Lecture 12
(26 Feb)

2 Notes

Lec 10 p. 42 Stirling (Corollary).

We also have:

**Thm (Stirling)**

\[
\frac{r'(z)}{r(z)} = \log z - \frac{1}{2z} + \sum_{k=1}^{R} \frac{(-B_{2k}}{2k}) \frac{z^{-2k}}{z^{2k}} + O_{R} \left( \frac{1}{|z|^{2R+1}} \right)
\]

as \( z \to \infty \) in \( \text{Arg } z \leq \pi - \delta \).

**PF**

Call all the \( \overline{z R + t} \) integral term in (42) Thm \( r(z) \).
Note \( r(z) \) is nicely analytic and \( r(z) = O(z^{-2R-1}) \)
by the Cor on (42). But:

\[
r'(z) = \frac{1}{2\pi i} \oint_{|w-z|=1} \frac{r(w)}{(w-z)^2} dw.
\]

Just use \( \text{Arg } z \leq \pi - 2\delta \) in place of \( \text{Arg } z \leq \pi - \delta \).

Done. \( \square \)

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About \( r'(z) \neq 0 \), Lec 10 p. 26. One can avoid Hurwitz's thm.

**Thm**

Let \( f_n(z) \to f(z) \) on \( |z|<R \) compacta. We assume
\( f_n, f \) are analytic. Let \( f_n(z) \neq 0 \) for all \( z \) and
\( f(z) \neq 0 \). Then: \( f(z) \neq 0 \).
Zeros of $f$ are isolated. Hence finite # on each $|z| \leq R - \varepsilon$.

Find $R_n \uparrow R$ so $f(R_n e^{i\theta}) \neq 0$.

Fix any $N$. Find $m, M > 0$ so $m \leq |f(R_n e^{i\theta})| \leq M$.

By uniform convergence,

$$\frac{m}{2} \leq |f_n(R_n e^{i\theta})| \leq 2M, \quad n \geq N.$$

Apply max mod principle to $f_n$ AND $1/f_n$. Get

$$|f_n(z)| \leq 2M, \quad n \geq N.$$

$$|\frac{1}{f_n(z)}| \leq \frac{2}{m}$$

on $|z| \leq R_N$. So,

$$\frac{m}{2} \leq |f_n| \leq 2M.$$

Let $N \to \infty$ to get $\frac{m}{2} \leq |f| \leq 2M$ on $|z| \leq R_N$.

Now let $N \to \infty$. Done. 

END OF NOTES

We then turned to the issue of the entire $f(s)$

$$\zeta_0(s) = s(s-1) \pi^{-5/2} \Gamma(\frac{5}{2}) \Gamma(s)$$

$$\zeta_0(s) = \zeta_0(1-s)$$

order 1, type $\infty$

and trying to get a product expansion over the zeros $\zeta_0$ trying to get a "Hadamard factorization" of $\zeta_0$ to justify Riemann's
unproved assertion. [from 1859]

Standard lemmas.

Lemma 1

$D = \text{simply-connected domain}$

Let $f = u + iv$ be analytic on $D_0$.
Then $u$ is harmonic on $D$ (i.e. $C^2$ and $u_{xx} + u_{yy} = 0$).

Conversely, given real-valued $u$ harmonic on $D_0$, we can cook up $v$, harmonic on $D_0$, so $F = u + iv$ is analytic on $D_0$.

Cor

Every harmonic $u$ on $D$ is actually $C^\infty$.

Lemma 2 (mean-value property)

Let $u$ be harmonic on $D$ (as above).
Let $|z - z_0| \leq R$ be contained in $D_0$.
Then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + R e^{i\phi}) \, d\phi$$

Lemma 3

$D$ as above. Let $g$ be analytic on $D$ and $g(z) \neq 0$. We can always find an analytic function $\phi(z)$ on $D$ such that $\exp(\phi) = g$.

[$\phi$ is unique up to $+2\pi i k$]
**Theorem (Jensen's formula)** — **Lemma 4**

D as above. Let \( \{ |z| \leq R \} \subseteq D \). Let \( f \) be analytic on \( D \), \( f \neq 0 \) on \( |z| = R \), \( f(0) \neq 0 \).

Then:

\[
\ln(f(0)) + \sum \ln \frac{R}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \ln(f(Re^{i\theta})) d\theta.
\]

Here \( a_j, \ldots, a_m \) are the zeros of \( f \) in \( 0 < |z| < R \), listed with multiplicity.

**Proof**

Wlog \( D = \{ |z| < R + \varepsilon \} \), \( \varepsilon \) tiny.

Wlog \( f \neq 0 \) on \( \{ R \leq |z| < R + \varepsilon \} \). Form analytic \( F(z) \):

\[
F(z) = f(z) \prod_{j=1}^m \frac{R^2 - a_j^2 z}{R(z - a_j)}.
\]

Get \( F = fF \) on \( |z| = R \), \( F(z) \neq 0 \) on \( |z| < R + \varepsilon \).

Apply Lemma 3 to get \( \log F(z) \). By Lemma 2 + 1)

\[
\ln(F(0)) = \frac{1}{2\pi} \int_0^{2\pi} \ln(f(Re^{i\theta})) d\theta.
\]

Done.

If \( f(0) = 0 \), people usually just pass to \( \frac{f(z)}{z^N} \).
Theorem (Lemma 5)

Let \( f \) be entire of order \( \leq \rho \). \( (f \not= 0) \)

Then, counting with multiplicity,

\[
n(r) \equiv N[\text{zeros of } f \text{ in } |z| \leq r] = O(r^{\rho + \varepsilon})
\]

for all \( r \) large. Here \( \varepsilon > 0 \).

Proof:

If \( f(0) = 0 \Rightarrow \text{pass to } g = \frac{f(z)}{z^N} \). \( g \) is still entire and has order \( \leq \rho \).

WLOG \( f(0) = 1 \). Know \( \ln M(R; f) \leq R^{\rho + \varepsilon} \), large \( R \).

Perturb \( R \) slightly to make \( f(Re^{i\theta}) \not= 0 \).

Apply Lemma 4 (Jensen):

\[
0 + \sum_{j=1}^{m} \ln \frac{R}{|a_j|} = -\frac{1}{2\pi} \int_{0}^{2\pi} \ln |f(Re^{i\theta})|d\theta \leq R^{\rho + \varepsilon}
\]

Hence:

\[
n(\frac{R}{2}) \ln 2 \leq R^{\rho + \varepsilon}
\]

\[
\Rightarrow n(r) = O(r^{\rho + \varepsilon}) \text{ all large } r. \quad \Box
\]
KEY THM (Lemma 6, Hadamard/Borel/Caratheodory)

I as above. \( f \) analytic on \( D \).
Suppose \( |z - z_0| \leq R_2 \leq D_0 \). Let \( f = \sum_{n=0}^{\infty} c_n (z - z_0)^n \)
on the closed disk.

Assume further that

\[
\text{Re } f(z) \leq M \quad \text{KEY}
\]
on the closed disk. Then:

(A) \( |c_n| \leq \frac{2}{R^n} \left( M - \text{Re } c_0 \right) \), \( n \leq 1 \)

(B) \( |f(z) - f(z_0)| \leq \frac{2R}{R - r} \left( M - \text{Re } c_0 \right) \), \( 1 \leq |z - z_0| \leq r < R \)

(C) \( \left| \frac{f^{(k)}(z)}{k!} \right| \leq \frac{2R}{(R - r)^{k+1}} \left( M - \text{Re } c_0 \right) \), \( k = 1, 2, \ldots \)

PF

See Ingham 50-51. Famous trick in this proof (starts 50 bottom).

\[ \sum_{n} \frac{1}{\lambda_n \gamma^\gamma} < \infty \quad \text{for each } \gamma > \beta. \]
**Pf**

Take \( \delta \) tiny. Look at \( \int_{|z|=\delta} r^{-\gamma} d\mu(r) \) and integrate by parts.

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**Corollary**

Let \( f \) be entire and \( f(0) \neq 0 \). Then:

\[
\sum \frac{1}{|a_n|^{p+\varepsilon}} < \infty, \text{ each } \varepsilon > 0.
\]

**Pf**

Lemma 5 + 7.  \( \square \)

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Thus, we always have (for \( f(z) \) entire)

\[
\sum \frac{1}{|a_n|^{\|p\|+1}} < \infty.
\]

We let

\[
p = \|p\| + 1
\]

(Do not confuse \( p \) with a prime!)

when we play with a given \( f_0 \).
When $p$ is a non-negative integer, following Weierstrass it is customary to define

$$
E(u; j, p) = \begin{cases} 
1 - u, & p = 0 \\
(1-u) \exp \left[ u + \frac{u^2}{2} + \cdots + \frac{u^p}{p} \right], & p \geq 1
\end{cases}
$$

Note that $E(z; j, p)$ is entire.

Take $|u| \leq \lambda < 1$. With some branch of $\log$

$$
\log E(u; j, p) = \log (1-u) + u + \cdots + \frac{u^p}{p}
$$

$$
= -\sum_{n=1}^{\infty} \frac{u^n}{n} + u + \cdots + \frac{u^p}{p}
$$

$$
= -\sum_{n=p+1}^{\infty} \frac{u^n}{n},
$$

Clearly

$$
|\log E(u; j, p)| \leq \frac{|u|^{p+1}}{1 - |u|}, \quad (p = 0 \text{ OK too}).
$$

Hence:

$$
\ln |E(u; j, p)| \leq \frac{|u|^{p+1}}{1 - \lambda}, \quad |u| \leq \lambda < 1.
$$
Given \( p \geq 0 \). Also given \( a_n \in \mathbb{C} - \{0\} \), 
\( a_n \to \infty \), and 
\[
\sum_{n} \frac{1}{|a_n|^{p+1}} < \infty \text{.}
\]

We call
\[
\prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, p\right)
\]
a canonical product of genus \( p \).

\section*{Theorem}

In the above, the canonical product of genus \( p \) converges uniformly and absolutely on \( \mathbb{C} \)-compacta. Hence it is an entire function with zeros exactly at \( \{a_n\} \).

\section*{Proof}

We use our standard reduction to the \"\( \sum_{k=1}^{\infty} \log(1+b_k(z)) \) theorem\".

Take \( K = \{ |z| \leq R \} \). Restrict attention to \( |a_n| > 1000R \). Hence, in products, each term has
\[
\left| \frac{z}{a_n} \right| < \frac{1}{1000} \quad \text{for } z \in K.
\]
Get:
\[ | \log \mathbb{E} \left( \frac{x}{an} \middle| p \right) | < \frac{1}{1 - \frac{1}{1000}} \]
\[ = (1.01) \left| \frac{x}{an} \right|^{p+1} \]
\[ \leq (1.01) \left( \frac{1}{1000} \right)^{p+1} \]
\[ \leq 0.002 \]

Therefore, the "log" is actually \( \log \).

And:
\[ | \log \mathbb{E} \left( \frac{x}{an} \middle| p \right) | \leq 0.002 \]
\[ | \mathbb{E} \left( \frac{x}{an} \middle| p \right) - 1 | \leq 0.01 \]

i.e. "\( |b_n(z)| \leq 0.01\)" (on \( K \)).

Must look at
\[ \sum_n | \log (1 + b_n(z)) | \]
on \( K \). This sum will be
\[ \leq \sum_n (1.01) \left( \frac{R}{|an|} \right)^{p+1} \]  \{by the above\} 

For all \( z \in K \). \Rightarrow all is OK. \( \Box \)
When we study canonical products, it is helpful to conceptualize them as

$$
\prod E\left( \frac{z}{an + p} \right) = \prod_{1 \leq |n| \leq 1000R} E\left( \frac{z}{an + p} \right)
$$

$$
\cdot \prod_{|n| > 1000R} E\left( \frac{z}{an + p} \right)
$$

over \( \{ |z| \leq R \} \).

\[\text{THIS PART IS NONZERO}\]

**THEOREM (preliminary factorization)**

Let \( f \) be entire. Let the order be \( p < \infty \).

Put \( p = \left\lfloor \frac{p}{2} \right\rfloor \). Let the zeros of \( f \) in \( \mathbb{C} - \{0\} \) be \( \{an \}_{n \neq 0} \). [This set could be empty.]

We then have:

$$
f(z) = z^N \exp \left[ \phi(z) \right] \prod_{an \neq 0} E\left( \frac{z}{an + p} \right)
$$

where \( \phi \) is some entire fun and where the product (if infinite) is abs + uniformly convex on \( \mathbb{C} \) compact.
\textbf{Pf}

Pass first to \( g(z) = \frac{f(z)}{z^N} \), as usual.

The fun \( g \) is entire, order \( \rho \), \( g(0) \neq 0 \).

Now review (9) and form

\[ h(z) = \frac{g(z)}{\prod_{a_n \neq 0} E(z, a_n, \rho)} \]

See (10) top. Get \( h(z) \neq 0 \) for all \( z \in \mathbb{C} \).

By Lemma 3 applied to \( h(z) \), we can write \( h = \exp(\phi(z)) \). Done.

Hadamard realized, in studying Riemann's work, that he needed some way of controlling \( \phi(z) \) using only information about \( \Re \phi(z) \).

This is what led to p. 6 Key Thm!
Hadamard's Factorization Theorem \( \approx 1893 \)

Given the situation of \( \phi(z) \) THM.

We then have that \( \phi(z) \) must be a polynomial of degree \( \leq \rho \).

(Recall that \( \rho = \Omega(1) \).)

In the case of \( \Theta_0(s) \), we had \( \rho = 1 \) and type \( \tau = \infty \). So, here,

\[
\Theta_0(s) = e^{As+B} \prod_{n} E\left(\frac{s}{\alpha_n}; 1\right).
\]

Lec II p. 31

(\text{current-day})

The proof of the HFT either follows an approach of \underline{Landau} or else one based on the so-called Poisson–Jensen formula [a very common identity used in Nevanlinna theory II]. The proof is \underline{function theory}, not number theory.
The Landau approach is remarkable for its simplicity. See:

Landau, Vorlesungen über Zahlentheorie,
Satz 423 (from 1927)

or


Ingham, pages 54(bottom) - 55(bottom)
is compressed, but follows Landau.

I presented the details of this in Lecture #12 and the first half of Lecture #13. I omit them here!