MATH 8280/5990 – SPRING 2016
INTROD. TO ANALYTIC NUMBER THEORY
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BRIEF TABLE OF CONTENTS

LECTURE 14: Classical "big oh" estimates for $\psi(x) - x$ and $\pi(x) - \mathrm{li}(x)$.

LECTURE 15: Preparations for deriving Riemann's explicit formula for $\psi_1(x)$; the sliding partial fraction development of $\zeta'(s)/\zeta(s)$; the function $S(T)$ and its use in counting (à la Riemann) the number of zeros ρ up to height T.

LECTURE 16: The explicit formula for $\psi_1(x)$; relation to the prime number theorem; standard estimates for $\psi(x) - x$ and $\pi(x) - \mathrm{li}(x)$ involving the "maximum abscissa" Θ.

LECTURES 17+18: The explicit formula for $\psi(x)$; applications to the size of $\psi(x) - x$.

LECTURES 19+20: The Perron summation formula with error term; the Möbius μ-function; an estimate for the summatory function $M(x)$; convergence of certain Dirichlet series involving $\mu(n)$; the Möbius inversion formula; elementary equivalence of $M(x) = o(x)$ and the prime number theorem.

LECTURE 21: Completion of the proof that PNT and $M(x) = o(x)$ are equivalent; the classical Dirichlet divisor problem bound; review of some basic properties of generalized Dirichlet series and "Dirichlet integrals"; Landau's theorem on singular points; elementary $\Omega_\pm$ estimates for $\psi(x) - x$ and $\pi(x) - \mathrm{li}(x)$ referencing $\Theta - \varepsilon$.

LECTURE 22: Preparations for Landau's proof of Hardy's theorem asserting an infinite number of zeros of $\zeta(s)$ along the critical line $\{\mathrm{Re}(s) = \frac{1}{2}\}$.

LECTURE 23: Landau's proof of the Hardy theorem. (Consult the Addendum in Lecture 24 for an improvement.)

LECTURE 24: Recollection of some complex function theory involving estimates of Phragmén-Lindelöf type; the Lindelöf μ-function for $\zeta(s)$; Littlewood's formula concerning the zero-counting function $N(\sigma; T_1; T_2)$.

LECTURE 25: More on Lindelöf's function $\mu(\sigma)$, also on generalized Dirichlet series; a quick rehash of Fourier transforms; development of a simple L_2 estimate for Dirichlet polynomials in the spirit of Hilbert's inequality.

LECTURE 26: Some van der Corput-type estimates of exponential sums; the 1921 Hardy-Littlewood theorem on approximating $\zeta(s)$ by its partial sums.

LECTURE 27: The Bohr-Landau theorem that all but an infinitesimal proportion of the zeros of $\zeta(s)$ are located within $\{|\mathrm{Re}(s) - \frac{1}{2}| < \varepsilon\}$.

LECTURE 28: D. J. Newman's quick proof of the PNT; attached, for reference, Landau's two-page 1932 *Göttinger Nachrichten* paper (giving the shortest known proof of the Wiener-Ikehara Tauberian theorem).

LECTURE 29: J. E. Littlewood's 1914 $\Omega_\pm$ estimate for $\psi(x) - x$ proved using a variant of Ingham's 1936 Fourier transform technique.

LECTURE 30: Connecting Euler's 1748 assertion that $\sum \frac{\mu(n)}{n} = 0$ to $M(x) = o(x)$ and the prime number theorem; some concluding remarks on $S(T)$ and Turing's Law (for checking the Riemann Hypothesis without ever leaving the line $\mathrm{Re}(s) = \frac{1}{2}$).

ADDENDUM A: Pertaining to Lecture 30 ("an alternate ending").

ADDENDUM B: Pertaining to Lecture 28.

CLOSING COMMENTS: Assorted remarks concerning Lectures 1–30.