Lecture 16

(11 March)

$\boxed{c > 1}$

We seek to use

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds \qquad (x \geq 1)$$

to get an explicit formula for $\psi_1(x)$. $\boxed{\text{Lec 7 p. 10}}$

We shall use an appropriate rectangle

$$R: [-\Delta, c] \times [-T, T]$$

and let $T \to \infty$, $\Delta \to \infty$.

Lemma $\boxed{\rho = \beta + i\gamma}$

(a) $\displaystyle\sum_{0<\gamma\leq T} \frac{1}{\gamma} = O(\ln^2 T)$ for $T \geq 2$.

(b) $\displaystyle\sum_{\gamma > T} \frac{1}{\gamma^2} = O\left(\frac{\ln T}{T}\right)$

Proof

We know $N(t) = \frac{t}{2\pi}\ln\left(\frac{t}{2\pi e}\right) + O(\ln t)$, $t \geq 2$, by Lec 15 p. 29. Recall that $N(t)$ is right continuous.

In both (a) and (b), WLOG $T \geq 1000$.

Write $N(t) = \frac{t}{2\pi} \ln\left(\frac{t}{2\pi e}\right) + R(t)$, $R(t)$ right continuous. For (a), get:

$$\sum_{0<\gamma\leq T} \frac{1}{\gamma} = O(1) + \int_2^T \frac{1}{t}\, dN(t)$$

$$= O(1) + \int_2^T \frac{1}{t}\left\{\frac{1}{2\pi} \ln \frac{t}{2\pi}\right\} dt \quad \leftarrow \text{Lec 15 p. 29}$$

$$+ \int_2^T \frac{1}{t}\, dR(t) \quad \leftarrow R(t) = O(\ln t)$$

$$= O(1) + O(\ln T) \int_2^T \frac{1}{t}\, dt$$

$$+ \frac{R(T)}{T} - \frac{R(2)}{2} - \int_2^T R(t) \frac{(-1)}{t^2}\, dt$$

$$= O(\ln^2 T) + \frac{O(\ln T)}{T} + O(1)$$

$$+ O(1) \int_2^T \frac{\ln t}{t^2}\, dt$$

$$= O(\ln^2 T) + O(1) \ln T \cdot \int_2^\infty \frac{1}{t^2}\, dt$$

$$= O(\ln^2 T). \quad \text{OK}$$

For (b),

$$\sum_{\gamma>T} \frac{1}{\gamma^2} = \int_T^\infty \frac{1}{t^2}\, dN(t)$$

{this is correct even if $T =$ some γ}

$$= \int_T^\infty \frac{1}{t^2}\left\{\frac{1}{2\pi}\ln\frac{t}{2\pi}\right\}dt$$

$$+ \int_T^\infty \frac{1}{t^2}\,dR(t)$$

$$= O(1)\int_{T/2\pi}^\infty \frac{\ln u}{u^2}\,du$$

$$+ \left.\frac{R(t)}{t^2}\right]_T^\infty - \int_T^\infty R(t)(-2)t^{-3}\,dt$$

$$= O(1)\int_{\mathcal{U}}^\infty \ln u\; d\left(\frac{1}{u}\right) \qquad \mathcal{U} \equiv \frac{T}{2\pi}$$

$$+ O\left(\frac{\ln T}{T^2}\right) + O(1)\int_T^\infty \frac{\ln t}{t^3}\,dt$$

$$= O(1)\left[\left[\frac{\ln u}{u}\right]_{\mathcal{U}}^\infty - \int_{\mathcal{U}}^\infty \frac{1}{u}\frac{1}{u}\,du\right]$$

$$+ O\left(\frac{\ln T}{T^2}\right) + O(1)\int_T^\infty \frac{\ln t}{t^3}\,dt \quad \leftarrow$$ we'll use parts again

$$= O(1)\frac{\ln T}{T} + O(1)\int_T^\infty \ln t\; d(t^{-2})$$

$$= O(1)\frac{\ln T}{T} + O(1)\left[\left[\frac{\ln t}{t^2}\right]_T^\infty - \int_T^\infty t^{-3}\,dt\right]$$

$$= O(1)\frac{\ln T}{T} + O(1)\frac{\ln T}{T^2} = O(1)\frac{\ln T}{T}. \quad \blacksquare$$

Lemma

For $m \geq 2$, we can always find some

$$T_m \in (m, m+1)$$

so that

$$\left| \frac{\zeta'(\sigma + iT_m)}{\zeta(\sigma + iT_m)} \right| \leq A_1 \ln^2 T_m \quad \text{for } -1 \leq \sigma \leq 2.$$

Here $A_1 =$ a suitable absolute constant.

Pf

WLOG $m \geq 1000$. $\quad (\ln 1000 = 6.90^+)$

By Lec 15 Thm p. 8, see also p. 29, we know:

$$N[m-2 \leq \gamma \leq m+2] = O(\ln m).$$

Write this as

$$N[m-2 \leq \gamma \leq m+2] \leq B \ln m.$$

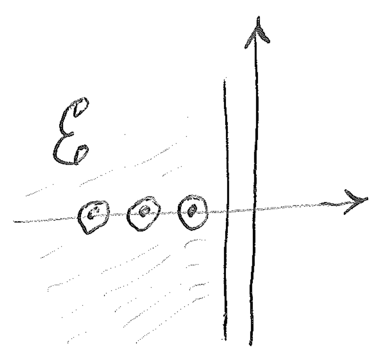
WLOG $B \geq 1$. Divide $(m, m+1]$ into $2[\![B \ln m]\!]$ equal left-open subintervals. Some interval must therefore contain NO γ. Let $T_m =$ midpoint of this subinterval. By construction,

$$|\gamma - T_m| \geq \frac{1}{4 \cdot 2 [\![B \ln m]\!]} \geq \frac{1}{8 B \ln m}$$

for all γ. Apply Lec 15, p. 13 (the partial fraction thm). With $t = T_m$, we clearly get

$$\frac{\zeta'}{\zeta}(\sigma + i T_m) = O(\ln T_m) + O(\ln m) \cdot O(\ln m)$$
$$= O(\ln^2 T_m)$$

for $-1 \leq \sigma \leq 2$. ∎



<u>Lemma</u>

Consider the domain

$$\mathcal{E} = \{\mathrm{Re}(s) < -1\} - \bigcup_{k=1}^{\infty} \{|s + 2k| \leq \tfrac{1}{2}\}.$$

We have

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq A_2 \ln(|s| + 10)$$

for $s \in \mathcal{E}$. Here A_2 = suitable absolute constant.

Pf

Recall the functional equation of $\xi(s)$, $\zeta(s)$.

Get:

$$\zeta(s) = \frac{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}\zeta(1-s)$$

$$= \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s)$$

$\Big\{$ but $\Gamma(s) = 2^{s-1}\pi^{-1/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)$ Lec 9 p. 30 (d) $\Big\}$

$$= \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)}{\pi^{1/2}2^{1-s}\Gamma(s)}\zeta(1-s)$$

$$= \pi^{s-1}2^{s-1}\frac{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)}{\Gamma(s)}\zeta(1-s)$$

$\Big\{$ but $\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right) = \dfrac{\pi}{\sin\pi\left(\frac{1}{2}-\frac{s}{2}\right)}$ Lec 9 p. 30 (c) $\Big\}$

$$= \pi^{s-1}2^{s-1}\frac{1}{\Gamma(s)}\frac{\pi}{\cos\frac{\pi s}{2}}\zeta(1-s)$$

$$\Rightarrow \zeta(1-s) = \pi^{-s}2^{1-s}\Gamma(s)\cos\frac{\pi s}{2}\cdot\zeta(s)$$

$\Downarrow$

Ingham p. 41

$$\boxed{\zeta(1-s) = 2\cdot(2\pi)^{-s}\cos\frac{\pi s}{2}\cdot\Gamma(s)\,\zeta(s)}\,.$$

Here s = any generic value in $\mathbb{C}$. Take logarithmic derivatives to get

$$-\frac{\zeta'}{\zeta}(1-s) = -\ln 2\pi - \frac{\pi}{2}\tan\frac{\pi s}{2} + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}$$

flip $s \hookrightarrow 1-s$

$\Downarrow$

$$-\frac{\zeta'(s)}{\zeta(s)} = -\ln 2\pi - \frac{\pi}{2}\operatorname{ctn}\frac{\pi s}{2} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} + \frac{\zeta'}{\zeta}(1-s)$$

$\Downarrow$

$$\boxed{\frac{\zeta'(s)}{\zeta(s)} = \ln 2\pi + \frac{\pi}{2}\operatorname{ctn}\frac{\pi}{2}s - \frac{\Gamma'}{\Gamma}(1-s) - \frac{\zeta'}{\zeta}(1-s)}\,.$$

Ingham 73

Recall that $\pi \operatorname{ctn} \pi z$ is periodic $z \to z+1$,

$$\pi \operatorname{ctn} \pi z = \lim_{N \to \infty} \sum_{-N}^{N} \frac{1}{z-n},$$

and

$$|\operatorname{ctn} \pi z + i| = O(e^{-2\pi y}) \quad \text{for } y \geqq 1.$$

similarly $y \leqq -1$

See Lec 9, pp. ③ (D) (A), ⑤ THM.

For $s \in \mathcal{E}_0$, p. ⑦ 2nd box gives:

$$\frac{\zeta'(s)}{\zeta(s)} = O(1) + O(1) + O(1) \left| \frac{\Gamma'}{\Gamma}(1-s) \right|$$

$$\boxed{\left| \frac{\zeta'(z)}{\zeta(z)} \right| \leqq \sum_{2}^{\infty} \frac{\Lambda(n)}{n^x}, \quad x \geqq 2}$$

$$= O(1) + O(1) \left| \operatorname{Log}(1-s) + O(1) \right|$$

$\boxed{\text{Stirling, Lec 12, p. ①}}$

$$= O(1) + O(1)\,|\ln|1-s||$$

note $|1-s| > 2$ on $\mathcal{E}$

$$\leq O(1) + O(1)\ln(|s|+10)$$

$|s| > 1$ on $\mathcal{E}$

$$\leq O(1)\ln(|s|+10),$$

as was to be proved. ∎

For our rectangle R on (1) we take

$$c = 2$$
$$\Delta = 2m+1, \quad m \text{ big}$$
$$T = T_m.$$

We know that

$$\psi_1(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds.$$

Here $x \geq 1$. Notice that

$$\left|\int_{2+iT_m}^{2+i\infty} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds\right| \leq \int_{T_m}^{\infty} \frac{x^3}{t^2}\,O(1)\,dt$$

$$= O(1)\frac{x^3}{T_m}.$$

Similarly for $\int_{2-i\infty}^{2-iT_m}$. Thus,

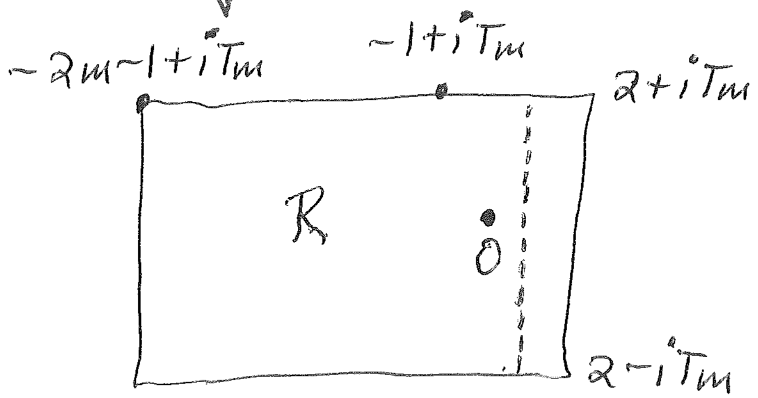
$$\psi_1(x) + O\left(\frac{x^3}{T_m}\right) = \frac{1}{2\pi i}\int_{2-iT_m}^{2+iT_m}.$$

By Cauchy Residue Thm, we have:

$$\frac{1}{2\pi i}\oint_{\partial R} \frac{x^{s+1}}{s(s+1)}\left[-\frac{\zeta'(s)}{\zeta(s)}\right] ds$$

$$= \text{Res}(\text{at } 1) + \text{Res}(\text{at } 0) + \text{Res}(\text{at } -1)$$

$$+ \sum_{k=1}^{m} \text{Res}(\text{at } -2k) + \sum_{|\gamma|<T_m} \text{Res}(\text{at } \rho)$$



$\Downarrow$

$$\frac{1}{2\pi i} \oint_{\partial R} \frac{x^{s+1}}{s(s+1)} \left[-\frac{\zeta'(s)}{\zeta(s)} \right] ds$$

Lec 9, p. 20

$$= \frac{x^2}{2} + x^1 \left[-\frac{\zeta'(0)}{\zeta(0)} \right] + x^0 \left[\frac{\zeta'(-1)}{\zeta(-1)} \right]$$

$$+ \sum_{k=1}^{m} (-1) \frac{x^{1-2k}}{(2k)(2k-1)}$$

$$+ \sum_{|\gamma| < T_m} (-1) \frac{x^{\rho+1}}{\rho(\rho+1)}$$

$\rho = \beta + i\gamma$
as usual
$0 < \beta < 1$

Note that:

$$LHS = \psi_1(x) + O\left(\frac{x^3}{T_m}\right)$$

$$+ \frac{1}{2\pi i} \int_{\substack{\text{horiz} \\ t = T_m}} + \frac{1}{2\pi i} \int_{\substack{\text{vertical} \\ \sigma = -2m-1}}$$

$$+ \frac{1}{2\pi i} \int_{\substack{\text{horiz} \\ t = -T_m}} .$$

See (10) bottom.

Apply 4 + 5 to [horiz, $t = T_m$]. Get:

$$\int_{\substack{\text{horiz} \\ t = T_m}} = O(1) \int_{-2m-1}^{-1} \frac{x^{1+\sigma}}{T_m^2} \ln m \, d\sigma$$

$$+ O(1) \int_{-1}^{2} \frac{x^{1+\sigma}}{T_m^2} \underline{\ln^2 m} \, d\sigma$$

$$\{ |x^{s+1}| = x^{\sigma+1} \text{ and } x \geqq 1 \}$$

uses $x \geqq 1$

$$= O(1) \frac{1}{m^2} (\ln m) \, O(m)$$

$$+ O(1) \frac{x^3}{m^2} \ln^2 m$$

$$= O(1) \frac{\ln m}{m} + O(1) x^3 \frac{\ln^2 m}{m^2}.$$

Similarly for [horiz, $t = -T_m$].

Apply 5 to [vertical, $\sigma = -2m-1$]. Get:

$$\int_{\substack{\text{vert} \\ \sigma = -2m-1}} = O(1) \int_{-T_m}^{T_m} \frac{x^{1+(-2m-1)}}{m^2} \ln m \, dt$$

$\boxed{x \geqq 1}$

$$\leqq O(1) \int_{-T_m}^{T_m} \frac{1}{m^2} \ln m \, dt$$

$$= O(1) \frac{\ln m}{m}.$$

We conclude that on 11 bottom:

$$\frac{1}{2\pi i} \oint_{\partial R} \frac{x^{s+1}}{s(s+1)} \left[-\frac{\zeta'(s)}{\zeta(s)} \right] ds$$

$$= \psi_1(x) + O\left(\frac{x^3}{T_m}\right)$$

$$+ O(1) x^3 \left(\frac{\ln m}{m}\right)^2 + O(1) \frac{\ln m}{m}.$$

Combining 11 top with 13 bottom, we get

$$\psi_1(x) + O(1)\frac{\ln m}{m} + O\left(\frac{x^3}{m}\right)$$

$$= \frac{x^2}{2} + Ax + B$$

$$+ \sum_{k=1}^{m} (-1) \frac{x^{1-2k}}{(2k)(2k-1)}$$

$$+ \sum_{|\gamma| < T_m} (-1) \frac{x^{\rho+1}}{\rho(\rho+1)}$$

$$\left\{ \begin{aligned} A &= -\frac{\zeta'(0)}{\zeta(0)} \\ B &= \frac{\zeta'(-1)}{\zeta(-1)} \end{aligned} \right\}$$

$$\left\{ \text{but } \sum \frac{1}{|\rho|^2} < \infty \right\}$$

$$\Downarrow \longleftarrow \boxed{\text{LET } m \to \infty}$$

$$\psi_1(x) = \frac{x^2}{2} + Ax + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)}$$

$$- \sum_{\text{all } \rho} \frac{x^{\rho+1}}{\rho(\rho+1)}$$

for <u>EACH</u> $x \geqq 1$, both series ABS conv.

$\boxed{\text{compare Lec 7 p. 10 thm}}$

Remark.

One definitely wants to keep $x \geq 1$. Indeed, for $0 < x < 1$, IE $\frac{1}{x} > 1$, we notice that

$$\sum_{k=1}^{\infty} \frac{x^{-2k}}{(2k)(2k-1)} = +\infty .$$

THM (Riemann's explicit formula for $\psi_1(x)$)

For each $x \geq 1$, we have

$$\psi_1(x) = \frac{x^2}{2} + Ax + B - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)} - \sum_{\text{all } \rho} \frac{x^{\rho+1}}{\rho(\rho+1)}$$

wherein $A = -\frac{\zeta'(0)}{\zeta(0)}$, $B = \frac{\zeta'(-1)}{\zeta(-1)}$.

Ingham 73

Pf

As above. See 14 bottom. ▨

Important Procedural Remark.

To keep things completely clear logically, notice that our proof of p. 15 THM technically only relied on $0 \leqq \beta \leqq 1$. IE, we did not need to know $\zeta(1+iy) \neq 0$.

To verify this, observe that certain "expungements" can be safely made:

Lec 6 pp. 6–8 Hadamard
(note that pp. 9–21 do not use 6–8)

Lec 7 pp. 5–9 (middle), 15–23 using $1/\zeta(z)$
(NOTE Lec 7 pp. 9 (bot) – 14 on $\psi_1(x)$ is OK)

Lec 8 pp. 1–4, 10–13 (top) on $\psi(x)$ + PNT
(NOTE Lec 9+10 is E–M, no $1/\zeta(z)$.)
(NOTE Lec 11 got functional eq for $\zeta(z)$, never needed $1/\zeta(z)$.)

In Lec 13, p. 4 THM, state only that $0 \leqq \text{Re}(\rho) \leqq 1$. Expunge $0 < \text{Re}(\rho) < 1$. Also on p. 5 in connection with HFT.

Expunge Lec 13 pp. 9 (bot) – 15, plus all of Lec 14 (related to zerofree regions).

With these expungements, a quick review shows that Lec 15 goes thru perfectly well — knowing only that $0 \leq \beta \leq 1$ and $\mathrm{Im}(\rho) \neq 0$.

Pages 1 – 15 above are then recovered without difficulty.

This being said, we can now get a "new" proof of the PNT as follows:

- Develop the explicit formula for $\psi_1(x)$ as on p. 15. (Note that this requires the CRT.)
- Do the Hadamard trick to get $\zeta(1+i\gamma) \neq 0$. See Lec 6, pp. 6 – 8.
- Use the functional equation of $\xi_0(z)$ to get $0 < \beta < 1$. See Lec 11, pp. 24 – 25, also 27.
- Choose R so big that $\sum_{|\gamma| > R} \frac{1}{|\rho|^2} < \varepsilon$.
- Exploit the explicit formula to get

$$\limsup_{x \to \infty} \left| \frac{\psi_1(x)}{x^2} - \frac{1}{2} \right| \leq 0 + 0 + 0 + \limsup_{x \to \infty} \left| \sum_{\rho} \frac{x^{\rho - 1}}{\rho(\rho+1)} \right|$$

(continued)

R held fixed

$$\leq 0 + \limsup_{x \to \infty} \sum_{|\gamma| \leq R} \frac{x^{\beta-1}}{|\rho||\rho+1|}$$

$$+ \limsup_{x \to \infty} \sum_{|\gamma| > R} \frac{x^{\beta-1}}{|\rho||\rho+1|}$$

$\{$ __but__ $|\rho+1| \geq |\rho|$ since $\beta \geq -\frac{1}{2} \}$

$$\leq 0 + \underset{\beta<1}{0} + \limsup_{x \to \infty} \sum_{|\gamma| > R} \frac{1}{|\rho|^2}$$

$$< \varepsilon .$$

- Hence $\psi_1(x) \sim \frac{x^2}{2}$ and we can repeat Lec 8 pp. 1–3. ■

This proof corresponds to Ingham 82 (middle).

Loosely Put:

Explicit Formula for $\psi_1(x)$ plus $\zeta(1+iy) \neq 0$, $y \in \mathbb{R}$, immediately implies the PNT.

It is now customary to define

$$\boxed{\Theta = \sup\{\mathrm{Re}(\rho)\}.}$$

The Riemann Hypothesis is equivalent to stating that $\Theta = \frac{1}{2}$. Obviously $\frac{1}{2} \leqq \Theta \leqq 1$.

THM

$$\psi_1(x) = \frac{x^2}{2} + O(x^{\Theta+1}) \quad \text{for large } x.$$

Pf

Obvious from p. 15 Thm since $\sum_\rho \frac{1}{|\rho|^2} < \infty$.

////

THM (Very Basic and Interesting)

$$\psi(x) = x + O(x^\Theta \ln^2 x)$$

$$\pi(x) = li(x) + O(x^\Theta \ln x).$$

Here $li(x) \equiv \int_2^x \frac{dt}{\ln t}$.

Corollary

Assume RH. Then:

$$\psi(x) = x + O(x^{\frac{1}{2}} \ln^2 x)$$

$$\pi(x) = li(x) + O(x^{\frac{1}{2}} \ln x).$$

These have never been improved.

Proof of Theorem

Know

$$\psi_1(x) = \frac{x^2}{2} + Ax + B + E(x) - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)},$$

$$E(x) \equiv -\sum_{k=1}^{\infty} \frac{x^{1-2k}}{(2k)(2k-1)}$$

by p. 15.

Note that $E(x) = b_1 x^{-1} + b_3 x^{-3} + b_5 x^{-5} + \cdots$ is a nice power series in x^{-1}.

Also know: Lec 14, p. 5

$$\frac{\psi_1(x) - \psi_1(x-h)}{h} \lessapprox \psi(x) \lessapprox \frac{\psi_1(x+h) - \psi_1(x)}{h} \quad (x \text{ large})$$

for all $1 \lessapprox h \leq \frac{x}{2}$ (say).

Look at upper part of the inequality.

$$\frac{\frac{(x+h)^2}{2} - \frac{x^2}{2}}{h} = x + \frac{h}{2}$$

$$\frac{A(x+h) + B - Ax - B}{h} = A$$

$$\frac{E(x+h) - E(x)}{h} = E'(x+\tilde{h}) \quad , \quad 0 < \tilde{h} < h$$

$$= O(x^{-2}) \quad \text{by Taylor series}$$

$$\left| \frac{(x+h)^{p+1} - x^{p+1}}{hp(p+1)} \right| \lesssim \frac{(x+h)^{\theta+1} + x^{\theta+1}}{h\gamma^2}$$

{very crude}

$$\lesssim \frac{(\text{constant})\, x^{\theta+1}}{h\gamma^2}$$

---

less crudely,

$$\left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)} \right| = \frac{1}{h} \left| \int_x^{x+h} \frac{u^\rho}{\rho} du \right|$$

$$\left\{ \begin{array}{l} \text{no ambiguity: } u^s \equiv \exp\{s \ln u\} \\ \qquad u > 1 \end{array} \right\}$$

$$\leqq \frac{1}{h} \frac{1}{|\rho|} \int_x^{x+h} u^\Theta du$$

$$\left\{ \tfrac{1}{2} \lessapprox \Theta \lessapprox 1 \right\}$$

$$\lessapprox \frac{1}{h} \frac{1}{|\gamma|} (x+h)^\Theta h$$

$$\leqq \frac{(\text{constant})\, x^\Theta}{|\gamma|}$$

Hence,

$$\left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)} \right| \lessapprox (\text{const}) \min\left[\frac{x^{\Theta+1}}{h\gamma^2}, \frac{x^\Theta}{|\gamma|} \right]$$

$$\leqq (\text{const}) \frac{x^{\Theta}}{|\gamma|} \min\left(\frac{x}{h|\gamma|}, 1\right)$$

$$= (\text{const}) \frac{x^{\Theta}}{|\gamma|} \begin{Bmatrix} 1 & \text{if } |\gamma| < \frac{x}{h} \\ \frac{x}{h|\gamma|} & \text{if } |\gamma| > \frac{x}{h} \end{Bmatrix}$$

We thus get:

$$\begin{aligned} \psi(x) \lesseqgtr\ & x + \frac{h}{2} + A + O(x^{-2}) \\ & + O(1) \sum_{|\gamma| < \frac{x}{h}} \frac{x^{\Theta}}{|\gamma|} \\ & + O(1) \sum_{|\gamma| > \frac{x}{h}} \frac{x^{\Theta+1}}{h\gamma^2} . \end{aligned}$$

The <u>lower</u> part of 20 bot will give similar; simply replace $x + \frac{h}{2}$ by $x - \frac{h}{2}$.

Get:

$$\psi(x) = x + O(h) + O(1) + O(x^{-2})$$
$$+ O(1) \sum_{|\gamma| < \frac{x}{h}} \frac{x^{\Theta}}{|\gamma|}$$
$$+ O(1) \sum_{|\gamma| > x/h} \frac{x^{\Theta+1}}{h\gamma^2} .$$

Here $1 \leq h \leq \frac{x}{2}$ and p. 1 LEMMA applies.

$$\psi(x) = x + O(h) + O(1)\, x^{\Theta} \ln^2\left(\frac{x}{h}\right)$$
$$+ O(1)\, x^{\Theta+1} \frac{1}{h} \frac{\ln(x/h)}{x/h}$$

$$= x + O(h) + O(1)\, x^{\Theta} \ln^2\left(\frac{x}{h}\right)$$
$$+ O(1)\, x^{\Theta} \ln\left(\frac{x}{h}\right) \leftarrow \boxed{\text{LIKE A MIRACLE}}$$

$$= x + O(h) + O(1)\, x^{\Theta} \ln^2\left(\frac{x}{h}\right)$$

We get $\psi(x) = x + O(x^{\Theta} \ln^2 x)$ with $h = 1$!

Do (related) calculus problem for safety:

$$\text{let } h \equiv \frac{x}{t}, \qquad 2 \lesssim t \lesssim x$$

$$h + x^{\theta} \ln^2\left(\frac{x}{h}\right) \equiv \frac{x}{t} + x^{\theta} \ln^2(t)$$

$\underline{\underline{\theta = 1}}$ $\Rightarrow$ $x\left[\frac{1}{t} + \ln^2 t\right]$ $\Rightarrow$ (const) x

minimum at $t = 2$

$\underline{\underline{\theta < 1}}$ (x large) $x^{\theta}\left[\frac{x^{1-\theta}}{t} + \ln^2 t\right]$

deriv of <u>bracket</u>:

$$-\frac{x^{1-\theta}}{t^2} + \frac{2\ln t}{t} < 0$$

iff

$$\frac{2\ln t}{t} < \frac{x^{1-\theta}}{t^2}$$

iff

$$2t\ln t < x^{1-\theta}$$

$$\Rightarrow t_{critical} \sim \frac{\frac{1}{2}x^{1-\theta}}{(1-\theta)\ln x}$$

$\Rightarrow$ <u>bracket</u> min is $\approx (1-\theta)^2 \ln^2 x$

$\Rightarrow$ OVERALL (const) $x^{\theta} \ln^2 x$.

We now continue via

$$\Pi(x) - li(x) = \frac{\psi(x) - x}{\ln x} + O(1) + \int_2^x \frac{\psi(t) - t}{t(\ln t)^2}\,dt$$

$$\Pi(x) = \pi(x) + \sum_{n=2}^{\infty} \frac{1}{n}\pi(x^{1/n})$$

à la Lec 14 pp. 8 + 7 + 10 (bottom)

$$\Downarrow$$

$$\pi(x) - li(x) = O\left(\frac{x^{1/2}}{\ln x}\right) + \frac{\psi(x) - x}{\ln x} + \int_2^x \frac{\psi(t) - t}{t(\ln t)^2}\,dt$$

$$|\pi(x) - li(x)| \lessapprox O\left(\frac{x^{1/2}}{\ln x}\right) + O(1)\,\frac{x^{\Theta}\ln^2 x}{\ln x} + O(1)\int_2^x \frac{t^{\Theta}\ln^2 t}{t(\ln t)^2}\,dt$$

$$
\begin{aligned}
&\leqq O(1)\, x^{\theta} \ln x \\
&\quad + O(1) \int_2^x t^{\theta - 1}\, dt \\
&\simeq O(1)\, x^{\theta} \ln x + O(1) \frac{1}{\theta} x^{\theta} \\
&\simeq O(1)\, x^{\theta} \ln x \,. \qquad \blacksquare
\end{aligned}
$$

2 HW problems

1 Prove rigorously that, for large x, the number of primes in $(1, x]$ **exceeds** that in $(x, 2x]$.

compare Lec 2 p. 20

2 Regarding Legendre and Ingham p. 2 (bottom). Prove that there is exactly one constant C such that

$$\left| \pi(x) - \frac{x}{\ln x - C} \right| = O\left(\frac{x}{\ln^3 x}\right)$$

and that value is 1.