Intro to Klyachko models for GLn

1. Whittaker models
2. Motivation for study of models
3. Klyachko models

$F$: finite or p-adic, $G = \text{GL}_n$, $U = \mathcal{G}(\mathfrak{g}, \ast) \leq G$

Field

1. Fix a nontrivial character $\psi$ of $F$. Define a character $\psi$ of $U$ by

$$\psi([u_{ij}]) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$$

Def: A Whittaker model of an irreducible repn $(\pi, V)$ of $G(F)$ is the image of an embedding

$$V \rightarrow \text{Ind}^G_{U(F)} \psi.$$

We've encountered Whittaker models for $\text{GL}_2$:

- $F$ finite field: $\text{Ind}^G_{U(F)} \psi$ multiplicity free
  - All higher diml irreps have model

- $F$ non-Archimedean, local field: $\text{Ind}^G_{U(F)} \psi$ multiplicity free
  - Irred. miss. irreps have Whittaker model

Can also define Whittaker models for automorphic repns

Leverage decomposition into local repns to deduce global results from local ones.
Uses for Whittaker models

- integral reps of L-functions w/ nice properties
- multiplicity free \Rightarrow Casselman- Shalika formula
- Fourier expansions of cusp forms

In general, not all irreducible admissible higher dim'l reps have Whittaker models.

2. General theme in study of L-functions, write down integral reps of L-functions w/nice properties to prove analytic continuation and functional equation.

Believed period integrals \& related to unique models lead to nice integrals related to L-functions (Piatetski-Shapiro, Furusawa-Shalika)

As said above, not everything has a Whittaker model, but we do still have a multiplicity free property for general \( n \)

Klyachko model generalizes the Whittaker model and a symplectic model in hopes of encompassing a wider range of reps.

3. I'll now follow a paper of Offen \& Sayag: "Global mixed periods and local Klyachko models for the general linear group".

Write \( n = r + 2k \).

Let

\[
H_{r,2k} = \{ (u^* t) eG_n : u \in U_r, h \in Sp_{2k} \}
\]

where \( Sp_{2k} = \{ g \in G_{2k} : t g(-w_k w^k) g = (-w_k w^k) \} \), \( w_k = (1, \ldots) \).

Extend \( \psi \) to a character of \( H_{r,2k} \) by \( \psi(u^* t) = \psi(u) \).

Def: The Kyachko model \( M_{r,2k} \) is

\[
M_{r,2k} = \text{Ind}_{H_{r,2k}(F)}^{G(F)} \psi.
\]

- \( M_{r,0} \) Whittaker model
- \( M_{0,n} \) even symplectic model
- \( M_{r,2k} \) is a mixed Whittaker-symplectic model

An irreducible admissible rep'n admits the model \( M_{r,2k} \) if it can be embedded in unispace.

- Kyachko models were first introduced by Kyachko in 84

over finite field: \( F \) finite field

\[
M = \bigoplus_{k \geq 0} M_{n-2k,2k}
\]

is a direct sum of all irreps each \( w \) multiplicity.

(Englin-Saxl '91)

M is a Gelfand model:

1. existence
2. disjointness
3. uniqueness
While success over finite fields, many want to investigate over p-adic.

- Heumos & Rallis were first to study of Klyachko models over p-adic field: $F_p$ p-adic

  disjointness ) Offen-Sayag '04
  uniqueness
  existence? :

Even in $GL_3$, exist irreducible admissible reps that admit no Klyachko model.

But every smooth irreducible unitary rep'n admits a Klyachko model.

Tadic's classification of unitary rep'n's:

$s$ : irreducible square integrable rep'n of $G_v$
$t \in \mathbb{Z}^+$

$s\left[ \frac{1}{2^t} \right] \times s\left[ \frac{3-1}{2^t} \right] \times \cdots \times s\left[ \frac{t-1}{2^t} \right] \text{ has a unique irreducible subrep'n } U(s, t)$.

Notation: $\rho[\alpha] = \det(1^\alpha \rho)$

$\sigma_1 \times \sigma_2$ rep'n of $G_{v+r_2}$ parabolically induced from $\sigma_1 \otimes \sigma_2$.

$B =$ collection of $U(s, t)$ and $U(s, t)[\alpha] \times U(s, t)[\beta]$ $0 < \alpha < \frac{t}{2}$
Theorem (Tadic '86): The unitary rep's are exactly of the form
\[ \sigma, x \cdots x \sigma_t \]
\[ \forall \sigma, \ldots, \sigma_t \in \mathcal{B}. \]

Often & Sayag use results on the purely symplectic model and highest derivatives of rep's to show the following:

**Theorem (often Sayag '07)**
\[ \bigl( U(\mathfrak{d}, 2m; \iota)[\alpha_i] \times \cdots \times U(\mathfrak{d}_q, 2m_{q_i})[\alpha_{q_i}] \bigr) \]
\[ \times \bigl( U(\mathfrak{d}_1, 2m_1; \iota)[\alpha_1] \times \cdots \times U(\mathfrak{d}_q, 2m_{q+1})[\alpha_q] \bigr) \]

admits the model \( M_r, 2k \) \( \forall \)
\[ r = r_1 + \cdots + r_q, \]
\[ k = m_1 r_1 + \cdots + m_q r_q + m_1 r_1' + \cdots + m_q r_q' \]

Klyachko models also studied over \( \overline{\mathbb{Q}} \in \mathfrak{C}. \)

Conjectured by Heumos '93 that irreducible unitary automorphic rep'n of \( \text{GL}_n(A_F), \) for a number field has a unique model.
Sources:

- "Models and periods for automorphic forms on $Gln"$ Heumos
- "Global mixed periods and local Klyachko models for the general linear group" Oflensky
- "A tour of $p$-adic representations theory of $Gl_2$" Kathy Weber ✓ Notes for Kathy's talk last semester on SNT site
- "Automorphic forms and representations" Bump