Representation Stability, Étale Cohomology and Combinatorics of Configuration Spaces over Finite Fields

Following Church-Ellenberg-Farb, Representation stability in cohomology and asymptotics for families of varieties over finite fields

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Definition

\[
\begin{align*}
\text{PConf}_n(F) &= \{(x_1, \ldots, x_n) \in F^n : x_i \neq x_j \text{ when } i \neq j\} \\
\text{Conf}_n(F) &= \text{PConf}(F)/S^n
\end{align*}
\]
Configuration Spaces: As Schemes

Let...

- $D_n$ be the space of monic degree-$n$ polynomials in $T$
- $\pi : \mathbb{A}^n \to D_n$ by $(x_1, \ldots, x_n) \mapsto (T - x_1) \ldots (T - x_n)$

$S^n$ acts $\mathbb{A}^n$ with $\pi(\sigma x) = \pi(x)$ for $\sigma \in S_n$. $D_n = \mathbb{A}^n/S_n$

**Definition**

- $\text{Conf}_n := D_n \setminus V(\Delta)$ where $\Delta$ is the discriminant
- $\text{PConf}_n := \mathbb{A}^n \setminus \bigcup_{i<j} V(x_j - x_i)$

Note $\pi : \text{PConf}_n \to \text{Conf}_n$
Configuration Spaces: A Warning

Conf_n \cong P\text{Conf}_n / S_n \text{ scheme-theoretically.}
Our Plan

- Recall $\text{PConf}_n(\mathbb{C})$ is an analytic manifold.
- $H^i(\text{PConf}_n(\mathbb{C}))$ is an $S_n$-representation...
- ...with maps between induced by $\text{PConf}_{n+1} \to \text{PConf}_n$
- We use the theory of FI-modules study $\chi H^i(\text{PConf}_n(\mathbb{C}))$ as $n \to \infty$ ...
- ...and use Grothendieck-Lefschetz relate it to combinatorics on $\text{Conf}_n(\mathbb{F}_q)$????
**Goal:** relate $H^*(\text{PConf}_n(\mathbb{C}))$ to $H^*(\text{Conf}_n(\mathbb{F}_q))$

**Problem:** Zariski topology and singular cohomology are not friends

**Solution:** Étale Cohomology

Following [Mil13], [Gro13].
Definition

A Grothendieck Topology on a category $\mathcal{C}$ consists of... for each $U \in \mathcal{C}$ a distinguished set of coverings $(U_i \to U)_{i \in I}$ such that

- various axioms are fulfilled
- which imitate the properties of $\mathcal{O}_\mathcal{P}(X)$

Such a category $\mathcal{C}$ equipped with a Grothendieck Topology is called a site.
The Étale Site

**Definition**
A morphism of varieties $f : X \to Y$ is Étale if it is smooth and unramified.
When $X$ and $Y$ are smooth, this is equivalent to inducing an isomorphism $T_x X \to T_y Y$ for each closed point $y \in Y$ and $x \in f^{-1}(y)$.

**Definition**
Let $\text{Et}(X)$ be the category of étale maps with target $X$.
Declare our coverings to be surjective families $(U_i \to U)_{i \in I}$.
Étale Cohomology

Definition

- An étale presheaf $\mathcal{F}$ on $X$ is a functor $\text{Et}(X)^{\text{op}} \to \text{Ab}$.
- An étale sheaf is an étale presheaf which satisfies site-theoretic analogues of the sheaf axioms.
- Denote the category of étale sheaves on $X$ by $\text{Sh}^{\text{ét}}(X)$
- $H^i_{\text{ét}}(X; \mathcal{F})$ is defined as $R^i(\Gamma)(\mathcal{F})$ for an étale sheaf $\mathcal{F}$
Let $\ell$ be a prime and $\mathbb{Z}/\ell^k$ the constant sheaf with value $\mathbb{Z}/\ell^k$.

**Definition ($\ell$-adic Cohomology)**

Define $H^i_{\text{ét}}(X; \mathbb{Z}_\ell) := \varprojlim H^i_{\text{ét}}(X; \mathbb{Z}/\ell^k)$

$H^i_{\text{ét}}(X; \mathbb{Q}_\ell) := H^i_{\text{ét}}(X; \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$
A Technical Note

Notation

► **Henceforth, when taking étale cohomology, X will be a variety defined over \( \mathbb{F}_q \).**

► **\( H^i_{\text{ét}}(X; \mathbb{Q}_\ell) \) will be shorthand for \( H^i_{\text{ét}}(X/\bar{\mathbb{F}}_q; \mathbb{Q}_\ell) \), with \( X/\bar{\mathbb{F}}_q \) denoting the base change of \( X \) to \( \bar{\mathbb{F}}_q \).**

► **\( X/\bar{\mathbb{F}}_q = X \times_{\text{Spec } \mathbb{F}_q} \text{Spec } \bar{\mathbb{F}}_q. \)**
Let $X$ be a nonsingular variety defined over $\mathbb{Z}$ and $G$ a finite Abelian group.

**Fact (nontrivial):** There exists a map

$$H^i_{\text{ét}}(X/\overline{\mathbb{F}}_q; G) \to H^i(X(\mathbb{C}); G)$$

**Theorem (Artin)**

*Under the conditions above, $H^i_{\text{ét}}(X/\overline{\mathbb{F}}_q; G) \to H^i(X(\mathbb{C}); G)$ is an isomorphism.*

*Taking limits and tensoring with $\mathbb{Q}_\ell$, $H^i_{\text{ét}}(X/\overline{\mathbb{F}}_q; \mathbb{Q}_\ell) \to H^i(X(\mathbb{C}); \mathbb{Q}_\ell)$ is an isomorphism as well.*
Let $Y$ be a compact topological space and $f : Y \to Y$.

**Theorem (Lefschetz Fixed-Point)**

$$\# \text{Fix}(f : Y \to Y) = \sum_{i \geq 0} (-1)^i \text{tr}(f^* : H^i(Y, \mathbb{Q}))$$

Grothendieck: apply this to $\text{Frob}_q : X_{/\overline{\mathbb{F}}_q} \to X_{/\overline{\mathbb{F}}_q}$ via étale cohomology.

**Recall:** $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is generated by $\text{Frob}_q$. $\implies |X(\mathbb{F}_q)| = \# \text{Fix}(\text{Frob}_q)$
**Theorem (Grothendieck-Lefschetz; [Gro77])**

*For any smooth projective variety $X$ over $\mathbb{F}_q$,*

$$|X(\mathbb{F}_q)| = \# \text{Fix}(\text{Frob}_q) = \sum_{i \geq 0} (-1)^i \text{tr}(\text{Frob}_q : H^i_{\text{ét}}(X; \mathbb{Q}_\ell))$$

*If $X$ is smooth but not projective, Poincaré duality implies*

$$|X(\mathbb{F}_q)| = q^{\dim X} \sum_{i \geq 0} (-1)^i \text{tr}(\text{Frob}_q : H^i_{\text{ét}}(X; \mathbb{Q}_\ell)^*)$$
**Grothendieck-Lefschetz: An Example**

**Theorem (Grothendieck-Lefschetz (Non-Projective))**

\[ |X(\mathbb{F}_q)| = q^{\dim X} \sum_{i \geq 0} (-1)^i \text{tr}(\text{Frob}_q : H^i_{\text{ét}}(X; \mathbb{Q}_\ell)^*) \]

**Example (|Conf_n(\mathbb{F}_q)|)**

**Fact:** \( \text{Frob}_q \) acts on \( H^i_{\text{ét}}(\text{Conf}_n; \mathbb{Q}_\ell) \) by multiplication by \( q^i \) and hence on \( H^i_{\text{ét}}(\text{Conf}_n; \mathbb{Q}_\ell)^* \) by \( q^{-i} \)

**Arnold:** \( H^i(\text{Conf}_n(\mathbb{C}); \mathbb{C}) = \mathbb{C} \) when \( i = 0, 1 \) and 0 otherwise

\[ \implies \text{tr}(\text{Frob}_q : H^i_{\text{ét}}(\text{Conf}_n; \mathbb{Q}_\ell)^*) = \begin{cases} 1 & i = 0 \\ q^{-1} & i = 1 \end{cases} \]

\[ \implies |\text{Conf}_n(\mathbb{F}_q)| = q^n(1 - q^{-1}) = q^n - q^{n-1} \]
Étale Fundamental Group

**Classical:** For $x \in X$, let $\text{Fib}_x$ be the functor $\text{Cov}(X) \to \text{Set}$ with $\text{Fib}_x(Y)$ defined for $\pi : Y \to X$ as $\pi^{-1}(x)$.

**Fact:** $\pi_1(X)$ acts transitively and faithfully on $\text{Fib}_x$ by *monodromy action*.

**Definition**

$$\pi_1^{\text{et}}(X, x) := \text{Aut}_{\text{Set}^{\text{G}_{\text{et}}}(X)}(\text{Fib}_{x}^{\text{et}}).$$
**Local Systems & $\ell$-adic Sheaves**

**Recall:** A local system (classically) is a locally constant sheaf of Abelian groups.

For $x \in X$, $L$ is an $\text{Aut}(A)$-local system if $L_x \cong A$.

**Fact:** There is an equivalence of categories between $\text{Aut}(A)$-local systems and representations $\pi_1(X) \to \text{Aut}(A)$.

**Definition**

For $G$ a topological group, an étale $G$-local system is a representation $\pi_1^{\text{et}}(X, x) \to G$.

An $\ell$-adic sheaf is an étale $\text{GL}_n(\overline{\mathbb{Q}}_\ell)$-local system.
The Analogy:

**Classically:** $GL_n(F)$-local systems $\leftrightarrow$ $F$-vector bundles with flat connection

**Étale:** $\ell$-adic sheaves are “like” vector bundles with flat connection.
Representation Stability over Finite Fields

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Introduction & Motivation

Ingredients: 
Étale Homotopy Theory

Ingredients: 
Grothendieck-Lefschetz

Ingredients: 
FI-Modules & Representation Stability

Statistics on \( \text{Conf}_n(\mathbb{F}_q) \) and the Braid Group
Theorem (G-L With Twisted Coefficients)

For an \(\ell\)-adic sheaf \(\mathcal{F}\) on projective \(X\),

\[
\sum_{x \in X(\mathbb{F}_q)} \text{tr}(\text{Frob}_q \mid \mathcal{F}_x) = \sum_i (-1)^i \text{tr}(\text{Frob}_q : H^i_{\text{ét}}(X; \mathcal{F})).
\]

In the non-projective case:

\[
\sum_{x \in X(\mathbb{F}_q)} \text{tr}(\text{Frob}_q \mid \mathcal{F}_x) = q^{\dim X} \sum_i (-1)^i \text{tr}(\text{Frob}_q : H^i_{\text{ét}}(X; \mathcal{F})^*).
\]
**Definition**

Let $\mathbf{FI}$ denote the category with objects $\text{Finite sets}$ and morphisms $\text{Injections}$. $\mathbf{FI}$ is equivalent to its skeletal subcategory with objects $\mathbf{n} := \{1, \ldots, n\}$.
**Definition**

- A FI-module over a commutative ring $R$ is a functor $V : \text{FI} \to \text{Mod}_R$. We denote $V(n) =: V_n$.

- More generally, a FI-[object] (resp. $\text{FI}^{\text{op}}$-[object]) $W$ is a functor $W : \text{FI} \to [\text{objects}]$ (resp. $\text{FI}^{\text{op}}$).

**Remark**

$\text{End}_{\text{FI}}(n) = S_n$. Hence, a FI-module defines a sequence of $S_n$ representations in a “coherent” manner.
**Example**

\[ V : \text{Fl} \to \text{Vect}_\mathbb{R} \text{ with } V_n = \langle x_1, \ldots, x_n \rangle \text{ and } V(\sigma) : x_k \mapsto x_{\sigma(k)} \text{ is a FI-Module.} \]

**Example**

\[ C : \text{Fl}^{\text{op}} \to \text{Top} \text{ with } C_n = \text{PConf}_n(\mathbb{C}) \text{ and for } \sigma : m \hookrightarrow n \]

\[ C(\sigma) : (z_1, \ldots, z_n) \mapsto (z_{i(1)}, \ldots, z_{i(m)}) \text{ is a FI}^{\text{op}}\text{-space.} \]

It follows that \( H^i \circ C \) is a FI-module!
**Definition**
A FI-module $V$ is finitely generated if there are finitely many elements $x_1, \ldots, x_n \in \bigcup_{i \geq 0} V_n$ such that each $V_n$ is generated by FI-images of the $x_i$.

**Remark**
Each of our examples are finitely-generated!


**Character Polynomials**

Recall that the conjugacy class of $\sigma \in S_n$ is determined by $(c_1(\sigma), c_2(\sigma), \ldots)$ where $c_i(\sigma) := \#i$-cycles in $\sigma$.

**Definition**

- A character polynomial $P$ is an element of the ring $\mathbb{Q}[X_1, X_2, \ldots]$ graded by $|x_i| = i$.

- We think of $P$ as giving a sequence of $S_n$-characters!
A sequence of $S_n$-characters $\{\chi_n\}$ is *given by the character polynomial* $P$ if for $\sigma \in S_n$, $\chi_n(\sigma)$ coincides with the class function $P_n : S_n \rightarrow \mathbb{Q}$ defined by $P_n(\sigma) = P(c_1(\sigma), c_2(\sigma), \ldots)$
Character Polynomials

A sequence of $S_n$-characters $\{\chi_n\}$ is \textit{eventually given by the character polynomial} $P$ if there exists $N$ such that for $n > N$ and $\sigma \in S_n$, $\chi_n(\sigma)$ coincides with the class function $P_n : S_n \to \mathbb{Q}$ defined by $P_n(\sigma) = P(c_1(\sigma), c_2(\sigma), \ldots)$.
Two Theorems on Character Polynomials

Theorem ([CEF14, 3.9])

*Given two character polynomials \( P, Q \in \mathbb{Q}[X_1, \ldots] \), \( \langle P_n, Q_n \rangle_{S_n} \) is independent of \( n \) when \( n \geq \deg P + \deg Q \).*

We denote \( \langle P, Q \rangle := \lim_{n \to \infty} \langle P_n, Q_n \rangle_{S_n} \).

Theorem ([CEF15, 3.3.4])

*Let \( V \) be a finitely generated FI-module over a field of characteristic zero and let \( \chi_V = \{\chi_n\} \) be its sequence of characters. \( \chi_V \) is eventually given by a unique character polynomial \( P_V \).*
Statistics on $\text{Conf}_n(\mathbb{F}_q)$

- Our focus: statistics depending on the length of irreducible factors in elements of $\text{Conf}_n(\mathbb{F}_q)$.
- Let $\chi : S^n \to \mathbb{Q}$ be a class function and $f \in \text{Conf}_n(\mathbb{F}_q)$. Let $R(f) = \{x \in \overline{\mathbb{F}}_q : f(x) = 0\}$.
- $\text{Frob}_q$ induces a permutation $\sigma_f$ on $R(f)$ (defined up to conjugacy).
- $\chi(f) := \chi(\sigma_f)$. 
Theorem ([CEF14, 3.7])

Let \( \chi \) be any class function \( S_n \to \mathbb{Q} \). Then,

\[
\sum_{f \in \text{Conf}_n(\mathbb{F}_q)} \chi(f) = \sum_{i} (-1)^i q^{n-i} \langle \chi, H^i(\text{PConf}_n(\mathbb{C})) \rangle
\]
Sketch.

- Restrict focus to $\chi$ irreducible
- $\text{PConf}_n \to \text{Conf}_n$ is a Galois cover with deck group $S_n$.
- (Non-trivial) yields $[\text{f.d. representations of } S_n] \cong [\text{f.d. local systems on } \text{Conf}_n \text{ trivial on } \text{PConf}_n]$
- Let $V$ correspond to $\chi$ and $\mathcal{V}$ be the corresponding local system.
- \[ \sum_{f \in \text{Conf}_n(\mathbb{F}_q)} \text{tr}(\text{Frob}_q : V_f) = \sum_{f \in \text{Conf}_n(\mathbb{F}_q)} \chi(f) \]
  
  \[ = q^n \sum_j (-1)^j \text{tr}(\text{Frob}_q : H^j_{\text{ét}}(\text{Conf}_n; \mathcal{V})^*) \]
Sketch (Cont.)

- Know Frob_q acts on $H^j_{\text{ét}}(\text{Conf}_n; \mathcal{V})^*$ by $q^{-i}$; just need $\dim_{\mathbb{Q}_\ell} H^j_{\text{ét}}(\text{Conf}_n; \mathcal{V})^*$. 
- Pull back $\mathcal{V}$ to $\tilde{\mathcal{V}}$ on $P\text{Conf}_n$

\[
H^j_{\text{ét}}(\text{Conf}_n; \mathcal{V})^* \cong (H^j_{\text{ét}}(P\text{Conf}_n; \tilde{\mathcal{V}}))^* S_n \\
\cong (H^j_{\text{ét}}(P\text{Conf}_n; \mathbb{Q}_\ell)^* \otimes \mathcal{V})^* S_n \\
\cong H^j_{\text{ét}}(P\text{Conf}_n; \mathbb{Q}_\ell)^* \otimes_{\mathbb{Q}_\ell[S_n]} \mathcal{V}
\]

- Rep. theory: $\dim(H^j_{\text{ét}}(P\text{Conf}_n; \mathbb{Q}_\ell)^* \otimes_{\mathbb{Q}_\ell[S_n]} \mathcal{V}) = \langle \chi, H^j_{\text{ét}}(P\text{Conf}_n; \mathbb{Q}_\ell) \rangle$

- Fact: Artin’s comparison map is an isomorphism of $S_n$-representations.
Let $P$ be a character polynomial and denote by
$$\langle P, H^i(\text{PConf}) \rangle = \lim_{n \to \infty} \langle P_n, H^i_{\text{et}}(\text{PConf}_n; \mathbb{Q}_\ell) \rangle.$$ 

**Theorem ([CEF14, 3.13])**

The following limit exists:
$$\lim_{n \to \infty} q^{-n} \sum_{f \in \text{Conf}_n(\mathbb{F}_q)} P(f) = \sum_{i=0}^{\infty} (-1)^i \frac{\langle P, H^i(\text{PConf}) \rangle}{q^i}.$$
The Braid Group

- $B_n$: the braid group on $n$ strands. $PB_n$: the pure braid group on $n$ strands
- $1 \to PB_n \to B_n \to S_n \to 1$ is exact
- **Recall:** $PConf_n(\mathbb{C})$ is a $K(\pi, 1)$ with $\pi = PB_n$. $Conf_n(\mathbb{C})$ is as well for $\pi = B_n$
- Thus, $H^i(PConf_n(\mathbb{C})) = H^i(PB_n)$. 
The Braid Group

Theorem ([CEF14, 4.1 & 4.3])

Let \( \chi \) be a \( S_n \)-character

\[
\sum_{f \in \text{Conf}_n(\mathbb{F}_q)} \chi(f) = \sum_{i} (-1)^i q^{n-i} \langle \chi, H^i(PB_n) \rangle
\]

Further, the inner product \( \langle P, H^i(PB_n) \rangle \) is independent of \( n \) for \( n \geq 2i + \deg P \) and

\[
\lim_{n \to \infty} q^{-n} \sum_{f \in \text{Conf}_n(\mathbb{F}_q)} P(f) = \sum_{i=0}^{\infty} (-1)^i \frac{\langle P, H^i(P\text{Conf}) \rangle}{q^i}
\]
**Example: Linear factors**

**Theorem ([CEF14, 4.4])**

The expected number of linear factors for a monic, squarefree degree-$n$ polynomial in $\mathbb{F}_q[t]$ approaches

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{q^i}$$

**Sketch.**

- Recall: $X_1(f) = c_1(\sigma_f)$ is the # of linear factors of $f$.
- **Fact:** when $i > 0$, $\langle X_1, H^i(P_n) \rangle$ is 0 when $n < i + 1$, 1 when $n = i + 1$ and 2 when $n > i + 1$.
- **Theorem** $\Rightarrow$

  $$\sum_{f \in \text{Conf}_n(\mathbb{F}_q)} X_1(f) = q^n - \frac{2}{q^{n-1}} + \frac{2}{q^{n-2}} + \cdots \pm 2q^2 \mp q$$

- Divide through by $|\text{Conf}_n(\mathbb{F}_q)| = q^n - q^{n-1}$, take a limit.
Thank you for listening!
References I

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References II
