Lecture 14 Synopsis

(4 Mar 2016)

The aim in this lecture was to develop some standard estimates for $\psi(x) - x$ and $\pi(x) - li(x)$ based on given zerofree regions for $\zeta(s)$. Ingham 60–67.

I began by recalling the Hadamard/Borel/Caratheodory lemma from Ingham 50.

I then turned to the development of Ingham's general estimate on $\frac{\zeta'(s)}{\zeta(s)}$ for a given zerofree region $\sigma > 1 - \eta(|t|)$. See Ingham 60–62.

$0 < \eta(t) \leqq \frac{1}{2}$

$\eta(t)$ decreasing on $[0, \infty)$, C^1

$\eta'(t) \to 0$ as $t \to \infty$

$\eta(t) \geqq \frac{1}{G \ln t}$ for t large (G = big constant)

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln^2 |t|) \quad \text{on} \quad \sigma > 1 - \alpha\eta(|t|)$$

for $|t|$ large and any $0 < \alpha < 1$.

One can then use $(c>1)$ Lec 7 p. 10

$$\psi_1(x) \approx \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]ds$$

$(x \geqq 1)$ and begin to do contour shifts over to the left, beyond $s=1$, using the Cauchy Residue Theorem. Here one wants to move the path of integration over to

$$\sigma = 1 - q\,\eta(|t|)$$

for a fixed $0<q<1$. {due to 1 bottom}

Ingham 62(bottom) – 63 gets

$$\bullet \quad \psi_1(x) \approx \frac{x^2}{2} + O\left[x^2 e^{-q\,\omega(x)}\right],$$ $\leftarrow$ $\boxed{\text{THM 21}}$

$$\omega(x) \equiv \min_{t \geqq e}\left[\eta(t)\ln x + \ln t\right].$$

$\uparrow$ I prefer "e" over Ingham's "1"

The introduction of $\omega(x)$ is slightly "slick". Classical estimates simply did "each $\eta(t)$"

separately, using <u>whatever</u> technique was natural.

Concerning $\omega(x)$, I noted:

Lemma

Keep $x \geqq 1$. Then:

$\boxed{\omega(1) = 1 \text{ clearly}}$

(a) $\omega(x)$ strictly $\uparrow$

(b) $\ln x - \omega(x)$ strictly $\uparrow$

(c) $1 < \omega(x) < 1 + \ln x$, $x > 1$.

Pf

Let $1 \leqq x_2 < x_1$. Let $\omega(x_2)$ "occur" for t_2, $\omega(x_1)$ "occur" for t_1. Get:

$$\omega(x_1) \leqq \eta(t_2)\ln x_1 + \ln t_2 \quad \text{a priori}$$

$$= \eta(t_2)\ln x_2 + \ln t_2 + \eta(t_2)[\ln x_1 - \ln x_2]$$

$$\leqq \omega(x_2) + \tfrac{1}{2}(\ln x_1 - \ln x_2) \qquad \text{see p. 1}$$

$$< \omega(x_2) + \ln x_1 - \ln x_2$$

$$\Rightarrow \omega(x_1) - \ln x_1 < \omega(x_2) - \ln x_2$$

$$\Rightarrow \ln x_1 - \omega(x_1) > \ln x_2 - \omega(x_2), \quad \text{i.e (b).}$$

Similarly :

$$\omega(x_2) \leqq \eta(t_1)\ln x_2 + \ln t_1 \quad \text{a priori}$$

$$< \eta(t_1)\ln x_1 + \ln t_1 = \omega(x_1)$$

$\Rightarrow$ $\omega(x)$ strictly $\uparrow$, i.e. (a) .

And,

$\omega(1) = 1$ by def

but, $\omega(x) \uparrow$ strictly, so $\omega(x) > 1 \quad (x > 1)$

and, $\ln x - \omega(x) \uparrow$ strictly, so $\ln x - \omega(x) > -1$
$(x > 1)$

IE, for $x > 1$,

$1 < \omega(x) < \ln x + 1$. This is (c). ■

<u>Theorem</u>

Given $\eta(|t|)$ as above. For large x, we have

$$\psi(x) = x + O[xe^{-\frac{\alpha}{2}\omega(x)}]$$

$$\pi(x) = li(x) + O[xe^{-\frac{\alpha}{2}\omega(x)}] .$$

Here $0 < \alpha < 1$.

Pf

Essentially like Ingham 64–65.

Keep $x \geqq 1000$ say. Take $0 < h < \frac{x}{2}$. Know

$$\frac{1}{h}\int_{x-h}^{x} \psi(u)\,du \leqq \psi(x) \leqq \frac{1}{h}\int_{x}^{x+h} \psi(u)\,du$$

$$\frac{\psi_1(x) - \psi_1(x-h)}{h} \leqq \psi(x) \leqq \frac{\psi_1(x+h) - \psi_1(x)}{h}$$

$$\left\{ \begin{array}{l} \text{here } x-h > \frac{x}{2} \geqq 500 \\ x+h < \frac{3}{2}x \end{array} \right\}$$

let's look at upper side first

$$\psi(x) \leqq \frac{\frac{1}{2}\left[(x+h)^2 - x^2\right] + O\left[(x+h)^2 e^{-4\omega(x+h)}\right] + O\left[x^2 e^{-4\omega(x)}\right]}{h}$$

$$\{\omega(u) \text{ strictly } \uparrow\}$$

$$\psi(x) \leqq x + \frac{h}{2} + \frac{O\left[x^2 e^{-4\omega(x)}\right] + O\left[x^2 e^{-4\omega(x)}\right]}{h}$$

$$\Rightarrow \psi(x) \leqq x + \frac{h}{2} + \frac{1}{h}\, O\left[x^2 e^{-4\omega(x)}\right]$$

next, do lower side; get

$$\psi(x) \gtrsim x - \frac{h}{2} - \frac{1}{h}\left[O(x^2 e^{-\alpha\omega(x)}) + O(x^2 e^{-\alpha\omega(\frac{x}{2})})\right]$$

$x - h > \frac{x}{2}$ and
$\omega(u) \uparrow$ strictly

$$\left\{\begin{array}{l} \ln u - \omega(u) \uparrow \text{ strictly } \Rightarrow \\ \ln x - \omega(x) > \ln\frac{x}{2} - \omega(\frac{x}{2}) \\ \omega(\frac{x}{2}) > \omega(x) - \ln 2 \\ e^{-\alpha\omega(\frac{x}{2})} < 2^{\alpha} e^{-\alpha\omega(x)} < 2e^{-\alpha\omega(x)} \end{array}\right\}$$

$$\Rightarrow \psi(x) \gtrsim x - \frac{h}{2} - \frac{1}{h} O[x^2 e^{-\alpha\omega(x)}] .$$

So,

$$\psi(x) = x + O(h) + O\left[\frac{1}{h} x^2 e^{-\alpha\omega(x)}\right] .$$

Put

$$h = \frac{x}{3} e^{-\frac{\alpha}{2}\omega(x)} \text{, } ~~say~~ .$$

cf. 3 Lemma (c)

Get:

$$\psi(x) = x + O\left[x e^{-\frac{a}{2}\omega(x)}\right].$$

Recall Lec 1 p. 4 middle. Then define:

$$\Pi(x) = \sum_{2 \leqq n \leqq x} \frac{\Lambda(n)}{\ln n} \qquad (x \geqq 2)$$

$\boxed{\text{Ingham p. 18}}$

$$= \sum_{p^m \leqq x} \frac{1}{m}$$

$$= \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots.$$

Note that

Lec 1 p. 5

$$x^{\frac{1}{M}} < 2 \qquad \text{for} \quad M = \left[\!\left[\frac{\ln x}{\ln 2} \right]\!\right] + 10.$$

Get:

$$\Pi(x) = \int_c^x \frac{1}{\ln t}\, d\psi(t) \qquad 1 < c < 2$$

$$= \frac{\psi(t)}{\ln t}\Big]_c^x - \int_c^x \psi(t)\, d\left(\frac{1}{\ln t}\right)$$

$$= \frac{\psi(x)}{\ln x} - 0 - \int_c^x \frac{\psi(t)(-1)}{(\ln t)^2}\frac{1}{t}\,dt$$

$$= \frac{\psi(x)}{\ln x} + \int_c^x \frac{\psi(t)}{t(\ln t)^2}\,dt \,.$$

Let $c \to 2$ to get

$$\pi(x) = \frac{\psi(x)}{\ln x} + \int_2^x \frac{\psi(t)}{t(\ln t)^2}\,dt \,.$$

Ingham 64 middle

Of course, we also have

$$li(x) = \int_2^x \frac{1}{\ln u}\,du$$

by our def (compare Ingham p. 3)

$$= \frac{x}{\ln x} - \frac{2}{\ln 2} - \int_2^x u\, d\left(\frac{1}{\ln u}\right)$$

$$= \frac{x}{\ln x} - \frac{2}{\ln 2} + \int_2^x \frac{u}{u(\ln u)^2}\,du \,.$$

So,

$$\boxed{\pi(x) - li(x) = \frac{\psi(x) - x}{\ln x} + \frac{2}{\ln 2} + \int_2^x \frac{\psi(t) - t}{t(\ln t)^2}\,dt}$$

which is a very useful identity, clearly.

We get: $\boxed{\beta \equiv \frac{\alpha}{2}}$ $\boxed{0<\beta<\frac{1}{2}}$

$$|\pi(x) - li(x)| \lesssim O(1) + \frac{O[xe^{-\beta\omega(x)}]}{\ln x}$$

by 7 top

$$+ \int_2^x \frac{O[te^{-\beta\omega(t)}]}{t(\ln t)^2} dt$$

{the implied constant needs inflation for small t}

$$\lesssim O[xe^{-\beta\omega(x)}] + O(1)$$

$$+ O(1) \int_2^x e^{-\beta\omega(t)} dt$$

$$\left\{ \begin{array}{l} 1 < \omega(t) < 1 + \ln t \quad \text{p. 3} \\ \hline xe^{-\beta\omega(x)} \gtrsim xe^{-\beta(1+\ln x)} \\ \qquad = e^{-\beta} x^{1-\beta} \end{array} \right\}$$

$$\lesssim O[xe^{-\beta\omega(x)}] + O(1) \int_2^x e^{-\beta\omega(t)} dt$$

$\{$ but $\ln u - \omega(u) \nearrow$ strictly, p. 3 $\}$

$$\lessapprox O[xe^{-\beta\omega(x)}] + \int_2^x O(1)\, e^{\beta(\ln t - \omega(t))} \frac{dt}{t^\beta}$$

$$\lessapprox O[xe^{-\beta\omega(x)}] + \int_2^x O(1)\, e^{\beta(\ln x - \omega(x))} \frac{dt}{t^\beta}$$

$$\lessapprox O[xe^{-\beta\omega(x)}] + O(1)\, x^\beta e^{-\beta\omega(x)} \left[\frac{t^{1-\beta}}{1-\beta}\right]_2^x$$

$$\lessapprox O[xe^{-\beta\omega(x)}] + O(1)\, x^\beta e^{-\beta\omega(x)} \frac{x^{1-\beta}}{1-\beta}$$

$0 < \beta < \frac{1}{2}$

$$\lessapprox O[xe^{-\beta\omega(x)}] .$$

So,

$$\Pi(x) - li(x) \approx O[xe^{-\frac{\alpha}{2}\omega(x)}] .$$

But

$$\Pi(x) - \pi(x) = \sum_{m=2}^{\infty} \frac{1}{m} \pi(x^{1/m}) \qquad \text{see } 7$$

$$= O\left[\frac{x^{1/2}}{\ln x}\right] + O[M x^{1/3}]$$

$$M = \left[\!\left[\frac{\ln x}{\ln 2}\right]\!\right] + 10$$

$$= O\left[\frac{x^{1/2}}{\ln x}\right] .$$

Hence, for large x,

$$\pi(x) - li(x) = O\left[x e^{-\frac{9}{2}\omega(x)}\right]$$

{noting 9 2 lines from bottom}.

$\blacksquare$

Example I

$$\eta(t) = \frac{1}{G \ln t}, \quad G \text{ big}, \quad t \gtrsim e$$

in accordance with Lec 13, p. 11 Thm.

$$\omega(x) = \min_{t \gtrsim e} \left\{ \frac{1}{G \ln t} \ln x + \ln t \right\} \qquad 2$$

Trivial calc problem with $u \gtrsim 1$ and

$$\frac{1}{G} \frac{\ln x}{u} + u .$$

Get

$$\min \approx 2\sqrt{\frac{\ln x}{G}} \qquad (x \text{ large}).$$

hence $\frac{\ln x}{G} \gtrsim 1$

So, by Lec 13 p. 11 and p. 4 Thm above,

$$\boxed{\begin{aligned} \psi(x) &= x + O\left[x e^{-c\sqrt{\ln x}}\right] \\ \pi(x) &= \mathrm{li}(x) + O\left[x e^{-c\sqrt{\ln x}}\right] \end{aligned}}$$

for suitably small $c > 0$.

The estimates in the box are the famous classical estimates of de la Vallée Poussin ~ 1899.

Example II

Lec 13 p. 4

Assume the Riemann Hypothesis, i.e. $\mathrm{Re}(\rho) = 1/2$ for all zeros of $\xi_0(s)$.

Here $\eta(t) = 1/2$.

$$w(x) = \min_{t \gtrsim e} \left\{ \frac{1}{2}\ln x + \ln t \right\} = \frac{1}{2}\ln x + 1.$$

2

In this situation, we get

$$\psi(x) = x + O\left[x e^{-\frac{\alpha}{2}\frac{1}{2}\ln x}\right]$$
$$= x + O\left[x^{1-\frac{\alpha}{4}}\right]$$
$$= x + O\left[x^{\frac{3}{4}+\varepsilon}\right] \qquad \begin{array}{l} \alpha = 1-4\varepsilon \\ \varepsilon = \frac{1}{4}(1-\alpha) \end{array}$$

$$\boxed{\pi(x) = li(x) + O\left[x^{\frac{3}{4}+\varepsilon}\right]} \, .$$

WE EXPECT AN EXPONENT MORE LIKE $\frac{1}{2}+\varepsilon$, NOT $\frac{3}{4}+\varepsilon$. (Under RH.)

To fix this, we must use a more refined technique. The idea on page 2 top is too crude! Not enough structure!!

Riemann recognized this fact. IE, a need for a more explicit formula for $\psi_1(x)$.