Math 8280/5990

Lecture 1 (20 Jan 2016)

These notes will be a kind of loose/informal diary of what was discussed in the lecture. They are <u>not</u> a textbook, nor will they strive for "completeness".

Analytic Number Theory = that part of number theory wherein results are obtained by use of functions and appropriate <u>analytic</u> techniques.

Traditionally, the "analytic techniques" have come from <u>complex analysis</u> (i.e., the theory of analytic functions). But, this focus has gradually been widened.

<u>Multiplicative A.N.T.</u> (loosely speaking) deals with aspects/functions which are intimately tied to prime numbers and the unique factorization property of the positive integers.

The first 2 lectures will use elementary analysis combined with arithmetic to get some nontrivial information about primes.

I assume the unique factorization thm as being known: i.e.,

$$n = p_1^{e_1} \cdots p_r^{e_r},$$

uniquely with primes $p_1 < \cdots < p_r$ and $e_j \geq 1$.

__Theorem 1 (Euclid)__

The number of primes is infinite.

__PF__

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Assume not. Let the full list be $p_1 < \cdots < p_m$.
Form $N = p_1 \cdots p_m + 1$ = integer. Factor N.
Therefore, some p_ℓ divides N. But,

$$\frac{N}{p_\ell} = p_1 \cdots \hat{p}_\ell \cdots p_m + \frac{1}{p_\ell}.$$

$\boxed{\wedge \text{ means omit}}$

This is not an integer (since $p_1 = 2$).
Contrad! ■

<u>Def:</u>

$$\pi(x) = N[p : p \leq x] \quad \text{for } x > 0.$$

By thm 1, $\pi(x) \to \infty$ as $x \to \infty$.

The idea of the proof of thm 1 can be used to prove $\pi(x) \geq \ln \ln x$. FOR $x \geq 2$ See Hardy–Wright, thm 10. This is <u>not</u> very interesting.

<u>Notation:</u>

$$f(t) = O[g(t)] \quad \text{as } t \to \infty$$

means $|f(t)| \leq M|g(t)|$ for $t \geq t_0$, <u>some</u> large t_0. But:

$$f(t) = O[g(t)] \quad \text{for } t \geq Q$$

means $|f(t)| \leq \mathfrak{M}|g(t)|$ for all $t \geq Q$.

In most situations, one can "get" the 2nd version merely by inflating M appropriately. This is why I wrote $\mathfrak{M}$.

Def:

$$\theta(x) = \sum_{p \leq x} \ln p \ , \quad \psi(x) = \sum_{p^m \leq x} \ln p \qquad (x > 0).$$

Empty sums are defined to be 0; hence $\theta(x) = \psi(x) = 0$ for $x < 2$.

$\boxed{\text{Notation used by Chebyshev!}}$

Obviously

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \cdots ,$$

where eventually $\theta(x^{1/N}) = 0$.

Thm 2

$$\psi(x) = \sum_{p \leq x} \left[\!\left[\frac{\ln x}{\ln p} \right]\!\right] \ln p \qquad (x \geq 2).$$

For $x < 2$, both sides are 0.

Proof

Choose any prime $p \leq x$. We seek its total contribution to the def. of $\psi(x)$. It will contribute a $\ln p$ so long as $p^m \leq x$, i.e $m \leq \frac{\ln x}{\ln p}$, i.e. $m \leq \left[\!\left[\frac{\ln x}{\ln p} \right]\!\right]$. Thus, we collectively get $\left[\!\left[\frac{\ln x}{\ln p}\right]\!\right] \ln p$. Now, just add over all relevant p. ∎

Thm 3

(a) $\psi(x) = \theta(x) + O[x^{\frac{1}{2}}(\ln x)^2]$, $x \geqq 2$;

(b) we get $\psi(x) = \theta(x) + O(x^{1/2})$ if we can somehow prove that $\theta(y) = O(y)$ for all $y \geqq 1$.

Pf

Need to estimate $\sum_{n=2}^{\infty} \theta(x^{1/n})$. But $\theta(x^{\frac{1}{n}}) = 0$ once $x^{1/n} < 2$, IE $n > \frac{\ln x}{\ln 2}$. Get:

$$\sum_{n=2}^{\infty} \theta(x^{1/n}) = \sum_{n=2}^{L} \theta(x^{1/n}), \quad L = \left[\frac{\ln x}{\ln 2}\right] + 10$$

$$\leqq (L-1)\,\theta(x^{1/2}) \leqq L \sum_{p \leqq x^{1/2}} \ln p$$

$$\leqq L \ln x \sum_{p \leqq x^{1/2}} 1 \leqq L \ln x \sum_{n \leqq x^{1/2}} 1$$

$$= O[x^{\frac{1}{2}}(\ln x)^2] \quad \{\text{very crudely}\}.$$

This proves (a). For (b), we need to look at $\sum_{n=3}^{\infty} \theta(x^{1/n}) = \sum_{n=3}^{L} \theta(x^{1/n})$.

A trivial modification shows that this sum is $O[x^{\frac{1}{3}}(\ln x)^2]$. Hence,

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + O[x^{\frac{1}{3}}(\ln x)^2]$$
$$= \theta(x) + O(x^{1/2})$$

by the hypothesis about $\theta(y)$. ▨

Thm 4 (Legendre – early 1800s)

Given $m \geqq 2$. We then have

$$m! = \prod p^{E_p}, \quad E_p = \sum_{j=1}^{\infty} [\![\frac{m}{p^j}]\!].$$

Proof

We use "baby set theory" (i.e., classes).

Write $m! = \prod_{k=1}^{m} k$. The only primes that "go into" k are clearly $\leqq m$.

With this being the case, temporarily fix any prime $p \leqq m$. Choose L so that $p^L \leqq m < p^{L+1}$.

Say that $k \in [1, m]$ is of "type j" when

$$k = p^j (\text{integer relatively prime to } p).$$

Here $j \geqq 0$. THINK UNIQUE FACTORIZATION THM.

Let $v_j = \operatorname{card}\{k \in [1, m] : k \text{ is of type } j\}$.

Notice that

$$\begin{aligned} v_0 + v_1 + v_2 + \cdots + v_L &= m && \text{type } 0, \ldots, L \\ v_1 + v_2 + \cdots + v_L &= \left[\!\left[\tfrac{m}{p}\right]\!\right] && 1, \ldots, L \\ v_2 + \cdots + v_L &= \left[\!\left[\tfrac{m}{p^2}\right]\!\right] && 2, \ldots, L \\ &\vdots \\ v_L &= \left[\!\left[\tfrac{m}{p^L}\right]\!\right]. && L \end{aligned}$$

Delete the first row and (then) add! Get:

$$v_1 + 2v_2 + 3v_3 + \cdots + Lv_L = \sum_{j=1}^{L} \left[\!\left[\tfrac{m}{p^j}\right]\!\right].$$

But, by construction, we clearly have

$$E_p = v_1 + 2v_2 + \cdots + Lv_L.$$

(Just think about this a second!)

Now, just slap together all the p^{E_p}. ∎

clearly any primes $> m$ get $E_p = 0$, exactly as they should

Chebyshev became interested in quotients of various factorials which turn out to be integers.

EG, the binomial coefficient $\binom{2n}{n} = \frac{(2n)!}{n!n!}$.

He sought to use standard Stirling-type estimates to get information about the number of primes in certain intervals.

Clearly, thm 4 p. 6 will be vital here.

We note:

BABY LEMMA

Let x be a positive rational (say, $\frac{m}{n}$ in lowest terms). The relation

$$x = p_1^{A_1} \cdots p_r^{A_r} = p_1^{B_1} \cdots p_r^{B_r} \qquad (p_1 < \cdots < p_r)$$

holds with $A_j \in \mathbb{Z}$ and $B_j \in \mathbb{Z}$ only if $A_j = B_j$. If $x \in \mathbb{Z}^+$ (i.e., $n = 1$), each A_j is necessarily $\geqq 0$.

Pf

Let G be giant. Notice that

$$(p_1 \cdots p_r)^G p_1^{A_1} \cdots p_r^{A_r} = (p_1 \cdots p_r)^G p_1^{B_1} \cdots p_r^{B_r} = \text{integer}.$$

By unique factorization thm, $G + A_j = G + B_j$. ■

The Baby Lemma says that unique factorization extends to $\mathbb{Q}^+$ in an obvious way.

It *also* implies that the numbers $\ln p_j$ are linearly independent over $\mathbb{Q}$.

Thm 5 (elementary integral calculus)

$m \geqq 2 \Rightarrow$

$$\ln(m!) = m \ln m - m + O(\ln m).$$

Pf

Suppose $y = f(x)$ is continuous, non-neg, and increasing on $1 \leqq x \leqq N+1$. By drawing a picture, clearly

$$f(1) + \cdots + f(N) \leqq \int_1^{N+1} f(t)\,dt \leqq f(2) + \cdots + f(N+1)$$

$$\Rightarrow \quad 0 \leqq \int_1^{N+1} f(t)\,dt - \sum_{j=1}^{N} f(j) \leqq f(N+1) - f(1).$$

Simply put $f(t) = \ln t$ to get:

$$0 \leq \int_1^{N+1} \ln t \, dt - \sum_{j=1}^{N} \ln j \leq \ln(N+1)$$

$$\left\{ \text{but } \int_1^x \ln t \, dt = x \ln x - x + 1 \, , \ x \geq 1 \right\}$$

$$\Downarrow$$

$$0 \leq (N+1)\ln(N+1) - N - \ln(N!) \leq \ln(N+1)$$

$$\Downarrow$$

$$(N+1)\ln(N+1) - N - \ln(N!) = \omega \ln(N+1)$$

for some $\omega \in [0,1]$

$$\Downarrow$$

$$\begin{aligned} \ln(N!) &= (N+1)\ln(N+1) - N - \omega \ln(N+1) \\ &= N\ln(N+1) + (1-\omega)\ln(N+1) - N \\ &= N\left[\ln N + \ln\left(1+\tfrac{1}{N}\right)\right] + O(\ln N) - N \\ &= N \ln N + O(1) + O(\ln N) - N \\ &= N \ln N - N + O(\ln N) \, . \quad \blacksquare \end{aligned}$$

Let

$$L(x) = \sum_{k \leqq x} \ln k \quad \text{for } x > 0 .$$

For $x < 2$, $L(x) = 0$. For $x \geqq 2$, $L(x) = \ln(\llbracket x \rrbracket !)$.

Thm 6

$$L(x) = x \ln x - x + O(\ln x) \quad \text{for } x \geqq 2 .$$

Pf

By inflating the constant, WLOG $x \geqq 100$.

Let $M \leqq x < M+1$. By Thm 5,

$$L(x) = L(M) = M \ln M - M + O(\ln M) .$$

The derivative of $t \ln t - t$ is $\ln t$. Hence $x \ln x - x$ differs from $M \ln M - M$ by at most $\ln(M+1)$. As such,

$$\begin{aligned} L(x) &= x \ln x - x + O(\ln(M+1)) + O(\ln M) \\ &= x \ln x - x + O(\ln x) . \quad \blacksquare \end{aligned}$$

Thm 7

Define (following von Mangoldt):

$$\Lambda(n) = \begin{cases} \ln p, & n = p^j \ (j \geq 1) \\ 0, & \text{otherwise} \end{cases} .$$

Here $n \geq 1$. We have:

(a) $\psi(x) = \sum_{n \leq x} \Lambda(n)$;

(b) $L(x) = \sum_{k \leq x} [\![\frac{x}{k}]\!] \Lambda(k)$; ← for $\log [\![x]\!]!$

(c) $L(x) = \psi(x) + \psi(\frac{x}{2}) + \psi(\frac{x}{3}) + \cdots$.

Proof

When $x < 2$, each of (a)(b)(c) just states $0 = 0$.
So, WLOG, $x \geq 2$.

Assertion (a) is now a tautology. OK

For (b), write $M \leq x < M+1$. By 11 (top), obviously

$$L(x) = L(M) = \log(M!) .$$

By Thm 4, Legendre-style,

$$L(M) = \sum E_p \ln p = \sum_p \left(\sum_{j=1}^{\infty} [\![\frac{M}{p^j}]\!] \right) \ln p$$

$$= \sum_{\text{all } k \leq M} [\![\frac{M}{k}]\!] \Lambda(k) .$$

The key issue now boils down to:

$$\text{if } p^j \leqq M \ (j \geqq 1), \ \underline{\text{why}} \ \underline{\text{is}} \ [\![\tfrac{x}{p^j}]\!] = [\![\tfrac{M}{p^j}]\!] \ ?$$

To verify this, let

$$\ell = [\![\tfrac{M}{p^j}]\!] \quad (\geqq 1).$$

We have

$$\frac{M}{p^j} = \ell + \phi \quad \text{with} \quad 0 \leqq \phi < 1.$$

Write $M = p^j(\text{integer}) + r$, $0 \leqq r \leqq p^j - 1$ à la Euclid.
Clearly,

$$0 \leqq \phi \leqq \frac{p^j - 1}{p^j}.$$

Also write $x = M + \theta$, $0 \leqq \theta < 1$. We then have:

$$\ell \leqq \frac{x}{p^j} = \frac{M+\theta}{p^j} = \frac{M}{p^j} + \frac{\theta}{p^j} = \ell + \phi + \frac{\theta}{p^j}$$

$$< \ell + \frac{p^j - 1}{p^j} + \frac{1}{p^j} = \ell + 1,$$

which gives $[\![\frac{x}{p^j}]\!] = \ell = [\![\frac{M}{p^j}]\!]$, as desired.
Accordingly:

$$L(x) = \sum_{k \leqq x} [\![\tfrac{x}{k}]\!] \Lambda(k).$$

OK

To prove (c), we start with $\sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$. cf. (a)

Take any *given* integer p^A with $A \geqq 1$. We ask: how many times does $\Lambda(p^A)$ assert "I am present" within $\sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$? This number will clearly be the largest ℓ so that $p^A \leqq \frac{x}{\ell}$. In other words, $\ell \leqq \frac{x}{p^A}$ or $\ell \leqq [\![\frac{x}{p^A}]\!]$. The collective contribution of p^A to $\sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$ will therefore be $[\![\frac{x}{p^A}]\!] \Lambda(p^A)$.

In view of (b), it is now evident that $\sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$ must reduce to $L(x)$. OK ∎

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about 1850

Chebyshev played with $\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$ and was therefore motivated to examine

$$L(x) - 2L\left(\frac{x}{2}\right).$$

N.B. Theorem 7(b) applies to $L(x) - 2L\left(\frac{x}{2}\right)$, but it is *easier* to use Thm 7(c).

Suppose for a moment that $x \geqq 4$. A quick calculation with Thm 6 gives

$$L(x) - 2L\left(\frac{x}{2}\right) = x(\ln 2) + O(\ln x).$$

By inflating the constant à la 3 (bottom), the same relation holds for $\underline{x \geqq 2}$.

On the other hand, by Thm 7 (c),

$$L(x) - 2L\left(\frac{x}{2}\right) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \psi\left(\frac{x}{4}\right) \pm \cdots$$

(where, as usual, the terms are eventually 0).

We therefore have:

$$x(\ln 2) + O(\ln x) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \psi\left(\frac{x}{4}\right) \pm \cdots$$

for all $x \geqq 2$.

THIS LAST RELATION CAN BE MANIPULATED!

Recall that $\psi(y)$ is non-neg and monotonically increasing. See Thm 7 (a).

Accordingly:

$$x(\ln 2) + O(\ln x) = \psi(x) - \left[\psi\left(\tfrac{x}{2}\right) - \psi\left(\tfrac{x}{3}\right)\right] - \cdots$$

$$\Downarrow$$

$$\psi(x) \geqq x(\ln 2) + O(\ln x). \qquad \blacksquare$$

At the same time,

$$x \ln 2 + O(\ln x) = \left[\psi(x) - \psi\left(\tfrac{x}{2}\right)\right] + \left[\psi\left(\tfrac{x}{3}\right) - \psi\left(\tfrac{x}{4}\right)\right] + \cdots$$

$$\Downarrow$$

$$\psi(x) - \psi\left(\tfrac{x}{2}\right) \leqq x \ln 2 + O(\ln x). \qquad \blacksquare$$

In both instances, one keeps $x \geqq 2$.

Take $x \geqq 1000$ (say) and make an iteration as follows:

$$\psi(x) - \psi\left(\tfrac{x}{2}\right) \lessapprox x \ln 2 + B \ln x \qquad \text{step } 1$$

$$\psi\left(\tfrac{x}{2}\right) - \psi\left(\tfrac{x}{4}\right) \lessapprox \tfrac{x}{2} \ln 2 + B \ln \tfrac{x}{2} \qquad \text{step } 2$$

$$\vdots$$

$$\psi\left(\tfrac{x}{2^r}\right) - \psi\left(\tfrac{x}{2^{r+1}}\right) \lessapprox \tfrac{x}{2^r} \ln 2 + B \ln \tfrac{x}{2^r} \qquad \text{step } r+1$$

$$\left\{ \begin{array}{l} \text{take } \tfrac{x}{2^r} \in [4, 8] \text{ for safety} \\ \text{hence } r = \dfrac{\ln x - \Omega}{\ln 2}, \quad \ln 4 \leqq \Omega \leqq \ln 8 \end{array} \right\}$$

ADD

$$\Downarrow$$

$$\psi(x) + O(1) \lessapprox 2x \ln 2 + \overset{B(r+1)\ln x}{\cancel{Br\ln x}}$$

$$\Downarrow$$

$$\psi(x) \lessapprox x(\ln 4) + \frac{B}{\ln 2}(\ln x)^2$$

$$\Downarrow$$

$$\psi(x) \leqq x(\ln 4) + O[\ln^2 x] .$$

To include $2 \leqq x < 1000$, one can inflate the constant.

Theorem A (Chebyshev $\approx$ 1850)

For $x \geqq 2$, we have:

$$x(\ln 2) + O(\log x) \leqq \psi(x) \leqq x(\ln 4) + O(\log^2 x) ;$$

$$x(\ln 2) + O(x^{1/2}) \leqq \theta(x) \leqq x(\ln 4) + O(x^{1/2}) .$$

Proof

The case of $\psi(x)$ was just done. OK

Since $0 \leqq \theta(y) \leqq \psi(y)$, clearly $\theta(y) = O(y)$ for all $y \geqq 1$. Recalling Thm 3(b), we have

$$\theta(x) = \psi(x) + O(x^{1/2}) ,$$

which then produces the inequality for $\theta(x)$. OK

Notice that

$$\pi(x) \ln x = \sum_{p \leq x} \ln x \gtrsim \sum_{p \leq x} \ln p = \theta(x) .$$

In light of this, theorem A assures us that

$$\pi(x) \gtrsim (\ln 2 - \varepsilon) \frac{x}{\ln x}$$

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for $x \gtrsim x_0(\varepsilon)$. Here $\varepsilon > 0$ is arbitrary.

(p. 3 line 5 is trivial in comparison.)

The celebrated <u>Prime Number Theorem</u>, which we seek to prove soon, states that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty .$$

We'll say more about $\pi(x)$ in Lecture 2.