Chain Rule for 2^{nd} Derivatives

$u = f[x(r,s), y(r,s)]$. Seek u_{rr}, u_{rs}, u_{ss}.

Solution

Let $A = A(x,y)$ be <u>any</u> function. By Chain Rule,

$$(*) \quad A_r = A_I x_r + A_{II} y_r = A_x x_r + A_y y_r$$

$$(**) \quad A_s = A_I x_s + A_{II} y_s = A_x x_s + A_y y_s \, .$$

We now use various choices for A.

$A = f = u$. Get:

$$u_r = f_I x_r + f_{II} y_r \qquad \text{by } (*) \, .$$

OK

Therefore,

$$(u_r)_r = (f_I x_r)_r + (f_{II} y_r)_r \qquad \text{addition rule}$$

$$= (f_I)_r x_r + f_I x_{rr} + (f_{II})_r y_r + f_{II} y_{rr} \qquad \text{prod rule}$$

say $A = f_I$ in $*$ say, $A = f_{II}$ in $*$

$$= [(f_I)_I x_r + (f_I)_{II} y_r] x_r + f_I x_{rr}$$
$$+ [(f_{II})_I x_r + (f_{II})_{II} y_r] y_r + f_{II} y_{rr}$$

$$= f_{II} x_r x_r + f_{I\,II} y_r x_r + f_I x_{rr}$$
$$+ f_{II\,I} x_r y_r + f_{II\,II} y_r y_r + f_{II} y_{rr}$$

{now clean this up}

$$u_{rr} = f_{\mathrm{I}\,\mathrm{I}}\, x_r^2 + 2 f_{\mathrm{I}\,\mathrm{II}}\, x_r y_r + f_{\mathrm{II}\,\mathrm{II}}\, y_r^2 + f_{\mathrm{I}}\, x_{rr} + f_{\mathrm{II}}\, y_{rr} \,.$$

OK

Next,

$$(u_r)_s = (f_{\mathrm{I}}\, x_r)_s + (f_{\mathrm{II}}\, y_r)_s$$ addition rule

$$= \underbrace{(f_{\mathrm{I}})_s}_{\text{say } A = f_{\mathrm{I}} \text{ in } **} x_r + f_{\mathrm{I}}\, x_{rs} + \underbrace{(f_{\mathrm{II}})_s}_{\text{say } A = f_{\mathrm{II}} \text{ in } **} y_r + f_{\mathrm{II}}\, y_{rs}$$ prod rule

$$= \left[(f_{\mathrm{I}})_{\mathrm{I}}\, x_s + (f_{\mathrm{I}})_{\mathrm{II}}\, y_s \right] x_r + f_{\mathrm{I}}\, x_{rs} + \left[(f_{\mathrm{II}})_{\mathrm{I}}\, x_s + (f_{\mathrm{II}})_{\mathrm{II}}\, y_s \right] y_r + f_{\mathrm{II}}\, y_{rs}$$

$$= f_{\mathrm{I}\,\mathrm{I}}\, x_s x_r + f_{\mathrm{I}\,\mathrm{II}}\, y_s x_r + f_{\mathrm{I}}\, x_{rs} + f_{\mathrm{II}\,\mathrm{I}}\, x_s y_r + f_{\mathrm{II}\,\mathrm{II}}\, y_s y_r + f_{\mathrm{II}}\, y_{rs}$$

{now clean this up}

$$u_{rs} = f_{\mathrm{I}\,\mathrm{I}}\, x_s x_r + f_{\mathrm{I}\,\mathrm{II}} (y_s x_r + x_s y_r) + f_{\mathrm{II}\,\mathrm{II}}\, y_s y_r + f_{\mathrm{I}}\, x_{rs} + f_{\mathrm{II}}\, y_{rs} \,.$$

OK

We must still do u_{ss}. This will be entirely similar to $(u_r)_r$, except switch role of r and s.

We thus get, by doing $(u_s)_s$,

$$\mathrm{I}\quad u_{ss} = f_{\mathrm{I}\mathrm{I}}\, x_s^2 + 2 f_{\mathrm{I}\mathrm{II}}\, x_s y_s + f_{\mathrm{II}\mathrm{II}}\, y_s^2 + f_{\mathrm{I}}\, x_{ss} + f_{\mathrm{II}}\, y_{ss}.$$

OK

Clearly, use of * and ** with various A is key. This procedure can be repeated for 3rd and higher derivatives of u. The answers become longer and longer.

NOTE: at the end, we typically go back and write

$f_{\mathrm{I}} = f_x$, $f_{\mathrm{II}} = f_y$ ‖ $f_{\mathrm{I}\mathrm{I}} = f_{xx}$, $f_{\mathrm{I}\mathrm{II}} = f_{xy}$, $f_{\mathrm{II}\mathrm{II}} = f_{yy}$.

E.G., we typically write

$$u_{rr} = \underline{f_{xx}}\, x_r^2 + 2 \underline{f_{xy}}\, x_r y_r + \underline{f_{yy}}\, y_r^2 + \underline{f_x}\, x_{rr} + \underline{f_y}\, y_{rr}.$$