Lecture 23 Synopsis

(13 April)

We use a series of Facts (in the writing style of Landau) to establish Hardy's theorem that

$$N_{critical}(T) \to \infty \quad \text{as} \quad T \to \infty.$$

Here $N_{critical}(T) = N[\rho : Re(\rho) = \frac{1}{2},\ 0 < Im(\rho) \leq T]$.

Fact 1

$T \geq 2$, $q \in \mathbb{R}$. Then

$$\left| \int_T^{2T} t^{\frac{1}{8}} e^{\frac{i}{2}(t \ln t + qt)} dt \right| = O(T^{5/8})$$

with an implied constant which is absolute.
(No dependence on q.)

Pf

Lec 22 p. 15 Lemma IV.

$G(t) = t^{1/8}$

$M = (2T)^{1/8}$

$$\left\{ \begin{aligned} 2F(t) &= t \ln t + qt \\ 2F'(t) &= 1 + \ln t + q \\ 2F''(t) &= \frac{1}{t} \end{aligned} \right.$$

$\Rightarrow r = \frac{1}{4T}$ for $[T, 2T]$

$$\frac{F'(t)}{G(t)} = \frac{1}{2}\,\frac{1 + q + \ln t}{t^{1/8}}$$

real

$$\frac{d}{dt}\left(\frac{A + \ln t}{t^{1/8}}\right) = \frac{t^{1/8}\,(t^{-1}) - (A + \ln t)\,\frac{1}{8}\,t^{-7/8}}{t^{1/4}}$$

$$= \frac{t^{-7/8}}{t^{1/4}}\left[1 - \frac{A + \ln t}{8}\right]$$

critical pt $\Leftrightarrow$ $8 = A + \ln t$ (etc)

so $\dfrac{F'(t)}{G(t)}$ has <u>AT MOST</u> ONE crit pt on $[T, 2T]$

$$\frac{M}{\sqrt{r}} = (\text{constant})\, T^{5/8}$$

Apply Lemma <u>IV</u> from Lec 22 either once or twice. ∎

NOTE:
<u>Analogous</u> Fact holds for $[T, T+H]$, any $H \in [1, T]$.

Recall

$$\xi(s) = G(s)\zeta(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\xi(s) = \xi(1-s)$$

à la Lec 11 eg 24 + 27. DEFINE:

following Landau

$$\boxed{f(s) = e^{-\frac{\pi i}{4}\left(s-\frac{1}{2}\right)}\xi(s)}$$

for $\mathrm{Im}(s) \geqq 1$.

Fact 2

(a) $f(s)$ is analytic on $\{\mathrm{Im}(s) \geqq 1\}$

(b) $|f(\sigma+it)| = e^{\frac{\pi}{4}t}|\xi(\sigma+it)|$

(c) $|f(1-\sigma+it)| = |f(\sigma+it)|$

(d) $f(\frac{1}{2}+it) = \text{real}$ for $t \in [1, \infty)$

(e) $\xi(\frac{1}{2}+it) = \text{real}$ for $t \in \mathbb{R}$.

Pf

Easy. ▨

Fact 3

Given any $-\infty < \sigma_1 < \sigma_2 < \infty$. We then have

$$|\Gamma(\sigma+it)| = \sqrt{2\pi}\,|t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|}\left(1+O\left(\frac{1}{|t|}\right)\right)$$

uniformly on $\{\sigma_1 \leq \sigma \leq \sigma_2,\ |t| \geq \frac{1}{10}\}$.

Pf

Standard corollary of Stirling's formula for $\mathcal{L}og\,\Gamma(\sigma+it)$. Lec 10 around 42. ∎

Recall that:

$$|\zeta(\sigma+it)| \lesssim \frac{e}{\delta(1-\delta)} t^{1-\delta} \qquad \sigma \geq \delta,\ t \geq 3$$

$$|\zeta(\sigma+it)| = O(\ln t) \qquad \sigma \geq 1-\frac{c}{\ln t},\ t \geq 3$$

$$|\zeta'(\sigma+it)| = O(\ln^2 t) \qquad \sigma \geq 1-\frac{c}{\ln t},\ t \geq 3$$

$$\mathcal{L}og\,\zeta(s) = O_\varepsilon(1) \quad \text{for } \sigma \geq 1+\varepsilon.$$

Here $0<\delta<1$, $c=$ small, $0<\varepsilon<\frac{1}{2}$. See Lec 6 9 20 4.

Fact 4

On $\{-\frac{1}{4} \leqq \sigma \leqq \frac{5}{4},\ t \geqq 1\}$, we have

$$|f(s)| = O(t^{1/2}) \,. \qquad \boxed{\text{CRUDE BOUND}}$$

Pf

3

By Fact 2(c), wlog $\frac{1}{2} \leqq \sigma \leqq \frac{5}{4}$. Apply p. 4 bottom with $\delta = \frac{1}{2}$. Get:

$$
\begin{aligned}
|f(\sigma+it)| &= c\, e^{\frac{\pi}{4}t} \left| \pi^{-\frac{s}{2}} \Gamma\!\left(\tfrac{s}{2}\right) \zeta(s) \right| \\
&\leqq c\, e^{\frac{\pi}{4}t} \left| \Gamma\!\left(\tfrac{\sigma}{2} + i\tfrac{t}{2}\right) \right| \left| \zeta(\sigma+it) \right| \\
&\leqq c\, e^{\frac{\pi}{4}t} \left(\tfrac{t}{2}\right)^{\frac{\sigma}{2}-\frac{1}{2}} e^{-\frac{1}{4}\pi t} \left| \zeta(\sigma+it) \right| \qquad \text{Fact 3} \\
&\leqq c \left(\tfrac{t}{2}\right)^{\frac{\sigma}{2}-\frac{1}{2}} t^{Q}
\end{aligned}
$$

where "c" can change from line to line and

$$Q \equiv \begin{cases} 1/2, & \sigma \leqq 1 \\ \frac{1}{100}, & \sigma > 1 \end{cases} .$$

The extreme exponents are $1/2$ and $\frac{1}{8} + \frac{1}{100}$, so we are done. ■

Fact 5

For $\sigma = \frac{5}{4}$ and $-\frac{1}{4}$, we have

$$|f(\sigma + it)| = O(t^{1/8})$$

in Fact 4.

Pf

on 5

Just review the proof and recall $|\zeta(s)| \leq \zeta(\sigma)$ whenever $\sigma > 1$. Get

$$|f(\tfrac{5}{4} + it)| = O(t^{1/8}).$$

Treat $\sigma = -1/4$ via Fact 2(c). ∎

Fact 6

On $\{-\frac{1}{4} \leq \sigma \leq \frac{5}{4},\ t \geq 1\}$, we actually have

$$|f(\sigma + it)| = O(t^{1/8})$$

for <u>any</u> σ.

Pf

This is an <u>immediate</u> consequence of Facts 4 + 5 when the <u>Phragmén</u> ~ <u>Lindelöf</u> <u>principle</u> for (general) analytic functions is applied. To avoid interruptions, we prove P-L in Lec 24.

$\blacksquare$

<u>Fact 7</u>

For $t \geqq \frac{1}{10}$ and some $\beta \in \mathbb{C}$ with $|\beta| = \sqrt{2\pi}$ we have:

$$\Gamma\left(\frac{5}{8} + it\right) = \beta e^{-\frac{\pi}{2}t} t^{\frac{1}{8}} e^{it \ln\left(\frac{t}{e}\right)} \left[1 + O\left(\frac{1}{t}\right)\right].$$

<u>Pf</u>

Kindergarten calculation with Stirling's formula; see Lec 10 around (42). $\blacksquare$

In what follows, we plan to compare

$$\int_T^{2T} \left|f\left(\tfrac{1}{2}+it\right)\right| dt \quad \text{with} \quad \left|\int_T^{2T} f\left(\tfrac{1}{2}+it\right) dt\right|.$$

{ Also similarly for $[T, T+H]$, $1 \leqq H \leqq T$. }

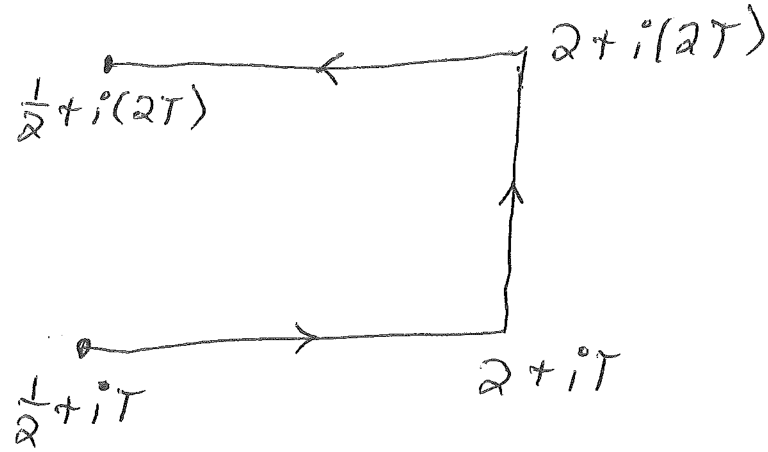
in Lec 24

Fact 8

$$\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(2T)} \zeta(s)\,ds = iT + O(T^{1/2}).$$

Pf

Cauchy integral theorem $\Longrightarrow$ use new path



Recall (4) bottom $\delta = 1/2$. Get

$$\left| \int_{\text{horizontal}} \zeta(s)\,ds \right| = O(T^{1/2}). \quad ////$$

Along $\sigma = 2$, we get the revised integral

$$\int_{2+iT}^{2+i(2T)} \left[1 + \sum_{n=2}^{\infty} n^{-2-it} \right] \underbrace{i\,dt}_{ds}$$

$$= i(2T - T) + \sum_{n=2}^{\infty} n^{-2}\, i \int_T^{2T} e^{-it\ln n}\, dt$$

$$= iT + \sum_{n=2}^{\infty} n^{-2}\, i \left[\frac{e^{-it\ln n}}{-i\ln n}\right]_T^{2T}$$

$$= iT + O(1) \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$$

$$= iT + O(1).$$

Adding things, we are done. ∎

$\left\{ \text{Note that } [T, T+H] \text{ gives } iH + O(T^{1/2}) \text{ insofar as } 1 \leqq H \leqq T. \right\}$

Fact 9

For large T, one has

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|\, dt > \frac{1}{2} T.$$

Pf

Trivial corollary of Fact 8. ∎

<u>Thm</u> Hardy 1914

$$N_{critical}(T) \to \infty \quad \text{as } T \to \infty .$$

In fact,

$$N_{crit}(T) \gtrsim c \ln T \quad \text{for } T \text{ large} .$$

<u>Pf</u>

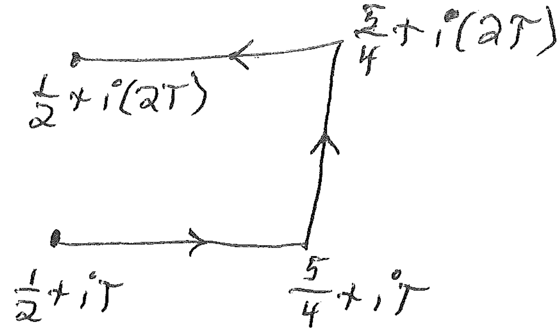
We study $f(s)$ on $[\frac{1}{2}, \frac{5}{4}] \times [T, 2T]$. Know

$$f(s) = e^{-\frac{\pi i}{4}(\sigma - \frac{1}{2})} e^{\frac{\pi t}{4}} \xi(s) . \qquad (3)$$

Apply CIT to

$$i \int_T^{2T} f(\tfrac{1}{2} + it)\, dt = \int_{\frac{1}{2}+i(T)}^{\frac{1}{2}+i(2T)} f(s)\, ds$$

$$=$$



.

By 6 Fact 6,

$$\int_{\text{horizontal}} f(s)\,ds = O(T^{1/8}).$$ //

For the $\left(\frac{5}{4}\right)$ contribution, note that

vertical

$$\int_{\frac{5}{4}+i(T)}^{\frac{5}{4}+i(2T)} f\left(\tfrac{5}{4}+it\right) \underbrace{i\,dt}_{ds}$$

has:

$$f\left(\tfrac{5}{4}+it\right) = e^{-\frac{3\pi i}{16}} e^{\frac{\pi t}{4}} \pi^{-\frac{1}{2}\left(\frac{5}{4}+it\right)} \cdot \Gamma\left(\tfrac{5}{8}+i\tfrac{t}{2}\right) \zeta\left(\tfrac{5}{4}+it\right)$$

$$= C e^{\frac{\pi t}{4}} e^{-i\frac{t}{2}\ln \pi}\, \Gamma\left(\tfrac{5}{8}+i\tfrac{t}{2}\right) \zeta\left(\tfrac{5}{4}+it\right)$$

complex and nonzero

changes from line to line

see 7 Fact 7

$$= C e^{\frac{\pi t}{4}} e^{-i\frac{t}{2}\ln \pi}\, \beta_1 e^{-\frac{\pi}{4}t} t^{1/8} e^{i\frac{t}{2}\ln\left(\frac{t}{2e}\right)} \cdot \left[1+O\left(\tfrac{1}{t}\right)\right] \cdot \zeta\left(\tfrac{5}{4}+it\right)$$

$$= C t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2e\pi}\right)} \cdot \left[1 + O\left(\frac{1}{t}\right)\right] \cdot \zeta\left(\frac{5}{4} + it\right)$$

$$= C t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)} \cdot \zeta\left(\frac{5}{4} + it\right) + O(t^{1/8}) \cdot O\left(\frac{1}{t}\right) \cdot \exp[O(1)]$$

(4) bottom

$$f = C t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)} \left\{ \sum_{n=1}^{\infty} n^{-\frac{5}{4} - it} \right\} + O(t^{-7/8}) .$$

Of course,

$$\int_{\frac{5}{4}+iT}^{\frac{5}{4}+i(2T)} O(t^{-7/8})\, dt = O(T^{1/8}) . \;/\!/\!/$$

We now need to focus on

$$C \int_T^{2T} \sum_{n=1}^{\infty} n^{-\frac{5}{4}} e^{-it\ln n}\, t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)}\, dt$$

$$= C \sum_{n=1}^{\infty} n^{-\frac{5}{4}} \int_T^{2T} t^{\frac{1}{8}} e^{i\frac{t}{2}\ln\left(\frac{t}{2\pi e n^2}\right)} dt$$

$$= C \sum_{n=1}^{\infty} n^{-\frac{5}{4}} O(T^{5/8}) \quad \text{by (1) Fact 1 !!}$$

$$= O(T^{5/8}).$$

It follows, by (11) + (12) + the above, that

$$\int_{\frac{5}{4}+iT}^{\frac{5}{4}+i(2T)} f(s)\, ds = O(T^{5/8}).$$

By (10) bottom + (11) top, we finally get:

$$i\int_T^{2T} f(\tfrac{1}{2}+it)\, dt = O(T^{1/8}) + O(T^{5/8})$$

$$= O(T^{5/8}).$$

Remark.

Landau uses Fact 6 for (11) line 2, i.e. $\int_{\text{horiz}} f(s)ds$.
Exploitation of the weaker Fact 4 produces $O(T^{1/2})$.
This is sufficient since

$$O(T^{5/8}) + O(T^{1/8}) + O(T^{1/2}) = O(T^{5/8}).$$

Phragmén–Lindelöf can thus be avoided.

$$\boxed{\int_T^{2T} f(\tfrac{1}{2}+it)\,dt = O(T^{5/8})}$$

$\left\{ \begin{array}{l} \text{with main contribution due} \\ \text{to } 13 \text{ lines } 1\sim 3 \text{ and} \\ \text{Fact 1} \end{array} \right\}$.

On the other hand, by Fact 2, on 3,

$$
\begin{aligned}
|f(\tfrac{1}{2}+it)| &= e^{\frac{\pi t}{4}} |\xi(\tfrac{1}{2}+it)| \\
&= e^{\frac{\pi t}{4}} \left| \pi^{-\frac{1}{2}(\frac{1}{2}+it)} \Gamma(\tfrac{1}{4}+i\tfrac{t}{2}) \zeta(\tfrac{1}{2}+it) \right| \\
&\geqq c\, e^{\frac{\pi t}{4}} \sqrt{2\pi}\, t^{\frac{1}{4}-\frac{1}{2}} e^{-\frac{\pi}{2}\frac{t}{2}} \left[1+O(\tfrac{1}{t})\right] |\zeta(\tfrac{1}{2}+it)| \\
&\geqq c\, t^{-1/4} |\zeta(\tfrac{1}{2}+it)|
\end{aligned}
$$

for t large. Hence:

$$
\begin{aligned}
\int_T^{2T} |f(\tfrac{1}{2}+it)|\,dt &\geqq c\,T^{-1/4} \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|\,dt \\
&\geqq c\,T^{-1/4}(T/2) \qquad \text{Fact 9 } 9 \\
&\geqq c\,T^{3/4} \quad \blacksquare
\end{aligned}
$$

Accordingly, for each large T,

$$\left| \int_T^{2T} f(\tfrac{1}{2}+it)\,dt \right| < \frac{1}{2} \int_T^{2T} |f(\tfrac{1}{2}+it)|\,dt .$$

As such, there must be some point in $(T, 2T]$ where the real-valued continuous function $f(\frac{1}{2}+it)$ undergoes a change of sign.

Remember that $f(s)$ is nicely analytic à la local Taylor series!

In other words: $(T, 2T]$ contains at least one odd order zero of $f(\frac{1}{2}+it)$.

By (3) top, hence likewise for $\zeta(\frac{1}{2}+it)$.

By studying the cases $T = 2^k$, we clearly get

$$N_{crit}(T) \to \infty$$

and, indeed,

$$N_{crit}(T) \gtrapprox c \ln T \quad (\text{all large } T).$$

∎

A moment's thought about p. 15 shows that we have actually proved:

$$\#\{\text{distinct } \rho : \mathrm{Re}(\rho) = \tfrac{1}{2},\ 0 < \mathrm{Im}(\rho) \leq T\}$$

$$\geq c \ln T .$$

Some further refinements were left for discussion in Lec 24.

[End of Lec 23]