (see next section)
3) Borel-Matsumoto: $\exists$ corresp. b'twn irreps $H_k$ and "smooth, admissible" irreps of $G$ w/ $k$-fixed vector $v$ ($k \cdot v = v \ \forall k \in K$).
4) So Hecke alg's. are a tool to understand the repn theory of reductive gps.

But what do Hecke algebras actually look like?
2) Presentations (Iwahori)

For this section, $G = GL_n$, but can be done for any Cartan type.

$H_k \cong X^*(T) \cong \mathbb{Z}^n$ (spherical Hecke algebra)

$H_B = \langle T_i, i = 1, \ldots, n-1 \rangle$

- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
- $T_i T_j = T_j T_i$, $i \neq j \pm 1$
- $T_i^2 = (q-1)T_i + q$

(finite Hecke algebra)

$H_\infty = \langle T_i, i = 0, \ldots, n-1 \rangle$

Same rels as for $H_0$, but indices mod $n$

(affine Hecke algebra)

Remarks

1) Not guaranteed a simple presentation of $H_k$ for other subgps. $K$, but ...

2) $H_{K_0}$ is commutative!
3) $H_B$ is finite dim, is a deformation of the group alg. of $S_n$ (finite Coxeter gps. in general):

If $g \mapsto 1$

$$H_B \mapsto C[S_n]$$

So repn theory of finite Hecke alg. relates to repn. theory of $S_n$.

4) Exact sequences:

$$1 \rightarrow \mathcal{P} \rightarrow \mathcal{K}^o \rightarrow K^o \rightarrow B(F_\mathfrak{g}) \rightarrow 1$$

$$\mapsto$$

$$0 \rightarrow H_{K^o} \rightarrow H_J \rightarrow H_B \rightarrow 0$$

So to understand $H_J$, want to understand $H_{K^o}, H_B$.

3) Quantum Schar-Weyl Duality

First, classical S-W duality:
Let $V = \mathbb{C}^n$ be the sta. repn of $G = \text{GL}_n(\mathbb{C})$.

Now take $V^k$ for $k \leq n$, and let $G$ act diagonally:

$g \cdot (v_1 \otimes \cdots \otimes v_k) = g \cdot v_1 \otimes \cdots \otimes g \cdot v_k$.

Let $S_k$ act on $V^k$ by permuting the factors:

$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$

These actions commute, and in fact are mutual centralizers.

**Schur-Weyl duality:** As a $(\text{GL}_n, S_k)$-bimod., $V^k$ decomposes as

$V^k = \bigoplus_{\lambda \vdash k} L^\lambda \otimes S^{\lambda}$,

where the $L^\lambda$ are (distinct) highest wt. repns, and the $S^\lambda$ are (distinct) Specht modules.
Now, let $V$ be std. repn. of the quantum gp. $U = U_q(\mathfrak{gl}_n)$, $q$ \(\neq\) root of unity, and let $U$ act on $V \otimes k$ by the coproduct map.

Since $U$ not "cocomm", we can't just permute the factors as before. Instead, we use the Yang-Baxter eqn. to define isomorphisms

$$R_i : V_1 \otimes V_i \otimes V_{i+1} \otimes V_k \cong V_1 \otimes V_{i+1} \otimes V_i \otimes V_k.$$ 

Thm (Jimbo '86): The alg. gen'd by the $R_i$ is isom. $H_B$ (for $GL_k$), and the $U$ and $H_B$ actions are mutual centralizers.

We have the decomps.

$$V \otimes k = \bigoplus_{\lambda \vdash k} L_\lambda^q \otimes S_\lambda^q,$$

where the $L_\lambda^q, S_\lambda^q$ are irrds, and deformations of the $L^\lambda, S^\lambda$. 
Remarks

1) Jimbo's results helped kick-start huge breakthroughs. One notable example: Jones' Fields Medal work on knot invariants.

2) This section only holds for $GL_n$, not a reductive group of any other type.

3) Not surprising that $U_q(02GL_n)$ is in S-W duality w/ a deformation of $C[Sn]$, but it is remarkable that this deformation turns out to be the Hecke algebra.

4) I am not aware of any "natural" (functorial) for remark 3, and in light of remark 2), might be hard to have a general result. Would be very interesting if such a result exists!