Lecture 27 Synopsis

(Wed, 27 Apr)

We seek to develop the famous thm of Bohr-Landau about zeros of $\zeta(s)$ to the right of $Re(s) = \frac{1}{2}$.

Recall, from Lec 24, (19) (20), that we had

$N(u; T_1, T_2)$ = # of zeros of $f(s)$ on $R \cup \partial R$ having abscissa $> u$ (and counted WITH multiplicity)

wherein $R = (\alpha, \beta) \times (T_1, T_2)$; $f(s)$ is analytic on $R \cup \partial R$, $f(\beta + it) \neq 0$, $f(\sigma + iT_1) \neq 0$, $f(\sigma + iT_2) \neq 0$. One defines $\phi(s) = \mathcal{L}og\, f(s)$ via horizontal analytic continuation starting at $\sigma = \beta$ insofar as $t \neq$ ordinate of a zero of f; otherwise by an obvious right continuity from above.

In this framework, we got Littlewood's formula

$$-\frac{1}{2\pi i} \oint_{\partial R} \phi(s)\, ds = \int_{\alpha}^{\beta} N(u; T_1, T_2)\, du$$

and, then, the simplified version in Lec 24, (25).

The Bohr – Landau theorem will arise by specializing $f(s)$ to be $\zeta(s)$ in the foregoing — and playing with appropriate α and β.

Not-too-surprisingly, matters will need to be looked at along the way <u>with the aid of</u> some of our previously obtained estimates.

<u>THEOREM</u> (basic form of Bohr – Landau thm) 1914

Consider $\zeta(s)$. For $T \geqq 2$, say, [not the ordinate of a zero of ζ], we have:

$$N(\tfrac{1}{2}+\varepsilon;\, h, T) = O_\varepsilon(T)$$

for each $\varepsilon > 0$. Here h is a tiny number such that $\zeta \neq 0$ on $\{\mathrm{Re}(s) \geqq 0,\ 0 \leqq \mathrm{Im}(s) \leqq h\}$; see Lec 11, p. 27 and Lec 13, pp. 6 (top) + 9 (top).

We stress here that, <u>in toto</u>, we have

$$N(0;\, h, T) \sim \frac{T}{2\pi} \ln\left(\frac{T}{2\pi e}\right)$$

by Lec 15, pp. 29 + 22 (box). As such, by the functional equation of ζ, only <u>0%</u> of the complex zeros of ζ can lie <u>outside</u> $|\mathrm{Re}(s) - \tfrac{1}{2}| \leqq \varepsilon$.

The relevant "PREVIOUSLY OBTAINED" estimates referenced on p. 2 are those found on

page 8 of Lec 15 (a priori upper bound)
page 13 of Lec 15 (the partial fraction thing)
page 14 of Lec 25 (L_2 estimate)
page 11 of Lec 26 (Hardy – Littlewood estimate).

In connection with the first <u>two</u> — from Lec 15 — we note the following:

FACT

Keep $T \geq 2$, say, and not the ordinate of a zero of ζ. Use the standard **UP AND ACROSS** definition of $\mathcal{L}og\, \zeta(s)$ beginning at some point $A \in \mathbb{R}$, $A >> 1$. Then:

$$\arg \zeta(\sigma + iT) = O(\ln T),$$

for all $-1 \leq \sigma \leq 2$.

Pf

by the Dirichlet series!

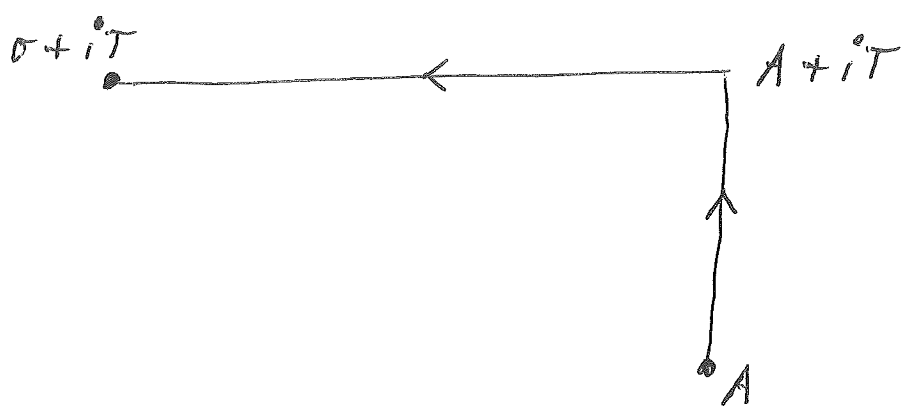
We know $|\operatorname{Log} \zeta(s)| = O_\eta(1)$, $\sigma \geq 1+\eta$.

hence also for $\operatorname{Arg} \zeta(s)$

Use

$$N[|\gamma - t| \leq 1] = O(\ln t)$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln t) + \sum_{\substack{\rho \\ |\gamma - t| \leq 1}} \frac{1}{s-\rho} \qquad \{-1 \leq \sigma \leq 2\}$$



$$\operatorname{Arg} \zeta(\sigma+iT) = \operatorname{Im} \int_A^\sigma \frac{\zeta'}{\zeta}(u+iT)\, du$$

$$+ \operatorname{Im} \operatorname{Log} \zeta(A+iT)$$

this is $O(2^{-A})$ by the Dirichlet series for $\operatorname{Log} \zeta$

$\{$we can let $A \to \infty\}$

$$\operatorname{Arg} \zeta(\sigma + iT) = -\operatorname{Im} \int_\sigma^\infty \frac{\zeta'}{\zeta}(u+iT)\,du$$

$$= -\operatorname{Im}\int_\sigma^2 \; - \operatorname{Im}\int_2^\infty \frac{\zeta'}{\zeta}(u+iT)\,du$$

$O(2^{-4})$ by the Dirichlet series for $-\frac{\zeta'}{\zeta}$

$$= -\operatorname{Im}\int_\sigma^2 \Big\{ O(\ln T) + \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \frac{1}{u+iT-\rho} \Big\} du$$

$$+ \; O(1)$$

$$= O(\ln T)$$

$$- \sum_{\substack{\rho \\ |\gamma - T| \leqq 1}} \Big\{ \operatorname{Arg}(2+iT-\rho) - \operatorname{Arg}(\sigma+iT-\rho) \Big\}$$

$$+ \; O(1)$$

Arg = ordinary principal value

$$= O(\ln T) + O(\ln T) + O(1)$$

$$= O(\ln T)$$

{ <u>much</u> <u>like</u> Lec 15, p. 28 } . ∎

By Littlewood's identity, Lec 24, p. 25, know:

$$\frac{1}{2} \leqq \alpha < \beta \leqq 2$$

$$\Downarrow$$

$$2\pi \int_\alpha^\beta N(\sigma; T_1, T_2)\, d\sigma$$

$$= \int_{T_1}^{T_2} \ln|\zeta(\alpha + it)|\, dt$$

$$- \int_{T_1}^{T_2} \ln|\zeta(\beta + it)|\, dt$$

$$+ O[(\beta - \alpha) \ln T_2] \longleftarrow \text{by FACT on } 3$$

at least for T_j which are not the ordinates of ζ-zeros. We'll keep $T_2 > T_1 \geqq 2$. Cf. also h on page 2.

Must focus on

$$\int_{T_1}^{T_2} \ln|\zeta(\beta+it)|\,dt$$

(since β will be taken $\geq \frac{3}{2}$ later). see 15 middle

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We propose to look first at

$$\int_{T/2}^{T} |\zeta(\sigma+it)|^2\,dt$$

TRICK

with $T \geq 1000$ and $\frac{1}{2} \leq \sigma \leq 2$.

Use H-L estimate from Lec 26 11, $\sigma_0 = \frac{1}{10}$.
Take $C = \pi$. We get:

$$\zeta(\sigma+it) = \sum_{n \leq T} n^{-\sigma-it} - \frac{T^{1-\sigma-it}}{1-\sigma-it} + O(T^{-\sigma})$$

for $t \in [\frac{1}{2}T, T]$.

So,

$$\zeta(\sigma+it) = \sum_{n \leq T} n^{-\sigma} n^{-it} + \frac{O(T^{1-\sigma})}{T} + O(T^{-\sigma})$$

$$= \sum_{n \leq T} n^{-\sigma} e^{-it\ln n} + O(T^{-\sigma})$$

for $\frac{1}{2}T \leq t \leq T$. We'll write this as

$$\underline{\underline{\zeta(\sigma+it) = \Sigma + \mathcal{R}}}.$$

Hence,

$$|\zeta(\sigma+it)|^2 = (\Sigma + \mathcal{R})(\overline{\Sigma} + \overline{\mathcal{R}})$$

$$= \Sigma\overline{\Sigma} + 2\,\mathrm{Re}(\mathcal{R}\overline{\Sigma}) + |\mathcal{R}|^2$$

$$= \Sigma\overline{\Sigma} + O(1)T^{-\sigma}|\Sigma| + O(1)T^{-2\sigma}$$

for $t \in [\frac{1}{2}T, T]$ and $\frac{1}{2} \leq \sigma \leq 2$.

Put:

$$\Sigma = \sum_{n \leq T} n^{-\sigma} e^{-i\lambda_n t} \quad \text{with}$$

$$\lambda_n \equiv \ln n \,,$$

and then use Lec 25 p. 14 (the L_2 estimate).
Here, of course, $n > m \Rightarrow$

$$\begin{aligned} \lambda_n - \lambda_m &= \ln n - \ln m \\ &= \frac{1}{\tilde{n}}(n-m) \,, \quad \tilde{n} \in (m, n) \\ &\geq \frac{1}{T} \,. \end{aligned}$$

We can apply Lec 25 14 with $\delta = \frac{1}{T}$. So,

$$\int_{T/2}^{T} |\Sigma|^2 dt = \frac{T}{2} \sum_{n \leq T} n^{-2\sigma} + \frac{O(1)}{1/T} \sum_{n \leq T} n^{-2\sigma}$$

{this *may* be improvable, but we *prefer* to stay with a crude bound!}

$$\Downarrow$$

$$\int_{T/2}^{T} |\Sigma|^2 \, dt = O(T) \sum_{n \leq T} n^{-2\sigma} \,.$$

Suppose now that $\sigma > \frac{1}{2}$. In that case, we go further and get

$$\int_{T/2}^{T} |\Sigma|^2 \, dt = O(T)\, \zeta(2\sigma)$$

$$= O(T) \frac{1}{2\sigma - 1} \,. \qquad \boxed{\sigma \in (\tfrac{1}{2}, 2]}$$

At the same time, (see 8 bottom)

$$\int_{T/2}^{T} T^{-\sigma} |\Sigma| \, dt \leq \left\{ \int_{T/2}^{T} T^{-2\sigma} dt \right\}^{\frac{1}{2}} \cdot \left\{ \int_{T/2}^{T} |\Sigma|^2 dt \right\}^{\frac{1}{2}}$$

{by Cauchy - Schwarz}

$$\leq \sqrt{T^{1-2\sigma}} \sqrt{\frac{O(T)}{2\sigma - 1}}$$

$$= O(1) \frac{T^{1-\sigma}}{\sqrt{2\sigma - 1}} = O(1) \frac{T^{1-\sigma}}{2\sigma - 1} \,.$$

Referring to 8 bottom again, we get

$$\int_{T/2}^{T} |\zeta(\sigma+it)|^2 dt$$

$$= \frac{O(T)}{2\sigma-1} + \frac{O(T^{1-\sigma})}{2\sigma-1} + O(T^{1-2\sigma})$$

for $\frac{1}{2} < \sigma \leqq 2$. Accordingly (via $2\sigma \geqq 1$):

$$\boxed{\int_{T/2}^{T} |\zeta(\sigma+it)|^2 dt \leqq \frac{O(1)}{\sigma - \frac{1}{2}} T}$$

for $\frac{1}{2} < \sigma \leqq 2$. The case $\sigma = \frac{1}{2}$ is <u>also</u> OK, but [obviously] not very informative.

Suppose NEXT that we only know $\sigma \geqq \frac{1}{2}$.

Observe that 10 line 1 is still usable.

Being totally crude, we can say:

$$\sum_{n \leqq T} n^{-2\sigma} \leqq \sum_{n \leqq T} \frac{1}{n} \leqq C + \ln T.$$

Hence,

$$\int_{T/2}^{T} |\Sigma|^2 \, dt = O(T \ln T)$$

$\{$ can then mimic 10 bot + 11 top $\}$

$$\Downarrow$$

$$\int_{T/2}^{T} |\zeta(\sigma+it)|^2 \, dt = O(T \ln T) + O(1) T^{1-\sigma} \sqrt{\ln T} + O(T^{1-2\sigma})$$

$$\boxed{\int_{T/2}^{T} |\zeta(\sigma+it)|^2 \, dt \leqq O(1) T \ln T}$$

for $\frac{1}{2} \leqq \sigma \leqq 2$.

We have obtained the above box, and 11 box, assuming $T \gtrapprox 1000$ (see 7).

THEOREM (standard a priori estimate)

For $T \geq 3$ and $\sigma \in [\frac{1}{2}, 2]$, we have:

$$\int_2^T |J(\sigma+it)|^2 dt = O(T) \min\{\ln T, \frac{1}{\sigma - \frac{1}{2}}\}.$$

PF

Matters are obvious for $3 \leq T \leq 10^6$. In fact, here,

$$\min\{\ln T, \frac{1}{\sigma-\frac{1}{2}}\} \geq \min\{\ln 3, \frac{1}{3/2}\}$$
$$= \frac{2}{3}$$

and it suffices to adjust the implied constant in $O(T)$.

For $T > 10^6$, choose l so that

$$\frac{T}{2^{l+1}} \leq 1000 < \frac{T}{2^l}.$$

Apply 11 box and 12 box to $\frac{T}{2^k}$ for $k \in [0, l]$.

Add! Get:

$$\int_{1000}^{T} |\zeta(\sigma+it)|^2\,dt = O(T)\min\left\{\ln T, \frac{1}{\sigma-\frac{1}{2}}\right\}.$$

To replace 1000 by 2, repeat the observation used for $3 \leqq T \leqq 10^6$. ▨

With more work, one can prove that:

$$\int_2^T |\zeta(\sigma+it)|^2\,dt \sim T\,\zeta(2\sigma), \quad \sigma > \frac{1}{2}$$

and

$$\int_2^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \sim T\ln T .$$

See Titchmarsh, <u>Theory of $\zeta(s)$</u>, first few sections in chapter 7. We <u>won't</u> need these more precise results.

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On p. 6 bottom, fix $T_1 \in [2, 2.5]$ to be well away from the ordinate of any ζ-zero.

> Since numerical work shows that
>
> $$\rho_1 = \frac{1}{2} + i[14.134725^+] ,$$ Lec 13 (5), Lec 22 (1)
>
> one can declare $T_1 = 2$. We prefer not to use this, however.

Keep $T = T_2 \geqq 3$ and then take

$$\frac{1}{2} < \alpha \leqq 1 \quad \text{and} \quad \beta = 2 \text{ (say)}.$$

One gets:

$$2\pi \int_\alpha^2 N(\sigma; T_1, T)\, d\sigma = \int_{T_1}^{T} \ln|\zeta(\alpha + it)|\, dt - \int_{T_1}^{T} \ln|\zeta(2+it)|\, dt + O(\ln T).$$

BABY LEMMA

(a) for $x \in \mathbb{R}$, $e^x \geqq 1 + x$;

(b) for $y \geqq 0$, $\ln y \leqq \frac{1}{2}(y^2 - 1)$.

$\boxed{\text{clearly very crude}}$

PF

We give 2 proofs of (a). The first notes that $g(x) = e^x$ is concave upward since $g'' > 0$. Hence $g(x)$ sits on or above the tangent line at each point x_0. Take $x_0 = 0$. Get: $g(x) \geqq 1 + x$ by inspection.

The 2nd proof is more boring. For $x > 0$, apply mean value thm to get

$$e^x - 1 = e^{\tilde{x}}(x - 0) = e^{\tilde{x}} \cdot x \quad , \quad 0 < \tilde{x} < x$$

$$e^x - 1 \geqq e^0 \cdot x = x . \quad \text{OK}$$

For $x < 0$, apply mean value theorem to get

$$1 - e^x = e^{\tilde{x}}(0 - x) = (-x)e^{\tilde{x}} \quad , \quad x < \tilde{x} < 0$$

$$1 - e^x \leqq (-x)e^0$$

$$e^x - 1 \geqq x . \quad \text{OK}$$

In (b), wlog $y > 0$. Put $y = e^u$ with $u \in \mathbb{R}$.

Must check that

$$u \lesseqqgtr \frac{1}{2}(e^{2u}-1)$$

$$2u \lesseqqgtr e^{2u}-1$$

$$e^{2u} \gtrapprox 1+2u \,.$$

But this is obvious by (a). ▥

By BABY LEMMA, then,

$$\int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt \lessapprox \frac{1}{2}\int_{T_1}^{T}\left(|\zeta(\sigma+it)|^2-1\right)dt$$

$$\lessapprox \frac{1}{2}\int_{T_1}^{T}|\zeta(\sigma+it)|^2\,dt$$

$$\lessapprox \frac{1}{2}\int_{2}^{T}|\zeta(\sigma+it)|^2\,dt$$

$$\leqq \frac{O(T)}{\sigma-\frac{1}{2}} \quad \text{by (13)}.$$

In addition,

$$\operatorname{Log} \zeta(2+it) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-2-it} \quad ;$$

$$\begin{aligned}
\int_{T_1}^{T} \ln|\zeta(2+it)|\,dt &= \operatorname{Re} \int_{T_1}^{T} \operatorname{Log} \zeta(2+it)\,dt \\
&= \operatorname{Re} \int_{T_1}^{T} \sum_{2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-2-it}\,dt \\
&= O(1) \sum_{2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-2} \left[\frac{e^{-it\ln n}}{-i\ln n} \right]_{T_1}^{T} \\
&= O(1) \sum_{2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-2} \frac{1}{\ln n} \\
&= O(1) \,. \quad //
\end{aligned}$$

Page 15 bottom then gives:

$$\begin{aligned}
2\pi \int_{\alpha}^{2} N(\sigma; T_1, T)\,d\sigma &\leq \frac{O(T)}{\alpha - \frac{1}{2}} + O(1) + O(\ln T) \\
&\leq \frac{O(T)}{\alpha - \frac{1}{2}} \quad // \,.
\end{aligned}$$

here $\alpha \in (\frac{1}{2}, 1]$

Write $\alpha = \frac{1}{2} + 2\omega$, $0 < \omega \leq \frac{1}{4}$. Notice that

$$N(\tfrac{1}{2} + 2\omega; T_1, T) \leq \frac{1}{\omega} \int_{\frac{1}{2}+\omega}^{\frac{1}{2}+2\omega} N(\sigma; T_1, T)\, d\sigma$$

since $N(\sigma; T_1, T)$ is monotonic decreasing in σ.

Thus:

$$N(\tfrac{1}{2} + 2\omega; T_1, T) \leq \frac{1}{\omega} \int_{\frac{1}{2}+\omega}^{2} N(\sigma; T_1, T)\, d\sigma$$

$$\leq \frac{1}{\omega} \frac{O(T)}{\omega} = \frac{4}{(2\omega)^2} O(T).$$

In other words,

$$\boxed{N(\alpha; T_1, T) \leq \frac{O(T)}{(\alpha - \frac{1}{2})^2}, \quad \frac{1}{2} < \alpha \leq 1}.$$

This <u>proves</u> the THM on p. 2. ▨

Because our use of 16(b) was so crude, one <u>suspects</u> that the foregoing box can be improved in the α-aspect.

cf. also 18 bottom

This is indeed correct. We claim, in fact, that

$$\int_{T_1}^{T} \ln|\zeta(\alpha+it)|\,dt = O(T)\ln\left(\frac{1}{\alpha-\frac{1}{2}}\right).$$

On 19, this leads to

$$\boxed{N(\alpha;T_1,T) \leqq O(T)\,\frac{1}{\alpha-\frac{1}{2}}\ln\left(\frac{1}{\alpha-\frac{1}{2}}\right)}.$$

There are 2 approaches to proving line 2.

Method I

By 18 (bottom), wlog $\alpha \in (\frac{1}{2}, \frac{3}{5}]$.

Under this hypothesis, we have $r \equiv \frac{1}{\alpha-\frac{1}{2}} \geqq 10$.

Notice that $\ln y \leqq \frac{1}{2\lambda}(y^{2\lambda}-1)$ for $0<\lambda\leqq 1$ [and $y \geqq 0$]. See 16. We'll keep $\lambda<1$. Get:

$$\int_{T_1}^{T} \ln|\zeta|\,dt \leqq \frac{1}{2\lambda}\int_{T_1}^{T}\left(|\zeta|^{2\lambda}-1\right)dt$$

$$\leqq \frac{1}{2\lambda}\int_{T_1}^{T}|\zeta|^{2\lambda}\,dt$$

$$\leqq \frac{1}{2\lambda}\left(\int_{T_1}^{T}|\zeta|^{2\lambda\frac{1}{\lambda}}dt\right)^{\lambda}\left(\int_{T_1}^{T}1\,dt\right)^{1-\lambda}$$

{ by Hölder's inequality } $\boxed{p=\frac{1}{\lambda} \quad q=\frac{1}{1-\lambda}}$

$$\leqq \frac{1}{2\lambda}\left(\frac{O(T)}{\sigma-\frac{1}{2}}\right)^{\lambda} T^{1-\lambda}$$

by 13

$$\leqq \frac{C}{\lambda}\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{\lambda} T$$

$$= \frac{C}{\lambda}\, r^{\lambda}\cdot T\,.$$

Put $\lambda = \frac{1}{\ln r}$. This is <u>admissible</u> since $r \geqq 10$.

Get:

$$\int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt \leqq C(\ln r)\,e\,T$$

$$\leqq O(T)\ln\left(\frac{1}{\sigma-\frac{1}{2}}\right).$$

OK

Method II

We use Jensen's inequality common in probability and measure theory.

To recall it, let $\Phi(v)$ be non-negative and convex on $\mathbb{R}$. (Hence Φ is automatically continuous.) Let Y be an extended real-valued random variable on a probability space $(\mathfrak{X}, \mu)$. Assume that $\mathbb{E}(Y)$ exists (i.e. that $Y \in L_1(\mu)$). Then:

$$\Phi(\mathbb{E}(Y)) \leqq \mathbb{E}(\Phi(Y)).$$

One simply approximates $\mathbb{E}(Y)$ by an obvious Riemann sum and uses

$$\left\{ \begin{array}{l} \Phi\left(\sum_{j=1}^{N} t_j v_j \right) \leqq \sum_{j=1}^{N} t_j \Phi(v_j) \\ \text{for } t_j \in [0,1] \text{ with } t_1 + \cdots + t_N = 1 \end{array} \right\}.$$

Put $\Phi(v) = \exp(v)$ to get

$$\exp\left(\frac{1}{H} \int_0^H f(t)\,dt \right) \leqq \frac{1}{H} \int_0^H \exp[f(t)]\,dt$$

anytime $f \in L_1[0, H]$.

A trivial specialization gives:

$$\exp\left(\frac{1}{T-T_1} \int_{T_1}^{T} 2 \ln |\zeta(\sigma+it)|\, dt \right)$$

$$\leqq \frac{1}{T-T_1} \int_{T_1}^{T} |\zeta(\sigma+it)|^2\, dt$$

$$\leqq \frac{1}{T-T_1} \frac{O(T)}{\sigma-\frac{1}{2}} \quad \text{by (13)}$$

$$\leqq \frac{O(1)}{\sigma-\frac{1}{2}} .$$

Accordingly:

$$\frac{1}{T-T_1} \int_{T_1}^{T} 2 \ln |\zeta(\sigma+it)|\, dt$$

$$\leqq \ln\left(\frac{B}{\sigma-\frac{1}{2}} \right) \qquad \left\{ \begin{matrix} \text{some} \\ B \geqq 1 \end{matrix} \right\}$$

$$\Downarrow$$

$$\int_{T_1}^{T} \ln |\zeta(\sigma+it)|\, dt \leqq O(T) \ln\left(\frac{1}{\sigma-\frac{1}{2}} \right)$$

for $\sigma \in (\frac{1}{2}, 1]$. OK

Incidentally, observe how 13 THM gets applied in both methods I + II. Switching to $\ln T$ in place of $\frac{1}{\sigma-\frac{1}{2}}$ produces the following:

method I

get $\frac{C}{\lambda}(\ln T)^{\lambda}\, T$ on 21 middle

$\Rightarrow$ take $\lambda = \frac{1}{\ln\ln T}$ (T giant)

$\Rightarrow \int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt = O(T\ln\ln T)$

uniformly for $\sigma \in [\frac{1}{2}, 1]$

method II

get $\leqq O(1)\ln T$ on 23 middle

$\Rightarrow \frac{1}{T-T_1}\int_{T_1}^{T} 2\ln|\zeta|\,dt \leqq \ln(B\ln T)$

$\Rightarrow \int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt = O(T\ln\ln T)$

uniformly for $\sigma \in [\frac{1}{2}, 1]$.

HENCE:

$$\boxed{\int_{T_1}^{T} \ln|\zeta(\sigma+it)|\,dt = O(T)\,\ln\left[\min\left(\ln T, \frac{1}{\sigma-\frac{1}{2}}\right)\right]}.$$

Closing Remark.

The estimate

$$N\left(\tfrac{1}{2}+\varepsilon\,;\,T_1,T\right) = O(T)\,\frac{1}{\varepsilon}\ln\left(\frac{1}{\varepsilon}\right)$$

mentioned on p. 20 (line 4) was obtained by Littlewood in 1924. Proc. Cambr. Philo. Soc. 22(1924)

It is possible to **expunge** the term $\ln\frac{1}{\varepsilon}$.

This was shown by A. Selberg around 1942 with the aid of some fundamentally new ideas. By use of a so-called mollifier method, Selberg was able to demonstrate that

$$\int_{\frac{1}{2}}^{2} N(\sigma\,;\,T_1,T)\,d\sigma = O(T).$$

Compare 15 (bottom) + 18 (middle).

See also Titchmarsh, Theory of $\zeta(s)$, around §9.24.