Lecture 24 Synopsis

(15 Apr)

Function theory – centered on max mod principle, Phragmén–Lindelöf principle, Lindelöf mu-function, Littlewood's formula for $\int_{\alpha}^{\beta} N(\sigma; T_1, T_2)\, d\sigma$.

Thm (Max Mod Principle)

Let D = bdd domain in $\mathbb{C}$. Let F be analytic on D. Let

$$\limsup_{z \to \xi} |F(z)| \leqq \mathcal{M}, \quad \underline{\text{all}}\ \xi \in \partial D.$$

Then:

$$|F(z)| \leqq \mathcal{M}, \quad \text{all } z \in D.$$

If equality ever holds, then $F(z) \equiv \mathcal{M} e^{i\theta}$ for some $\theta \in \mathbb{R}$.

Pf

As in function theory, with standard use of

$$F(z_0) = \frac{1}{2\pi} \int_0^{2\pi} F(z_0 + re^{i\varphi})\, d\varphi$$

for $0 < r < \text{dist}(z_0, \partial D)$. ▨

Thm (Phragmén-Lindelöf)

Let D = bdd simply-connected domain. Let F be analytic on D. Let $|F| \leqq G$ for some big constant G. Let

$$\overline{\lim_{z\to\xi}}\, |F(z)| \leqq M, \quad \text{all } \xi \in \partial D \sim \{a_1, \dots, a_m\}.$$

Then:

$$|F(z)| \leqq M, \quad \text{all } z \in D.$$

Pf

EG $m = 1$. $z - a_1 \neq 0$ on D. Construct single-valued branch $\log(z-a_1)$. Also $(z-a_1)^\varepsilon$. Let $F_\varepsilon = F\cdot\left(\frac{z-a_1}{R}\right)^\varepsilon$. Here $R = 2\,\mathrm{diam}(D)$.

Note $\overline{\lim_{z\to a_1}}\, |F_\varepsilon| = 0$. And $\overline{\lim_{z\to\xi}}\, |F_\varepsilon| \leqq M\cdot 1$. Hence $|F_\varepsilon(z)| \leqq M$. Fix any $z \in D$. Get

$$|F(z)| \leqq M \left|\frac{R}{z-a_1}\right|^\varepsilon.$$

Let $\varepsilon \to 0$. ▥

(Simply-connected D taken for maximal simplicity in the proof.)

Counterexample if no G exists.

$$F(z) = \exp\left(\frac{i}{z}\right), \quad D = \{|z|<1, y>0\}$$

$$\mathcal{M} = e^{1}, \quad a_1 = \{0\}$$

$$e^{\frac{1}{y}} \to \infty \quad \text{as} \quad y \to 0^{+}$$

a, b finite

Fact

Given $E = \{a<x<b, y>0\}$. Let F be analytic on E. Let $|F(z)| \leq G$.

Let $\overline{\lim_{z \to \xi}} |F(z)| \leq \mathcal{M}$, all $\xi \in \partial E \cap \mathbb{C}$.

Then $|F(z)| \leq \mathcal{M}$ on E.

Pf

Apply p. ② after passing to a change of variable $\mathfrak{z} = \frac{1}{z+c}$, with c big enough to have $c+a>0$. The new domain $E_{\mathfrak{z}}$ is bounded. (c = real !!!) ▨

Fact

Let $E = \{-\frac{\pi}{2} < x < \frac{\pi}{2},\ y > 0\}$. The fcn $w = \sin z$ maps E in a **1-1** way onto $\{\operatorname{Im}(w) > 0\}$. ∂E corresponds to $\mathbb{R}$ in a nice fashion.

Proof

Look at the formula

$$\sin(x+iy) = \sin x \cdot \cosh y + i\cos x \cdot \sinh y .$$

Use standard fcn theory. ■

Note that:

$$F(z) = e^{-i\sin(z)} \qquad (z \in E)$$

has $|F(z)| > 1$, although $\lim\limits_{z \to \xi} |F(z)| = 1$, each $\xi \in \partial E \cap \mathbb{C}$. Also, for fixed x in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have:

$$|F(x+iy)| = e^{\cos x \sinh y}$$

and $\cos x \cdot \sinh y \sim \cos x \cdot \frac{1}{2} e^{y}$.

$(y \to \infty)$

<u>Thm</u> (compare Ingham p. 95) (classical P-L thm)

Let $E = \{a_1 < x < a_2,\ y > 0\}$. Let F be analytic on E; let

$$\overline{\lim_{z \to \xi}}\ |F(z)| \leqq \mathcal{M}, \quad \underline{\text{all}}\ \xi \in \partial E \cap \mathbb{C};$$

$$|F(x+iy)| \leqq \mathcal{C}\, \underline{\exp}\left[e^{cy}\right], \text{ some } \mathcal{C},$$

$$\text{some } 0 < c < \underline{\underline{\frac{\pi}{a_2 - a_1}}}.$$

Then:

$$|F(z)| \leqq \mathcal{M} \text{ on } E.$$

<u>Pf</u>

wlog $a_1 = -\frac{\pi}{2}$, $a_2 = \frac{\pi}{2}$. Take $c < b < 1$.

Study

$$F_\varepsilon(z) \equiv F(z)\, e^{i\varepsilon \sin(bz)} \quad \text{on } E.$$

By formula for $\sin(x+iy)$ on ④, get

$$|F_\varepsilon(z)| = |F(z)|\, e^{-\varepsilon \cos(bx)\sinh(by)}$$

$$\overline{\lim_{z \to \xi}}\ |F_\varepsilon(z)| \leqq \mathcal{M} \cdot 1$$

but

$$e^{cy} - \varepsilon\left(\cos b\tfrac{\pi}{2}\right)\sinh(by) \to -\infty$$

as $y \to +\infty$

$$\Downarrow$$

$|F_\varepsilon| \rightrightarrows 0$ as $y \to \infty$

$$\Downarrow$$

$|F_\varepsilon| \leqq$ <u>some</u> G on E

and $\overline{\lim_{z\to\xi}}\, |F_\varepsilon| \leqq \mathcal{M}$, all $\xi \in \partial E \cap \mathbb{C}$.

Apply ③. Get $|F_\varepsilon(z)| \leqq \mathcal{M}$ on E,

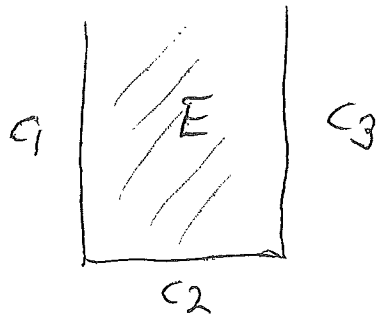
so

$$|F(z)| \leqq \mathcal{M} e^{\varepsilon \cos(bx)\sinh(by)}, \text{ each } z.$$

Let $\varepsilon \to 0^+$. Get $|F(z)| \leqq \mathcal{M}$. ▨

Corollary

$E \approx \{a_1 < x < a_2,\ y > 0\}$. F analytic on E. Let $|F(x+iy)| = O(e^{Gy})$, some giant G. Let $\overline{\lim}_{z \to \xi} |F|$ be bdd à la



Then:

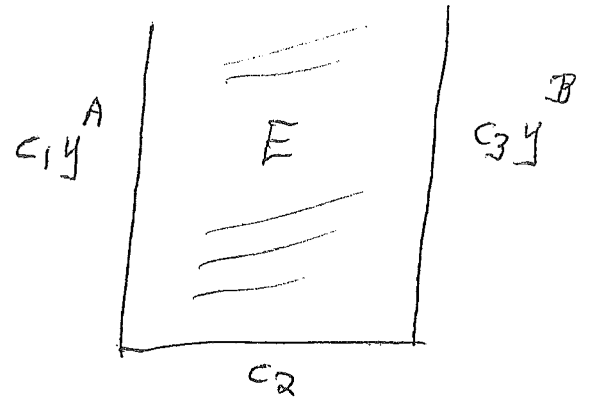
$$|F(z)| \leqq \max\{c_1, c_2, c_3\} \quad \text{on } E.$$

Thm (convexity thm)

Given $E = \{a<x<b,\ y>y_0\}$ with a $y_0 > 0$. Let F be analytic on E and have

$$|F(x+iy)| = O(e^{Gy}), \quad G = \text{giant}.$$

Let $\overline{\lim\limits_{z\to\xi}}\, |F(z)|$ be bounded àla sizes



$$\left\{ \begin{array}{c} A \in \mathbb{R} \\ B \in \mathbb{R} \\ \text{negative OK} \\ c_j > 0 \end{array} \right\}.$$

We can then find a constant $\mathcal{M}$ depending in an <u>explicit</u> way on

$$\{E,\ A,\ B,\ \max\{c_1, c_2, c_3\}\}$$

such that

$$|F(x+iy)| \leqq \mathcal{M}\, y^{A\left(\frac{b-x}{b-a}\right) + B\left(\frac{x-a}{b-a}\right)}.$$

Pf (sketch)

WLOG $c_1 = c_2 = c_3 = 1$ and $a = 0$, $b = 1$.

Introduce (on E)

$$\mathrm{Log}(-iz) = \mathrm{Log}\, z - i\frac{\pi}{2}.$$

Look at

$$g(z) = \exp\left[(A(1-z) + Bz)\, \mathrm{Log}(-iz) \right].$$

Write, for $0 < x < 1$, $y > y_0$,

$$\mathrm{Log}(y - ix) = \ln y + \mathrm{Log}\left(1 - \frac{ix}{y}\right)$$
$$= \ln y - \frac{ix}{y} + O\left(\frac{1}{y^2}\right).$$

Get $[0 < x < 1,\ y > y_0]$:

$$|g(x+iy)| = y^{A(1-x) + Bx} \exp\left[\underset{\substack{\uparrow \\ \text{depends on} \\ A,\, B,\, E}}{O(1)}\right].$$

Form

$$H(z) \equiv \frac{F(z)}{g(z)} \quad \text{on } E.$$

$$\overline{\lim_{z \to \xi}}\ |H(z)| \lesssim \text{some } \beta , \qquad \xi \in \partial E \cap \mathbb{C}$$

while

$$|H(x+iy)| \lesssim \frac{O(1) e^{Gy}}{y^{A(1-x)+Bx} \exp[O(1)]}$$

$$\lesssim O(1) e^{2Gy} \quad \text{on} \quad E$$

$$\Downarrow$$

$$|H(z)| \lesssim \beta , \text{ all } z \in E$$

$$|F(z)| \lesssim \beta |g(z)|$$

$$|F(z)| \lesssim y^{A(1-x)+Bx} \beta \exp[O(1)] . \quad \blacksquare$$

Lindelöf mu-fcn. ← 1908

Let $F(z)$ be analytic on

positive

$$E_0 = \{ \alpha < x < \beta , y > y_0 \} .$$

Assume:

$$|F(x+iy)| \lesssim O(e^{Gy}) \quad \text{on} \quad E_0 .$$

(Some giant G.)

We define, for $\alpha < x < \beta$,

$$\mu(x) = \inf\{\omega : |F(x+iy)| = O(y^{\omega})\}.$$

Here we allow $\mu(x) = \pm\infty$ in an obvious sense.

Tautologically, for each x,

$$\mu(x) = \overline{\lim_{y\to\infty}} \frac{\ln|F(x+iy)|}{\ln y}.$$

NOTE:

$$\alpha = -1, \quad \beta = 1, \quad y_0 = 1$$

$$F(z) = e^{-iz^2}$$

$$|F(z)| = e^{2xy}$$

$$|F(x+iy)| \leqq e^{2y} \quad \text{on } E_0$$

$$\mu(x) = \left\{ \begin{array}{ll} -\infty, & -1 < x < 0 \\ 0, & x = 0 \\ +\infty, & 0 < x < 1 \end{array} \right\}.$$

Fact

Suppose that $\mu(x) < +\infty$ for *all* $x \in (\alpha,\beta)$. If $\mu(x_0) = -\infty$ for *some* $x_0 \in (\alpha,\beta)$, we must then have $\mu(x) \equiv -\infty$ on (α,β).

Pf

Simply apply p. ⑧ THM with appropriate a, b, A, B and let one of A or B tend *incrementally* to $-\infty$. ▨

Thm (convexity of μ)

Given F on E_0 as above. Assume that $-\infty < \mu(x) < +\infty$ for each $x \in (\alpha,\beta)$. The fcn $\mu(x)$ is then *convex* on (α,β); i.e.,

$$\mu[(1-t)x_1 + t x_2] \leq (1-t)\mu(x_1) + t\mu(x_2)$$

for $t \in [0,1]$ and $x_1 < x_2$ in (α,β).

Pf

Easy consequence of p. ⑧ THM. ▨

It is a standard thm of basic analysis that every (finite) convex fcn $\phi(x)$ on (α, β) is automatically continuous.

Let $F(z) = \zeta(z)$.

The Euler-Maclaurin development (in the style of Euler) given in Lec 9 p. ⑲ immediately shows that $\underline{\mu(x) < +\infty}$ for every $x \in \mathbb{R}$.

Cf. Lec 5 pp. ⑩ (line 5) + ⑫ (thm) for $x > 0$.

Recall $\operatorname{Log} \zeta(z)$ in Lec 6, pp. ③ + ④, in connection with

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}} \quad , \quad \operatorname{Re}(z) > 1 .$$

Clearly:

$$\operatorname{Log} \zeta(z) = O_\varepsilon(1) \quad \text{for } x \geqq 1+\varepsilon .$$

Hence: $-A_\varepsilon \leqq \ln|\zeta(x+iy)| \leqq A_\varepsilon$ here.

Fact

F(z) on ⑩

Given $\zeta(z)$. We have $-\infty < \mu(x) < +\infty$ for all $x \in \mathbb{R}$. In fact: $\mu(x) = 0$ for all $x > 1$.

Pf

Obvious by p. ⑬ and the Fact on ⑫. ■

Now exploit $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and $\xi(s) = \xi(1-s)$ à la Lec 11 p. ㉔.

Recall:

$$|\Gamma(\sigma+it)| = \sqrt{2\pi}\,|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\left(1+O\left(\frac{1}{|t|}\right)\right)$$

for any $\sigma_1 \leq \sigma \leq \sigma_2$ and $|t| \geq 1$. See Lec 23 p. ④ Fact 3; also Lec 10 p. ㊷ for Stirling.

Get:

$$\pi^{-\frac{\sigma}{2}} \left|\Gamma\left(\frac{\sigma}{2}+i\frac{t}{2}\right)\right| \left|\zeta(\sigma+it)\right|$$

$$= \pi^{-\frac{1-\sigma}{2}} \left|\Gamma\left(\frac{1-\sigma}{2}-i\frac{t}{2}\right)\right| \left|\zeta(1-\sigma-it)\right|$$

$$\Downarrow$$

{ by ⑭ line −4 }

$$\pi^{-\frac{\sigma}{2}} \sqrt{2\pi} \left(\frac{t}{2}\right)^{\frac{\sigma}{2}-\frac{1}{2}} e^{-\frac{\pi}{4}t} \left|\zeta(\sigma+it)\right|$$

$$\sim \pi^{-\frac{1-\sigma}{2}} \underset{\sqrt{2\pi}}{\uparrow} \left(\frac{t}{2}\right)^{\frac{1-\sigma}{2}-\frac{1}{2}} e^{-\frac{\pi}{4}t} \left|\zeta(1-\sigma+it)\right|$$

{compare Lec 23 p. ⑤}

$$\left|\zeta(\sigma+it)\right| \sim c(\sigma)\, t^{\frac{1}{2}-\sigma} \left|\zeta(1-\sigma+it)\right|$$

as $t \to +\infty$.

THM

For $F(s) = \zeta(s)$, we have

$$\mu(\sigma) = \mu(1-\sigma) + \frac{1}{2} - \sigma .$$

Pf

As above. ▨

By ⑬ (top), ⑭ (top), ⑮ THM, we get:

$$\mu(\sigma) = \left\{ \begin{array}{ll} 0, & \sigma \geqq 1 \\ \frac{1}{2}-\sigma, & \sigma \leqq 0 \end{array} \right\} .$$

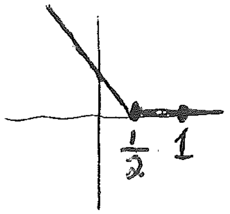
Application of p. ⑫ THM then gives:

$$\mu(\sigma) \leqq \frac{1}{2} - \frac{\sigma}{2} \quad \text{for } 0<\sigma<1 .$$

The exact value of $\mu(\sigma)$ at any given $\sigma \in (0,1)$ remains a mystery.

1908

Lindelöf has conjectured that



$$\mu(\sigma) = \left\{ \begin{array}{ll} \frac{1}{2}-\sigma, & 0<\sigma<\frac{1}{2} \\ 0, & \frac{1}{2} \leqq \sigma < 1 \end{array} \right\} .$$

It is known that the Riemann Hypothesis (Lec 16, p. ⑲, $\Theta = \frac{1}{2}$) implies Lindelöf's

Lec 14, p. ⑫

conjecture. See, e.g., Titchmarsh's book on $\zeta(s)$.

Fact

Lindelöf's conjecture is equivalent to proving that $\mu(\frac{1}{2}) = 0$.

Pf

Clearly Lindelöf $\Rightarrow \mu(\frac{1}{2}) = 0$.

Now suppose $\mu(\frac{1}{2}) = 0$. By convexity ⑫ and $\mu(\sigma) = 0$ when $\sigma > 1$, we get $\mu \leqq 0$ on $[\frac{1}{2}, 1]$.

If we had $\mu(x_0) < 0$ for some $x_0 \in (\frac{1}{2}, 1]$, application of ⑫ again would give

$$\mu(x_0) < 0, \quad \mu(2) = 0 \Rightarrow \mu(\tfrac{3}{2}) < 0.$$

Contrad!! Hence $\mu = 0$ on $[\frac{1}{2}, 1]$.

By p.⑮ THM, get $\mu = \frac{1}{2} - \sigma$ on $[0, \frac{1}{2}]$. Hence all is OK. ▨

The best that is currently known is that $\mu(\frac{1}{2})$ is at most a specific fraction somewhat less than $\frac{1}{6}$.

It has sometimes been claimed that $\mu(\frac{1}{2}) \lessapprox \frac{1}{8}$, but this has never panned out [i.e., proven to be correct]. The conventional wisdom is that achieving even this would be a "major advance".

∼

Now we turn to Littlewood's formula.

Let $(\alpha, \beta) \times (T_1, T_2)$ be a given rectangle. We'll call it R. Let $f(s)$ be analytic on $R \cup \partial R$. Let $f(\beta + it) \neq 0$. Also let

$$f(\sigma + iT_1) \neq 0, \quad f(\sigma + iT_2) \neq 0.$$

We are completely happy if f vanishes at some points of $\{\sigma = \alpha\}$. $\{t \neq T_1, T_2\}$

Begin by defining a single-valued branch of $\log f(s)$ on a narrow open set containing $\{\sigma = \beta, T_1 \leq t \leq T_2\}$. For t-values **not** matching the **ordinate** of a zero of $f(s)$ on $R \cup \partial R$, define $\phi(s) \equiv \operatorname{Log} f(s)$ by horizontal analytic continuation starting with $\log f(s)$. Compare Lec 15 p. ㉕.

Once that is done, we then use continuity FROM **ABOVE** wrt t to take care of the ordinates of f-zeros. {Note that this makes good sense even for $\sigma = \alpha$.}

THM (Littlewood)

Given R, f as above. Let *

$N(u; T_1, T_2)$ = # of zeros of $f(s)$ on $R \cup \partial R$ having abscissa $\geq u$ (and counted WITH multiplicity).

We then have:

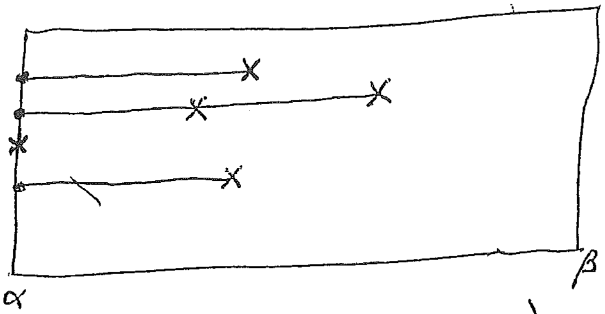
$$-\frac{1}{2\pi i} \oint_{\partial R} \phi(s)\, ds = \sum_{j=1}^{N} [\, \mathrm{Re}(\rho_j) - \alpha\,]$$

$$= \int_{\alpha}^{\beta} N(\sigma; T_1, T_2)\, d\sigma$$

using an obvious ρ_j notation for the zeros of f.

Pf

Make the connected open set R' by drawing



in an obvious way. The x's corr to ρ_j.

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* Note that $N(u; T_1, T_2)$ is right continuous.

Write

$$f(s) = f_0(s)(s-\rho_1)\cdots(s-\rho_N)$$

$$\left\{ \begin{array}{c} f_0(s) \text{ analytic and nonzero} \\ \text{on } R \cup \partial R \end{array} \right\}.$$

The branch $\mathcal{L}og\, f_0(s)$ is uniquely defined on $R \cup \partial R$ once it is "started" on $\sigma = \beta$.

Let us agree that the standard principal value $Log\, z$ has $-\pi < Arg(z) \leqq \pi$. Then:

$$Log(-q) = \lim_{\varepsilon \to 0^+} Log(-q + i\varepsilon)$$

for every $q > 0$.

There is no loss of generality in presupposing that

$$\overset{\phi(s)}{\mathcal{L}og\, f(s)} = \mathcal{L}og\, f_0(s) + \sum_{j=1}^{N} Log(s-\rho_j)$$

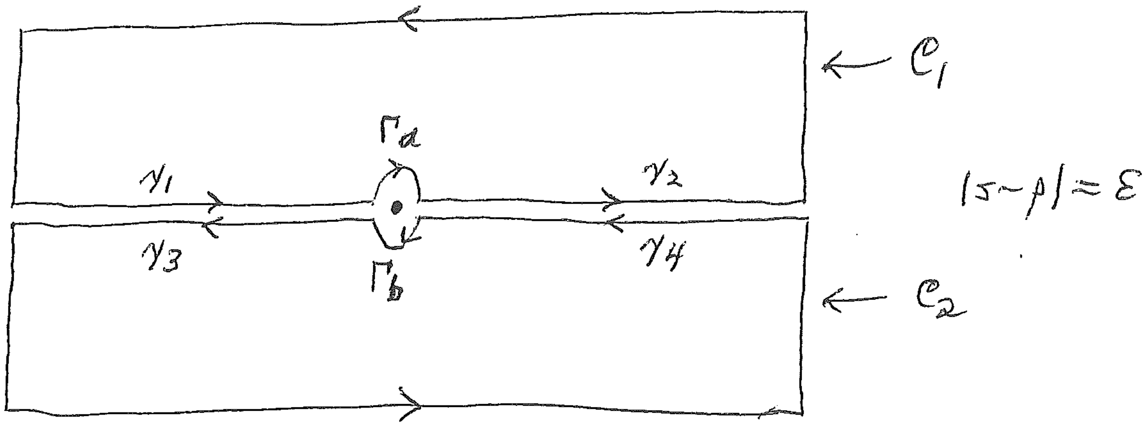
first along $\sigma = \beta$, then throughout $\underline{\underline{R'}}$.

Naturally, along $\partial R'$, one must be more careful [utilizing, e.g., the continuity from above idea].

also in $Log\, z$

Take just one zero ρ_j and drop the j.

For simplicity, take $\alpha < \mathrm{Re}(\rho) < \beta$. The case $\mathrm{Re}(\rho) = \alpha$ is an easy adaptation.



$$\int_{\mathcal{C}_1} + \int_{\gamma_1} + \int_{\Gamma_a} + \int_{\gamma_2} \mathrm{Log}(s-\rho)\,ds = 0$$

{by CIT}

$$\int_{\mathcal{C}_2} + \int_{\gamma_3} + \int_{\Gamma_b} + \int_{\gamma_4} \mathrm{Log}(s-\rho)\,ds = 0$$

$$\left| \int_{\Gamma_a} \mathrm{Log}(s-\rho)\,ds \right| \leq \int_{\Gamma_a} \left[\ln\frac{1}{\varepsilon} + 2\pi \right] |ds|$$

$$= O\left(\varepsilon \ln\frac{1}{\varepsilon}\right) \to 0$$

$$\left| \int_{\Gamma_b} \mathrm{Log}(s-\rho)\,ds \right| = O\left(\varepsilon \ln\frac{1}{\varepsilon}\right) \to 0$$

similarly

Obviously:

$$\int_{\gamma_2} + \int_{\gamma_4} = 0 \qquad (\mathrm{Arg}(s-\rho) = 0).$$

But,

$$\int_{\gamma_1} \mathrm{Log}(s-\rho)\,ds = \int_{q}^{\mathrm{Re}(\rho)-\varepsilon} [\ln|s-\rho| + \overset{\swarrow}{i\pi}]\,d\sigma$$

$$\int_{\gamma_3} \mathrm{Log}(s-\rho)\,ds = -\int_{q}^{\mathrm{Re}(\rho)-\varepsilon} [\ln|s-\rho| \underset{\uparrow}{-\,i\pi}]\,d\sigma$$

$$\Rightarrow \int_{\gamma_1} + \int_{\gamma_3} = 2\pi i\,[\mathrm{Re}(\rho) - q] + O(\varepsilon).$$

Hence, collectively, we get:

$$\int_{C_1} + \int_{C_2} + 2\pi i\,[\mathrm{Re}(\rho) - q] = o(1)$$

$$\Downarrow$$

$$\oint_{\partial R} \mathrm{Log}(s-\rho)\,ds = -2\pi i\,[\mathrm{Re}(\rho) - q].$$

This will hold for each ρ_j.

Of course, by CIT,

$$\oint_{\partial R} \mathcal{L}og\, f_0(s)\,ds = 0 .$$

Adding produces:

㉑ line −6

$$\oint_{\partial R} \mathcal{L}og\, f(s)\,ds$$

$$= -2\pi i \sum_{j=1}^{N} \left[\mathrm{Re}(\rho_j) - \sigma \right]$$

OR

$$-\frac{1}{2\pi i} \oint_{\partial R} \phi(s)\,ds = \sum_{j=1}^{N} \left[\mathrm{Re}(\rho_j) - \sigma \right] .$$

(OK)

If one writes

$$N(\sigma; T_1, T_2) = \sum_{\substack{\text{each} \\ \rho_j}} N_{\rho_j}(\sigma; T_1, T_2)$$

in an obvious way, we clearly get

$$\int_\alpha^\beta N(\sigma; T_1, T_2)\,d\sigma = \sum_j \left[\mathrm{Re}(\rho_j^\circ) - \alpha\right].$$

Here, of course, one can suppress any ρ_j° having $\mathrm{Re}(\rho_j^\circ) = \alpha$. ▨

Corollary (Littlewood)

$$2\pi \int_\alpha^\beta N(\sigma; T_1, T_2)\,d\sigma$$

$$= \int_{T_1}^{T_2} \ln|f(\alpha+it)|\,dt - \int_{T_1}^{T_2} \ln|f(\beta+it)|\,dt$$

$$- \int_\alpha^\beta \arg f(\sigma+iT_1)\,d\sigma + \int_\alpha^\beta \arg f(\sigma+iT_2)\,d\sigma,$$

wherein $\arg f$ comes from $\operatorname{Log} f(s)$ àla ⑲.

Pf

Use ⑳ and take the appropriate real part. ▨

$$\boxed{\mathrm{Re}\left[\, i \oint_{\partial R} \phi(s)\,ds \right]}$$

Addendum

(a remark about Lec 23)

I commented that the technique of Lec 23 actually gives $\geq$ (const) T^{ω} zeros on the critical line for some small ω. I claim that $\omega \approx 1/8$ works.

More precisely, I claim that:

$$\#\{\text{online } \zeta\text{-zeros with } U < \gamma \leq 2U\}$$
$$\gtrsim (\text{small constant})\, U^{1/8}$$

once U is large enough.

Let H be any number in $[T^{\frac{51}{100}}, T]$. Keep T large. Note that Lec 23, Fact 1, holds equally well for

$$\int_T^{T+H}.$$

Lec 23 Facts 2 - 7 require no change. On ⑦ (bottom) of Lec 23, look at

$$\int_T^{T+H} |f(\tfrac{1}{2}+it)|\,dt \quad \text{vs.} \quad \left|\int_T^{T+H} f(\tfrac{1}{2}+it)\,dt\right|.$$

Analog of Fact 8 is

$$\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(T+H)} J(s)\,ds \approx iH + O(T^{1/2}).$$

$\uparrow$ note role of $n=1$

See (also) Lec 23 p. ⑨ middle. The analog of Fact 9 is:

$$\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|\,dt > \tfrac{1}{2}H$$

once T is large enough.

On Lec 23 pp. ⑩ ~ ⑪, use $[\frac{1}{2}, \frac{5}{4}] \times [T, T+H]$.

On ⑫, get

$$\int_{\frac{5}{4}+iT}^{\frac{5}{4}+i(T+H)} O(t^{-7/8})\,dt = O(HT^{-7/8})$$
$$= O(T^{1/8})$$

since $H \leqq T$. On ⑬ line 3, get $O(T^{5/8})$ again. Hence, on ⑭ top,

$$\int_T^{T+H} f(\tfrac{1}{2}+it)\,dt = O(T^{5/8}).$$

On ⑭ (bottom), we get

$$\int_T^{T+H} |f(\tfrac{1}{2}+it)|\,dt \geqq c T^{-1/4} \int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|\,dt$$
$$\geqq c T^{-1/4} (H/2)$$
$$\geqq c_2 H T^{-1/4}.$$

Observe, however, that

$$T^{5/8} \leqq c_2 H T^{-1/4}$$

anytime

$$H \geqq \frac{1}{c_2} T^{7/8} .$$

This suggests keeping

(*) $$H \geqq G T^{7/8}$$

for some **giant** constant G. Doing so clearly produces

$$\left| \int_T^{T+H} f(\tfrac{1}{2}+it)\,dt \right| < \frac{1}{2} \int_T^{T+H} |f(\tfrac{1}{2}+it)|\,dt$$

once T is large enough.

Hence, **under (*)**, we find at least ONE true change of sign for $f(\frac{1}{2}+it)$ in $[T, T+H]$. See Lec 23 ⑮ (lines 3~5).

All this being said, let U (now) be large and take:

$$H = G(2U)^{7/8} .$$

Let

$$U_n = U + nH, \qquad 0 \le n \le [\![\tfrac{U}{H}]\!] .$$

Look at the disjoint intervals

$$(U_{n-1}, U_n] \qquad (n \ge 1) .$$

We clearly get *at least* $[\![\frac{U}{H}]\!]$ true changes of sign of $f(\frac{1}{2}+it)$ $[\![$hence, *distinct* *zeros* $]\!]$ on $(U, 2U]$. This number clearly exceeds

$\boxed{[\![\frac{U}{H}]\!]}$ (small constant) $U^{1/8}$.

$\boxed{\text{OK}}$

$\boxed{T^{\omega}}$ A review of this proof shows that a *similar* estimate holds for a wider class of Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad (a_1 \neq 0)$$

having functional equation similar to that of $\zeta(s)$. The *total* number of zeros will still be $\sim$ (constant) $T \ln T$. And the existence of an Euler product will **NOT** be required.

Going back to $\zeta(s)$, I also noted that with a much harder proof, A. Selberg proved ☆

$$N_{crit}(T) > (\text{tiny constant})\, T \ln T \,.$$

(1942)

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In the early 1970s, N. Levinson used a different [but related] approach to get

$$> \frac{1}{3}\left(\frac{T}{2\pi} \ln \frac{T}{2\pi e}\right) .$$

Conrey pushed this to

$$> 40\% \left(\frac{T}{2\pi} \ln \frac{T}{2\pi e}\right) .$$

☆ Hardy and Littlewood reached $> cT$ in 1921.