Whitaker functions
& Demazure operators
Source: Brubaker, Bump, & Licata

1. Iwahori Whit. funcs
2. Computing w/ operators \( \rightarrow \) rep'n of Hecke alg
3. Connection to geometry of flag varieties.

Iwahori Whit. funcs

Notation: \( G = \text{GL}_n(Q_p) \), \( \hat{G} = \text{GL}_n(C) \)
\( T = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix} \), \( B = \begin{pmatrix} \lambda \ast \\ 0 \end{pmatrix} \), \( N = \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix} \)
\( O = Z_p \), \( P = <p> \), \( q_p = |O|/p \)
\[ J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

The principal series rep's of \( G \) are of the form

\[ \Pi = \text{Ind}^G_B (\tau) \]

Fix a char. \( \psi : N \to \mathbb{C} \)

A Whittaker functional is a linear map

\[ \Omega_\tau : \text{Ind}^G_B (\tau) \to \mathbb{C} \]

s.t.

\[ \Omega_\tau (\Pi(n)f) = \psi(n) \Omega_\tau (f) \]
Let \( \text{M}(\tau) := \text{Ind}_{B}^{G}(\tau)^{J} \)

The standard basis \( \{ \Phi_{w} \}_{w \in W} \) of \( \text{M}(\tau) \) consists of characteristic functions on \( J \)-double cosets and is indexed by \( W := S_{n} \).

We want to calculate:

The Iwahori Whittaker functions:

\[
W_{J, \Phi_{w}}(g) := \Omega_{\gamma}(\pi(g) \Phi_{w})
\]
2. Demazure - Jusztig

A dual connection:

\[ z \in \hat{T}(C) \leftrightarrow \text{chars of} \ T(\mathbb{Q}_p) \]

\[ z = \left( \begin{array}{c} z_1 \\ \vdots \\ z_n \end{array} \right) \leftrightarrow J_z \left( \begin{array}{c} t_1 \\ \vdots \\ t_n \end{array} \right) \]

\[ = \prod z_i \cdot \text{ord}(t_i) \]

\[ \chi \text{ char of } \hat{T}(C) \leftrightarrow a_x \in T(\mathbb{Q}_p) \]

\[ \chi(\begin{array}{c} z_1 \\ \vdots \\ z_n \end{array}) = z_1^x \cdot z_n^x \leftrightarrow a_x = \left( \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \end{array} \right) \]

Suffices to calculate \( W \)'s \( D_n a_x \)
W's will be fncs in $\Theta(\hat{\tau})$.

For each $s_i \in W$, the \textit{Demazure operator} $D_i$ on $\Theta(\hat{\tau})$.

\begin{equation*}
D_i f(z) := (1 - q^{-1} \frac{z_{i+1}}{z_i}) D_i
\end{equation*}

Also consider:

\begin{equation*}
J_i = D_i - 1
\end{equation*}
Thm A To any $w \in W$ & dominant $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$

\[ \mathcal{H}_{\mathcal{F}}(a_{-1}) = (\star) \underbrace{\mathcal{J}_w(\mathbb{Z})}_{\text{Further, } \mathcal{T}_i \rightarrow \mathcal{T}_i \text{ gives a rep' of the finite Hecke alg } \mathcal{H}_{q_i} \text{ on } \Theta(q_i)} \]

**Remarks** The $\mathcal{J}_w$ come from analyzing the effect of intertwining ops $\mathcal{A}_w : \text{Ind}_B^G(\mathfrak{k}_\mathbb{Z}) \rightarrow \text{Ind}_B^G(\mathcal{J}_w \mathfrak{k}_\mathbb{Z})$ on $\mathfrak{z}$
They are closely related to Demazure - Lusztig ops which are derived as endomorphisms on equiv. K-theory of flag variety.

3. Connection to geometry

\[ X := \hat{G}/\hat{B} \times_{\text{var}} (\text{flags implied}) \]

\[ Y_w := BwB/B \quad \text{is a Schubert cell} \]

\[ X_w := \overline{BwB}/B = UY_u \quad u \leq w \]

is a Schubert variety.
Most $X_w$'s are singular

Let $w = s_{h_1} \cdots s_{h_r}$ be a reduced decomp of $w$

The Bolt-Samelson variety is a certain quotient

$$P_{h_1}^B \times \cdots \times P_{h_r}^B = Z_w$$

parabolic

E.g. $P_1 = BvB_sB$

$$=(*)$$

$$& \rightarrow \quad Z_w \rightarrow X_w \text{ is a resolution}$$
of singularities, w/ constant fibers $F_u$ over each $Y_u, u \leq w$.

**Thm B**

$$D_w = \sum_{u \leq w} P_{w,u}(q^{-1}) T_u$$

Where $P_{w,u}$ is the Poincaré polynomial of $F_u$

$$\text{poly in } q^{-1}, \text{w/ } n^{th} \text{ coeff } = H^{2n}(F_u)$$

i.e., the relationship between $Z_w$ & $Y_u$ is the same as that between $D_w$ & $T_u$. (cool!)
Further remarks:

This realization of Whittaker functions also gives efficient proofs of:

- Casselman-Shalika formula
- Demazure character formula

& can be used to show that $\mathcal{W}_{\tau, \tau'}(a, s)$ is a specialization of non-symmetric Macdonald polynomials.

(in type) Hecke alg rep'n \(\rightarrow\) R-matrix for a quantum group

\(\text{a la Andy's talk}\)

\(\text{a la Ben's class; also see his most recent paper}\)

lattice model