2263 for Wed, Nov 27

Time is short. So I was going to (just) show 3 <u>examples</u> of surface integrals.

(I) Let $\mathcal{S}$ be the portion of the **cylinder** $x^2+y^2=4$ located between $z=0$ and $z=H$.

Calculate:

$$\iint_{\mathcal{S}} x^2y^2\,d\sigma \text{ (or } dS\text{)}.$$

Could be some MASS!

Sol.

Take parameters via $\left\{ \begin{array}{l} x = 2\cos\theta \\ y = 2\sin\theta \\ z = z \end{array} \right\}$ i.e. θ, z.

Have:

$$\vec{r} = \langle 2\cos\theta, 2\sin\theta, z\rangle \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le z \le H \end{array}.$$

Know

$$d\sigma = |\vec{r}_\theta \times \vec{r}_z|\,d\theta\,dz \qquad \text{\S 17}\ \boxed{6}.$$

Get:

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} i & j & k \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= i(2\cos\theta) - j(-2\sin\theta) + k(0)$$ ②

$$= \langle 2\cos\theta, 2\sin\theta, 0 \rangle$$

$$\Rightarrow |\vec{r}_\theta \times \vec{r}_z| = 2$$

$$\Rightarrow d\sigma = 2\, d\theta\, dz.$$

So, compare 1123 $\boxed{2}$

$$\text{integral} = \iint_S x^2 y^2 \underline{\underline{2\, d\theta\, dz}}$$

$$= \iint_{D_{\theta z}} (2\cos\theta)^2 (2\sin\theta)^2\, 2\, d\theta\, dz$$

$$= \int_0^{2\pi}\int_0^H 8\cdot 4\cos^2\theta \sin^2\theta\, dz\, d\theta$$

$$\{ \sin 2\theta = 2\sin\theta\cos\theta \}$$

$$= 8\int_0^{2\pi} \sin^2(2\theta)\, d\theta \cdot \int_0^H 1\, dz$$

$$= 8\int_0^{2\pi} \frac{1-\cos 4\theta}{2}\, d\theta \cdot H$$

$$= 8\pi H.$$

P Q R

II Let $\vec{F} = \langle y, 2x, z \rangle$. Let S be the portion of the cone $z = \sqrt{x^2+y^2}$ between $z = 0$ and $z = 1$. Compute the upward flux of $\vec{F}$ thru S without using "fancy" parameters.

Sol.

Just stay with $x, y,$ and $z = f(x,y) = \sqrt{x^2+y^2}$. See 1130 $\boxed{10}$. We take our unit normal to be:

$$\frac{-f_x \vec{i} - f_y \vec{j} + (1)\vec{k}}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

1127 $\boxed{5}$

There is thus a leading "+" in box $\boxed{10}$.

$z = 1 = \sqrt{x^2+y^2}$

$$\text{Flux} = + \iint_{D_{xy}} \Big[\overset{\boxed{P}}{y}(-f_x) + \overset{\boxed{Q}}{2x}(-f_y) + \overset{\boxed{R}}{z}(1) \Big] \, dx\,dy$$

$$\left\{ f_x = \frac{x}{\sqrt{x^2+y^2}}, \; f_y = \frac{y}{\sqrt{x^2+y^2}}, \; z = \sqrt{x^2+y^2} \right\}$$

$$= \iint_{\{x^2+y^2 \leq 1\}} \left[\frac{-3yx}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} \right] dx\,dy$$

{ this double integral is easy !!
we can use polar coords à la chap 15 }

$$= \int_0^1 \int_0^{2\pi} \left[\frac{-3r^2 \sin\theta \cos\theta}{r} + r \right] \underline{\underline{r}}\, d\theta\, dr$$

$$= [0] + \int_0^1 \int_0^{2\pi} r^2\, d\theta\, dr$$

THE FLUX

$$= 2\pi \left[\frac{r^3}{3} \right]_0^1 = \frac{2\pi}{3}.$$

III (Do) Same problem as II, but using some "more complicated" parameters.

CAUTION: In 1129 box $\boxed{9}$, he uses

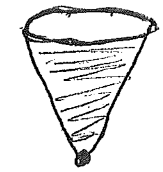
$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

IF THIS $\vec{n}$ does not point the desired way, you <u>flip</u> $\vec{n}$ to $-\vec{n}$.

IE use $-\iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\,dv$

Sol.

$z = \sqrt{x^2+y^2}$ suggests $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases}$



$\boxed{r, \theta = \text{parameters}}$

with $0 \le r \le 1$, $0 \le \theta \le 2\pi$. So!

$\vec{r} = \langle r\cos\theta, r\sin\theta, r \rangle$.

Let's see what $\dfrac{\vec{r}_\theta \times \vec{r}_r}{|\vec{r}_\theta \times \vec{r}_r|}$ is!!

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} i & j & k \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$$

$$= i(r\cos\theta) - j(-r\sin\theta) + k(-r\sin^2\theta - r\cos^2\theta)$$

$$\vec{r}_\theta \times \vec{r}_r = \langle r\cos\theta, r\sin\theta, -r \rangle$$

This vector is pointing in "<u>negative</u> z" direction. IE, so is

$$\frac{\vec{r}_\theta \times \vec{r}_r}{\underbrace{|\vec{r}_\theta \times \vec{r}_r|}_{r\sqrt{2}}}. \implies \text{WE MUST FLIP.}$$

We simply say to ourselves:

$$\text{Flux} = - \iint_{D_{r\theta}} \vec{F} \cdot (\vec{r}_\theta \times \vec{r}_r)\, dr\, d\theta .$$

KEY

Get:

$$\text{Flux} = - \iint_{D_{r\theta}} \langle y, 2x, z \rangle \cdot \langle r\cos\theta, r\sin\theta, -r \rangle\, dr\, d\theta$$

$$= \iint_{D_{r\theta}} \langle y, 2x, z \rangle \cdot \langle -r\cos\theta, -r\sin\theta, r \rangle\, dr\, d\theta$$

$$= \iint_{D_{r\theta}} \langle r\sin\theta, 2r\cos\theta, r \rangle \cdot \langle -r\cos\theta, -r\sin\theta, r \rangle \cdot dr\, d\theta$$

$$= \int_0^1 \int_0^{2\pi} \left[-3r^2 \sin\theta \cos\theta + r^2 \right] d\theta\, dr$$

$$= [0] + \int_0^1 \int_0^{2\pi} r^2\, d\theta\, dr$$

$$= 2\pi \left[\frac{r^3}{3} \right]_0^1 = \frac{2\pi}{3} .$$

THE FLUX

It is curious that our (final) integrals are nearly the same as line 4 on (4). This is often the case!! VERY REASSURING.