0.0.1 Dirichlet L-functions

- Dirichlet (1837) proved there are infinite number of primes in an arithmetic sequence \( b, b+m, b+2m, \ldots \) by using Dirichlet L-series \( \sum_{n>0} \frac{\chi(n)}{n^s} \), where

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}
\]

- **Definition** Dirichlet character mod \( m \) \( \chi : \mathbb{Z} \to \mathbb{C} \) has conditions:
  1. \( \chi(n+m) = \chi(n) \quad \forall n \in \mathbb{Z} \)
  2. \( \chi(km) = \chi(k)\chi(m) \quad \forall k, m \in \mathbb{Z} \)
  3. \( \chi(n) \neq 0 \Leftrightarrow \gcd(n, m) = 1 \)
  4. **principal**: \( \chi_0(n) = 1 \Leftrightarrow \gcd(n, m) = 1 \)
  5. **trivial**, ie mod 1 \( \chi(n) = 1 \forall n \in \mathbb{Z} \)

also \( \chi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^* \) extended to \( \mathbb{Z}/m\mathbb{Z} \) by \( \chi(n) = 0 \) for \( \gcd(m, n) > 1 \)

- Has an Euler product

\[
\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}
\]

- Tried to follow Legendre, but failed until he started using analytic techniques:
  - Dirichlet made use of

\[
\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx
\]

and \( s \to 1^+ \) in form of a well known identity

\[
\int_0^1 x^{k-1}\log^\rho \left( \frac{1}{x} \right) dx = \frac{\Gamma(1+\rho)}{k^{1+\rho}}
\]

where \( k > 0 \) is constant, \( \rho > 0 \) has \( \rho \to 0 \).
  - Used complex analysis and the Euler product
  - but did not need analytic continuation.
  - Seems to use \( \chi \to \) roots of unity but also needs \( \chi(n) = 0 \) when \( p \mid n \) to eliminate a lot of terms of \( \sum \chi(n)/n^s \) to show that

\[
\sum \frac{1}{q^{1+\rho}} \to \infty \text{ as } \rho \to 0
\]

where \( q = np + m \)

- Eisenstein proved analytic continuation and functional equation for a Dirichlet series related to \( \zeta \).
• Ernst Kummer (1839,40) introduced $\zeta$ of a cyclotomic field to investigate class number of these fields following Dirichlet

• Riemann (1859) used Poisson summation

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x \xi} \, dx$$

to show analytic continuation and functional equation of $\zeta$ which is the Dirichlet series with trivial character:

$$\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(1 - s)$$

• Dedekind (1893) extended $\zeta$ to arbitrary number fields of an algebraic extension $K/\mathbb{Q}$ using trivial $\chi$. Dedekind

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \prod_{p}(1 - N(p))^{-1}$$

$a$ non-zero ideal in ring of integers $\mathcal{O}_K$ of $K$ and $p$ is prime ideal, $N$ is index $[\mathcal{O}_K : \mathfrak{a}] = |\mathcal{O}_K/\mathfrak{a}|$.

– Proven by Hecke (1917) to have meromorphic continuation and functional equation.

• Examples
  - The twisted mean square and critical zeros of Dirichlet L-functions
  - An explicit lower bound for special values of Dirichlet L-functions
  - Several expressions of Dirichlet L-functions at Positive integers
  - On asymptotic properties of the generalized Dirichlet L-functions
  - Simultaneous nonvanishing of Dirichlet L-functions and twists of Hecke-Maass L-functions in the critical strip
  - Explicit bounds on exceptional zeroes of Dirichlet L-functions

  - investigation of Dirichlet L-functions of Diaphontine numbers?! (very irrational?!)

0.0.2 Hecke L-functions

• A generalization of the Dirichlet L-function and in particular a generalization of Dedekind $\zeta$

$K$ number field,

$v$ non-archimedean place

$\mathcal{O}_K$ ring of integers of $K$,

$\mathfrak{p} \subset \mathcal{O}_K$ prime ideal

$N\mathfrak{p}$ number of elements in finite field $\mathcal{O}_K/\mathfrak{p}$

$|x|_v = |x|_\mathfrak{p} = (N\mathfrak{p})^{-\text{ord}_\mathfrak{p}(x)}$ for $x \in K$

For real embedding $\sigma : K \to \mathbb{R}$ for archimedean $v$ $|x|_v = |\sigma(x)|$. 
Leads to Hecke character (Grössencharacter) \( \chi_v : K^* \to \mathbb{C}^* \):

\[
\chi(x) = \prod_v \chi_v(x)
\]

with conditions:

1. \( x \in K \subset K_v^* \) implies
   \[
   \chi(x) = 1 \quad \text{product formula}
   \]
2. all but finite number of \( \chi_v \) be unramified, ie, trivial on \( \{ x \in K_v^* \mid |x|_v = 1 \} \)
3. For unramified place \( v \) corresponding to \( p \), \( \chi(p) = \chi_v(\varpi_v) \) for uniformizer \( \varpi \in K \)
4. Ordinary ideal \( a \subset \mathcal{O}_K \) only included in \( \sum \) if product of unramified primes

- Hecke L-function (1916)

\[
L_K(s, \chi) = \sum_a \frac{\chi(a)}{(Na)^s} = \prod_p \left( 1 - \chi(p)(Np)^{-s} \right)^{-s}
\]

where \( a \), ideals of \( \mathcal{O}_K \) are products of prime ideals corresponding to places where \( \chi_v \) is unramified.

- \( \chi \) trivial, ie., \( \chi_v = 1, \forall v \) \( L(s, \chi) \) is Dedekind \( \zeta \) of \( K \): \( \sum (Na)^{-s} \). Furthermore, \( K = \mathbb{Q} \) becomes Riemann \( \zeta \).

- If \( \chi \) is finite order \( L_K(s, \chi) \) becomes Dirichlet L-function.

- Hecke: express L-function in terms of generalized \( \theta \)-function, which led to deriving analytic cont., functional equation, boundedness in vertical strips

### 0.1 Modular forms

- Hecke (1936) expanded L-functions into area of Modular forms: theta series:

\[
\theta(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}
\]

holomorphic in \( \mathfrak{H} \), has

\[
\theta\left( \frac{-1}{\tau} \right) = C(\tau^{1/2}) \theta(\tau), \quad \theta(\tau + 2) = \theta(\tau)
\]

Is a modular form of weight \( k = 1/2 \) period \( \lambda = 2 \), \( C \) condition for group generated by \( \tau \mapsto \tau + 2 \) and \( \tau \mapsto -\frac{1}{\tau} \), ie has Taylor expansion

\[
f(\tau) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi in\tau}{\lambda}}
\]

which implies holomorphic at \( \infty \). (\( a_0 = 0 \) ⇒ cuspform.)
• Hecke: sequence \( a_0, a_1, \ldots \subset \mathbb{C} \) \( a_n = O(n^d) \), for some \( d > 0 \). Given \( \lambda > 0, k > 0, C = \pm 1 \), define:

\[
\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

\[
\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \phi(s)
\]

\[
f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi in\tau}
\]

led to

• **Theorem** (Hecke’s Converse Thm 1936) Following are equivalent:

  1. \( \Phi(s) + \frac{a_0}{s} + \frac{Ca_0}{k-2s} \) is an entire function bounded in vertical strips and satisfies functional equation \( \Phi(s) = C\Phi(k-s) \)

  2. \( f \) is a weight \( k \) modular form \( Mf_m(k, \lambda, C) \), period \( \lambda \), multiplier condition \( C \)

• Connects modular forms and L-series/functions, (leads to Wiles discoveries including Fermat’s thm)

• Maass forms (1949): non-holomorphic modular forms that are eigenfunctions of Laplacian.

### 0.2 Automorphic forms, Eisenstein Series

\[
E_s(z) = \sum_{\gamma \in (P \cap \Gamma) \backslash \Gamma} \text{Im}(\gamma z)^s
\]

\( \text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R}) / \text{SO}(2) = \Gamma \backslash \mathcal{H} \), \( P \) parabolic, eg., upper triangular. Continues \( \xi \)

\[
\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad \xi(s) = \xi(1-s)
\]

• Selberg (1962) mero ctn for \( E_s : s(1-s)\xi(2s) \cdot E_s \) has analytic ctn to entire fcn of \( s \). Fcnl eqn:

\[
\xi(2s)E_s = \xi(2-2s)E_{1-s}
\]

**Characteristics:**

1. simple pole at \( s = 1 \) with residue \( 3/\pi \).

2. \( \text{in} 0 < \text{Re}(s) < 1/2 \) poles at \( \rho/2 \) where \( \rho \) is non-trivial zero of \( \zeta(s) \).

• Lots of ways to use Eisenstein series to generate integral representations of L-functions with Euler products, use analytic characteristics of Eisenstein series (analytic continuation, functional equation)
Colin de Verdière (1982,3) Meromorphic continuation of Eisenstein involves distribution theory including Sobolev spaces, Friedrichs self-adjoint extension of a restriction of a symmetric unbounded operator, eg., the Laplacian
\[
\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]
cuspforms are smooth, rapid decay, Eisenstein series is smooth moderate growth.

- Constant term of Eisenstein series
  \[ c_p E_s(z) = \int_0^1 E_s(z + t) \, dt \]
  \[ c_p E(x + iy) = y^s + \frac{\xi(2s - 1)}{\xi(2s)} y^{1-s} \]

- Rankin-Selberg method \( f, g \) cuspforms w/ F-series
  \[ f(z) = \sum_{n>0} a_n e^{2\pi inz} \]
  then
  \[ \int_{P \backslash \mathbb{H}} y^s f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2} = (4\pi)^{-(s+2k-1)} \Gamma(s + 2k - 1) \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^{s+2k-1}} \]

- pullbacks of Eisenstein series, eg., Rankin triple product:
  \[ \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2 \hookrightarrow \text{Sp}_{6 \times 6} \]
  holomorphic cuspforms of weight \( 2k \) for \( \text{SL}_2(\mathbb{Z}) \): \( f, \varphi, \psi \)
  \[ \int \int \int (E \cdot \iota)(z_1, z_2, z_3) f(z_1) \varphi(z_2) \overline{\psi(z_3)} (y_1 y_2 y_3)^{2k-2} \, dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 \]
  \[ = \Gamma's \times \zeta's(\text{constant with} \pi) \times L_{f, \varphi, \psi}(s + 4k - 1) \]

- Iwasawa-Tate wraps everything up in the adele’s. Garrett MFM notes looks at \( \zeta \), Dirichlet L-function in terms of adeles/ideles, eg., \( \chi \) is a character on \( \mathbb{J}/k^\times \)

0.3 Some informal references

- (Garrett):
  - [http://www.math.umn.edu/~garrett/m/v/basic_rankin_selberg.pdf](http://www.math.umn.edu/~garrett/m/v/basic_rankin_selberg.pdf)
  - (Garrett) Colin de Verdière meromorphic continuation of Eisenstein series: [http://www.math.umn.edu/~garrett/m/v/cdv_eis.pdf](http://www.math.umn.edu/~garrett/m/v/cdv_eis.pdf)
  - [http://www-users.math.umn.edu/~garrett/m/v/pseudo-cuspforms.pdf](http://www-users.math.umn.edu/~garrett/m/v/pseudo-cuspforms.pdf)
• Gelbart, Stephen S.; Miller, Stephen D.  
Riemann’s zeta function and beyond. (English summary)  

• Dirichlet, Peter Gustav Lejeune  
There are infinitely many prime numbers in all arithmetic progressions with first term and difference coprime  
arXiv:0808.1408v2 [math.HO]

• Ireland, Kenneth F; Rosen, Michael I  
A Classical Introduction to Modern Number Theory  
Graduate texts in mathematics ; 84. 2nd ed.. New York : Springer-Verlag 1990

• Bruinier, Jan H (Jan Hendrik),  
The 1-2-3 of Modular Forms : Lectures at a Summer School in Nordfjordeid, Norway  
Universitext; SpringerLink (Online service),Berlin : Springer 2008