Lecture 28

(Fri, Apr 29)

A short presentation of D. J. Newman's proof of PNT. It's fun. And it's slick.

{ Bak and Newman, Complex Analysis, chap 19, 3rd ed.
Also: Monthly 87 (1980) 693–696. }

Known Facts

1. $\zeta(s) = \sum_1^\infty n^{-s}$ analytic $re(s) > 1$
2. $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ nice convergence $re(s) > 1$ Lec 6 (3)
3. $\zeta(s) - \frac{1}{s-1}$ analytic $re(s) > 0$ Lec 5 (6) + (10)
4. $\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$ near $s = 1$ Lec 17 (40)
5. $\operatorname{Log} \zeta(s) = \sum_{n=2}^\infty \frac{\Lambda(n)}{\log n} n^{-s}$ $re(s) > 1$ Lec 6 (4)
6. $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^\infty \frac{\Lambda(n)}{n^s}$ $re(s) > 1$ Lec 6 (6)
7. $\zeta(s) \neq 0$ on $Re(s) \geq 1$ Lec 6 (6) (7)

Newman likes

$$\phi(s) \equiv \sum_p \frac{\ln p}{p^s}, \quad re(s) > 1.$$

Obviously,

$$-\frac{\zeta'(s)}{\zeta(s)} = \phi(s) + \underbrace{\sum \frac{\ln p}{p^{2s}} + \sum \frac{\ln p}{p^{3s}} + \cdots}$$

and the underlined fcn is analytic on $Re(s) > \frac{1}{2}$.

Indeed,

$$(\ln p)\sum_{n=2}^{\infty} p^{-n\sigma} = (\ln p)\frac{p^{-2\sigma}}{1-p^{-\sigma}}$$

$$\leqq (\ln p)\frac{p^{-2\sigma}}{1-2^{-\sigma}} \qquad \sigma \geqq 1+\varepsilon$$

etc etc.

Let $E(s)$ mean a fcn analytic on $\mathrm{Re}(s) > \frac{1}{2}$.
Not necessarily the same one each time...

$$\boxed{-\frac{\zeta'(s)}{\zeta(s)} = \phi(s) + E(s).}$$

FACT 1

Write

$$\zeta(s) = (s-1)^{-1}\left[1+\gamma(s-1) + O(s-1)^2\right] \qquad \text{near } s=1.$$

Take logarithmic derivative. Get

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \left(\gamma + O(s-1)\right).$$

$$\boxed{\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + O(s-1) \quad \text{near } s=1.}$$

FACT 2

see Lec 17 p. 42

Recall

$$\psi(x) \equiv \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \ln p \qquad (x > 1)$$

$$= \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \cdots$$

with

$$\theta(x) = \sum_{p \leq x} \ln p \quad .$$

See Lec 1 4 12.

Additional Known Fact for $x \geq 2$

8. $$c_1 x \leq \psi(x) \leq c_2 x \,, \quad c_3 x \leq \theta(x) \leq c_4 x \qquad (c_j > 0)$$

$$\psi(x) = \theta(x) + O(x^{1/2})$$

"Chebyshev"

See Lec 1 4 5 16 – 18.

$$\boxed{\begin{gathered} \psi(x) = \theta(x) + R(x) \\ R(x) = O(x^{1/2}) \\ x \geq 1 \,. \end{gathered}}$$

FACT 3

Recall

$$\mathrm{re}(s) > 1 \quad \Rightarrow \qquad (\psi(x) = 0,\ x < 2)$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty x^{-s}\, d\psi(x)$$

$$= \left[x^{-s}\psi(x) \right]_1^\infty - \int_1^\infty \psi(x)\, d(x^{-s})$$

$$= 0 + s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx$$

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx$$

and $$\frac{1}{s-1} = \int_1^\infty \frac{x}{x^{s+1}}\, dx$$

$$\boxed{-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\psi(x) - x}{x^{s+1}}\, dx \qquad \mathrm{re}(s) > 1 .}$$

FACT 4

37 line 7

See Ingham 18(17), 91(8) and Lec 8 11.

But,

$$\psi(x) - x = \theta(x) - x + R(x) \qquad \text{see (3), Fact 3}$$

and

$$\int_1^\infty \frac{R(x)}{x^{s+1}}\,dx = \text{analytic on } Re(s) > \tfrac{1}{2}$$
$$(\text{since } R(x) = O(\sqrt{x})) \ .$$

So,

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\theta(x) - x}{x^{s+1}}\,dx + E(s)$$

à la (2)

FACT 5

$$\int_1^\infty \frac{\theta(x) - x}{x^{s+1}}\,dx = -\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} + E(s)$$

on $Re(s) > 1$ and we get:

(a) LHS has a meromorphic continuation to $Re(s) > 1/2$

(b) LHS has **no** pole at $s = 1$

(c) LHS has **no** poles on $\{Re(s) \geqq 1\}$.

(a) is obvious by 1.

(b) is easy by 2 (bottom) ; (c) then follows by 1. $\blacksquare$

Next:

$$\int_1^\infty \frac{\theta(x)-x}{x^s}\frac{dx}{x} = \int_0^\infty \frac{\theta(e^v)-e^v}{e^{sv}}\,dv$$

$$\left\{ \begin{array}{l} x = e^v \\ v = \ln x \end{array} \right\}$$

shift $s \to s+1$

$$\Rightarrow \text{get } \int_0^\infty \frac{\theta(e^v)-e^v}{e^{(s+1)v}}\,dv$$

$$= \int_0^\infty e^{-sv}\left[\frac{\theta(e^v)}{e^v} - 1\right] dv .$$

<u>FACT 6</u>

$$\int_0^\infty e^{-sv}\left[\frac{\theta(e^v)}{e^v}-1\right] dv$$

(a) is analytic on $\mathrm{Re}(s) \gtrsim 0$

(b) is meromorphic on $\mathrm{Re}(s) > -\frac{1}{2}$

(c) <u>secretly</u> has poles at $\rho - 1$, where

$$\xi_0(\rho) = 0 \ , \quad \xi_0(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\!\left(\frac{s}{2}\right)\zeta(s)$$

à la Lec 13 4 5 .

On the other hand, following Newman,

$$\phi(s) = \int_1^\infty u^{-s}\, d\theta(u), \quad re(s) > 1 \qquad \begin{pmatrix}\theta(u)=0\\ u<2\end{pmatrix}$$

$$= u^{-s}\theta(u)\Big]_1^\infty - \int_1^\infty \theta(u)\, d(u^{-s})$$

$$= s\int_1^\infty \frac{\theta(u)}{u^{s+1}}\, du$$

$$= s\int_0^\infty e^{-sv}\theta(e^v)\, dv$$

but

$$\frac{s}{s-1} = s\int_0^\infty e^{-sv} e^v\, dv$$

$$\Downarrow$$

$$\phi(s) - \frac{s}{s-1} = s\int_0^\infty e^{-vs}\left[\theta(e^v) - e^v\right] dv$$

$$\frac{\phi(s)}{s} - \frac{1}{s-1} = \int_0^\infty e^{-sv}\left[\theta(e^v) - e^v\right] dv \qquad re(s) > 1$$

$$\boxed{\frac{\phi(s+1)}{s+1} - \frac{1}{s} = \int_0^\infty e^{-sv}\left[\frac{\theta(e^v)}{e^v} - 1\right] dv \qquad re(s) > 0}$$

FACT 7

same fcn
as in Fact 6

DEFINITION

$$g(s) \equiv \frac{\phi(s+1)}{s+1} - \frac{1}{s} \;;$$

$$f(v) \equiv \frac{\theta(e^v)}{e^v} - 1 \qquad (v \geq 0) \,.$$

By 7, Fact 7, know

$$g(s) = \int_0^\infty e^{-sv} f(v)\,dv\,, \qquad re(s) > 0$$

in the style of a Laplace transform.

6 Fact 6 allows us to better understand g.

THEOREM

g and f as above. Then:

(i) $g(s) = \int_0^\infty e^{-sv} f(v)\,dv$, $re(s) > 0$

(ii) $f(v)$ is bounded and <u>piecewise</u> C^∞

(iii) $g(s)$ is meromorphic on $Re(s) > -\frac{1}{2}$

BUT HAS NO POLES on $Re(s) \geq 0$.

Proof

(i) as above. (ii) by Chebyshev on 3.

(iii) see Fact 6. ∎

NEWMAN'S GENERAL THM

Let $f(v)$ be ANY bounded, piecewise continuous fcn on $[0,\infty)$. Let

$$g(s) = \int_0^\infty e^{-sv} f(v)\,dv, \quad \mathrm{re}(s) > 0.$$

ASSUME THAT g extends to a single-valued analytic function on a connected open set slightly bigger than $\mathrm{Re}(s) \geqq 0$. (Call it g again.) Then:

$$\int_0^\infty f(v)\,dv \text{ exists and equals } g(0).$$

Pf

$\int_0^\infty f(v)\,dv$ means $\lim_{T\to\infty} \int_0^T f(v)\,dv$!

To convey the function theory flavor, switch to $z = x+iy$ instead of s.

Let

$$g_T(z) = \int_0^T e^{-zv} f(v)\,dv, \quad T > 0.$$

The fcn $g_T(z)$ is entire for each T. [Simply view as a standard limit of Riemann sums. Recall Lec 3 18.]

$\underset{(N \to \infty)}{\wr}$

Must prove:

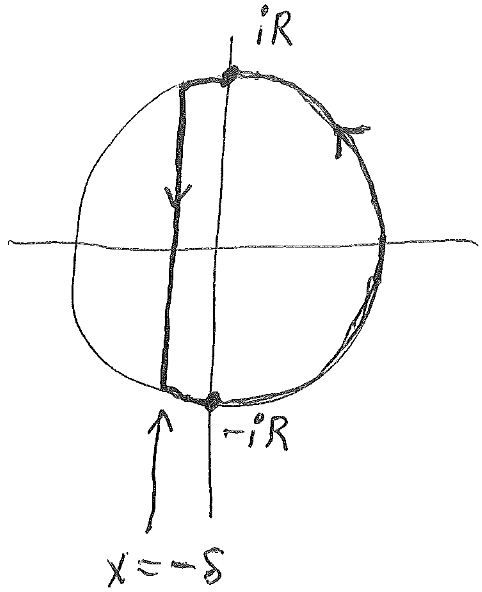
$$\lim_{T \to \infty} g_T(0) = g(0).$$

Take R giant and <u>freeze</u> it!

Select a tiny $\delta > 0$ (depending on R) so that $g(z)$ is nicely analytic on

$$\{ |s| \leqq R \} \cap \{ x \geqq -\delta \}.$$

⟦ That means on a slightly bigger open set! ⟧



$C =$ heavy path

$C_+ =$ portion with $x > 0$

$C_- =$ portion with $x < 0$

Apply Cauchy integral formula to

$$\underbrace{[\,g(z)-g_T(z)\,]\,e^{zT}\left(1+\frac{z^2}{R^2}\right)}_{\text{KEY}} .$$

Newman's Trick

Get:

$$g(0)-g_T(0)=\frac{1}{2\pi i}\oint_C [g-g_T]\,e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} .$$

We estimate RHS in several steps.

First over C_+. Let $B=\sup_{v\geq 0}|f(v)|$.

On C_+,

$$|g(z)-g_T(z)| = \left|\int_T^\infty e^{-zv}f(v)\,dv\right|$$

$$\leq B\int_T^\infty e^{-xv}\,dv$$

$$= B\,\frac{e^{-xT}}{x}$$

$$|e^{zT}| = e^{xT}$$

$$\left|1+\frac{z^2}{R^2}\right| = \left|1+\frac{z^2}{z\bar{z}}\right| = \left|1+\frac{z}{\bar{z}}\right|$$

$$= \frac{|\bar{z}+z|}{R}$$

$$= \frac{2|x|}{R} = \frac{2x}{R}$$

$$|g-g_T|\,|e^{zT}|\,\left|1+\frac{z^2}{R^2}\right| \leq \frac{2B}{R}$$

$$\Downarrow$$

$$\left|\frac{1}{2\pi i}\int_{C_+}(g-g_T)\,e^{Tz}\left(1+\frac{z^2}{R^2}\right)dz\right|$$

$$\leq \frac{1}{2\pi}\int_{C_+}\frac{2B}{R}\frac{|dz|}{R}$$

$$= \frac{1}{2\pi}\frac{2B}{R}\frac{\pi R}{R} = \frac{B}{R}\,. \quad /\!/\!/$$

For C_-, we write:

$$I_1 = \frac{1}{2\pi i} \int_{C_-} g(z)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

$$I_2 = \frac{1}{2\pi i} \int_{C_-} g_T(z)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} .$$

g_T is entire 10.

For I_2, by deformation of contour, note that

$$I_2 = \frac{1}{2\pi i} \int_{iR}^{-iR} [\,\ldots\,] \frac{dz}{z}$$

$\begin{pmatrix} \text{left half} \\ |z| = R \end{pmatrix} \leftarrow$ see 10

here, on $|z| = R$, have

$$|g_T(z)| \leq \int_0^T e^{-vx} |f(v)|\, dv \qquad \text{9 bot}$$

$\{x < 0\}$

$$\leq B \int_0^T e^{v|x|}\, dv$$

$$= B\, \frac{e^{T|x|} - 1}{|x|} \leq B\, \frac{e^{T|x|}}{|x|}$$

$$|e^{zT}| = e^{xT} = e^{-|x|T}$$

$$\left|1+\frac{z^2}{R^2}\right| = \left|1+\frac{z^2}{z\bar{z}}\right| = \left|1+\frac{z}{\bar{z}}\right|$$

$$= \frac{2|x|}{R}$$

take <u>product</u> to get

$$|[\cdots]| \leq B\frac{e^{T|x|}}{|x|}e^{-|x|T}\frac{2|x|}{R} = \frac{2B}{R}$$

$$\Downarrow$$

$$\left|\frac{1}{2\pi i}\int_{\substack{\text{left half} \\ |z|=R}} [\cdots]\frac{dz}{z}\right| \leq \frac{1}{2\pi}\frac{2B}{R}\pi = \frac{B}{R}. \quad //$$

I_2

Must now do I_1. We'll do Newman's method first and, then, note an alternate reasoning.

interesting

R is <u>frozen</u>, as is δ. 10

Look at the integrand

$$e^{zT} g(z)\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\,dz$$

on curve C_-. Each chunk

$$\left\{\, g(z),\ 1+\frac{z^2}{R^2},\ \frac{1}{z} \,\right\}$$

is bounded <u>by something</u>. So is e^{zT};

$$|e^{zT}| = e^{xT} \leqq e^0 = 1 .$$

Switch now to a parametric representation of C_-, say $z = z(\lambda)$, $0 \leqq \lambda \leqq 1$, $\lambda \uparrow$.

Get new integral

$$\int_0^1 \mathcal{B}(\lambda)\, e^{z(\lambda)T}\, z'(\lambda)\,d\lambda$$

$\uparrow$ continuous + bounded

can now apply an elementary bounded convergence thm for Riemann integrals, since

$$\left| e^{z(\lambda)T} \right| = e^{x(\lambda)T}, \quad \lambda \in [0,1]$$

$$\leqq 1$$

AND

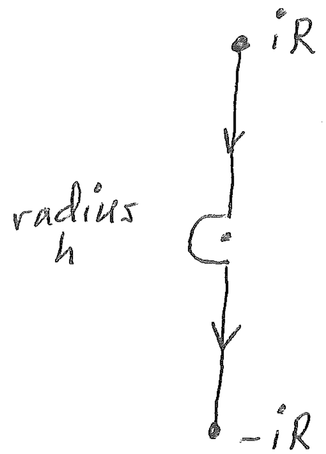
$$\lim_{T\to\infty} e^{x(\lambda)T} = 0 \quad \text{pointwise on } 0<\lambda<1.$$

In fact, this last limit is <u>uniform</u> on each $[\varepsilon, 1-\varepsilon]$. Get:

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{C_-} g(z)\left(1+\frac{z^2}{R^2}\right)e^{zT}\,\frac{dz}{z} = 0. \,/\!/$$

A highly suggestive <u>alternate</u> approach to I_1 goes as follows.

Take $h > 0$ microscopic. Make a new path $C_-(h)$ à la



By the extended (limit) form of the CIF, we have

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_{C_+} (g - g_T)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

$$+ \frac{1}{2\pi i} \int_{C_-(h)} (g - g_T)\, e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

see 9

anytime g is ONLY known to be continuous on $\{x \geq 0\}$ <u>and</u> <u>analytic</u> <u>near</u> <u>$z = 0$</u>.

The I_2 part of the $C_-(h)$ integral again gives $\Theta \frac{B}{R}$, $|\Theta| \leq 1$. See 13 14.

For the I_1 portion, use $C_-(h)$ as given:

$$\frac{1}{2\pi i}\int_{iR}^{ih} g(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \longleftarrow I_{11}$$

$$+\ \frac{1}{2\pi i}\int_{\substack{|z|=h \\ \text{left}}} g(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \longleftarrow I_{12}$$

$$+\ \frac{1}{2\pi i}\int_{-ih}^{-iR} g(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \longleftarrow I_{13}$$

Note:

$$I_{11} = (\text{const})\int_h^R g(iy)\, e^{iyT}\left(1-\frac{y^2}{R^2}\right)\frac{1}{y}\, dy$$

$$= o(1) \quad \text{by } \underline{\text{Riemann}}-\underline{\text{Lebesgue}} \text{ lemma}$$

$$\{h, R \text{ fixed}\}$$

$I_{13} = o(1)$ similarly

$I_{12} = o(1)$ by a mimic of 15 (bot) + 16

$\{h > 0 \text{ fixed}\}$

So,

$$I_1 = o(1) \, . \; //$$

End of Alternate Approach!

Remember $R =$ giant, but fixed.

Get:

$$\limsup_{T \to \infty} \left| \frac{1}{2\pi i} \int_C (g - g_T) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right|$$

$$= \limsup_{T \to \infty} |g(0) - g_T(0)|$$

$$\leq \frac{B}{R} + \frac{B}{R} + 0 \quad \text{by } (12), (14), (16) \text{ or line 2 above}$$

$$= \frac{2B}{R} \, .$$

Since R is arbitrary, deduce that

$$\limsup_{T \to \infty} |g(0) - g_T(0)| = 0 \, . \quad \blacksquare$$

Corollary

$$\int_1^\infty \frac{\theta(x) - x}{x^2}\, dx \quad \text{is convergent.}$$

Pf

Recall 7 (bottom) + 8. Then apply Newman's general thm. Get

$$\int_0^\infty \left[\frac{\theta(e^v)}{e^v} - 1 \right] dv \quad \text{converges}$$

$$\{ x = e^v, \ v = \ln x \}$$

$$\int_1^\infty \frac{\theta(x) - x}{x} \frac{dx}{x} \quad \text{converges.} \quad \blacksquare$$

FACT

Suppose $H(x)$ is piecewise continuous on $[1,\infty)$. Suppose $H(x)\nearrow$. Suppose

$$\int_1^\infty \frac{H(x)-x}{x^2}\,dx \quad \text{converges}$$

as an improper integral. Then

$$H(x) \sim x \quad \text{as} \quad x \to \infty .$$

Pf

Suppose $H(x) \geqq \lambda x$ frequently as $x \to \infty$ for some $\lambda > 1$. Notice that, AT SUCH x,

$$H(u) \geqq \lambda x \quad \text{on} \quad [x, \lambda x] \qquad (\text{by } H\nearrow)$$

$$H(u) - u \geqq \lambda x - u \quad \text{here}$$

$$\int_x^{\lambda x} \frac{H(u)-u}{u^2}\,du \geqq \int_x^{\lambda x} \frac{\lambda x - u}{u^2}\,du$$

put $u = xw$

$$= \int_1^{\lambda} \frac{\lambda x - xw}{x^2 w^2}(x\,dw)$$

$$= \int_1^{\lambda} \frac{\lambda - w}{w^2}\,dw > 0 .$$

This violates

$$\left| \int_{y_1}^{y_2} \frac{H(u)-u}{u^2}\,du \right| < \varepsilon$$

for all $y_2 \geqq y_1 \geqq y_\varepsilon$.

Now let $H(x) \leqq \eta x$ frequently as $x \to \infty$ for some $\eta < 1$. Look AT SUCH x.

$$H(u) \geqq \eta x \quad \underline{\underline{\text{on}}} \quad [\eta x, x] \qquad (\text{by } H \uparrow)$$

$$H(u) - u \geqq \eta x - u \qquad \text{here}$$

$$\int_{\eta x}^{x} \frac{H(u)-u}{u^2}\,du \geqq \int_{\eta x}^{x} \frac{\eta x - u}{u^2}\,du$$

put $u = xw$

$$= \int_{\eta}^{1} \frac{\eta x - xw}{x^2 w^2}(x\,dw)$$

$$= \int_{\eta}^{1} \frac{\eta - w}{w^2}\,dw$$

$$= -\int_{\eta}^{1} \frac{w-\eta}{w^2}\,dw < 0.$$

This violates the Y_1, Y_2 condition above.

So,

$$H(x) \sim x. \quad \blacksquare$$

<u>Corollary</u> (PNT)

$$\theta(x) \sim x.$$

<u>Pf</u>

Combine 20 + 21. $\blacksquare$

<u>REMARKS.</u>

$\boxed{1}$ Clearly, a <u>very</u> nice proof! ☺

$\boxed{2}$ It is reasonable to conjecture Newman (actually) began with 4 box, the FACT on 21, and Landau, Gött. Nachr. 1932 [attached below].

$\boxed{3}$ Various <u>extensions</u> of the THM on 9 have been made based on the <u>idea</u> of 17, 18, 19 top.

$\boxed{4}$ We'll <u>return</u> to page 9 THM a bit later, in a comment about lecture 30.

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Göttinger Nachr. $\boxed{1932}$

pp. 525–527

$\boxed{\text{This theorem is essentially the Wiener – Ikehara Tauberian theorem (when you let } \lambda \to \infty).}$

Über Dirichletsche Reihen.

Von

Edmund Landau.

Vorgelegt in der Sitzung am 25. November 1932.

No complex variables; only harmonic analysis.

Durch Weiterführung der N. Wienerschen Methode bewiesen Herr Heilbronn und ich[1]) den

Satz: *Es gibt zwei für $\lambda > 0$ definierte positive Funktionen $P_1(\lambda)$ und $P_2(\lambda)$ mit*

$$\lim_{\lambda=\infty} P_1(\lambda) = \lim_{\lambda=\infty} P_2(\lambda) = 1$$

und folgender Eigenschaft.

Die Dirichletsche Reihe

note

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad \underline{a_n \geqq 0},$$

konvergiere für $\sigma > 1$.

(Trivialerweise ist also,

$$e^{-y} \sum_{\lambda_n \leqq y} a_n = H(y) \quad \text{für} \quad y \gtreqless 0$$

gesetzt[2]),

$$f(s) = s \int_{-\infty}^{\infty} H(y) e^{-y(s-1)} dy \quad \text{für} \quad \sigma > 1.)$$

Für $|t| \leqq 2\lambda$, $\sigma = 1 + \varepsilon$, $\varepsilon > 0$, sei bei $\varepsilon \to 0$ gleichmäßig in t

$$h_\varepsilon(t) = f(s) - \frac{1}{s-1} \to h(t).$$

Dann ist

$$P_1(\lambda) \geqq \overline{\lim_{y=\infty}}\, H(y) \geqq \underline{\lim_{y=\infty}}\, H(y) \geqq P_2(\lambda).$$

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1) *Bemerkungen zur vorstehenden Arbeit von Herrn Bochner* (Mathematische Zeitschrift, im Druck). Wegen aller historischen Bemerkungen verweise ich auf diese Arbeit. MZ 37 (1933) 10–16

2) Für $y_2 \geqq y_1$ ist also $H(y_2) \geqq H(y_1)\, e^{y_1 - y_2}$.

Landau, Collected Works vol. 9

Ich teile hier einen noch mehrfach vereinfachten Beweis unseres Satzes mit, bei dem ich o. B. d. A. $\lambda_1 = 0$ annehmen darf.

Hilfssatz:

$$\lim_{y=\infty} \int_{-\infty}^{\lambda y} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv = \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2}\, dv = \pi. \quad ^{3)}$$

Beweis: Für $\varepsilon > 0$ ist

$$\frac{1}{2} \int_{-2\lambda}^{2\lambda} e^{yti}\left(1 - \frac{|t|}{2\lambda}\right) \frac{h_\varepsilon(t) - 1}{1 + \varepsilon + ti}\, dt$$

$$= \frac{1}{2} \int_0^\infty (H(u) - 1)\, e^{-\varepsilon u}\, du \int_{-2\lambda}^{2\lambda} \left(1 - \frac{|t|}{2\lambda}\right) e^{t(y-u)i}\, dt$$

$$= \int_0^\infty H(u)\, e^{-\varepsilon u} \frac{\sin^2 \lambda (y-u)}{\lambda (y-u)^2}\, du - \int_0^\infty e^{-\varepsilon u} \frac{\sin^2 \lambda (y-u)}{\lambda (y-u)^2}\, du.$$

$\varepsilon \to 0$ ist links und im Subtrahendus rechts, also im Minuendus rechts unter dem Integralzeichen ausführbar.

$$\frac{1}{2} \int_{-2\lambda}^{2\lambda} e^{yti}\left(1 - \frac{|t|}{2\lambda}\right) \frac{h(t) - 1}{1 + ti}\, dt$$

$$= \int_0^\infty H(u) \frac{\sin^2 \lambda (y-u)}{\lambda (y-u)^2}\, du - \int_0^\infty \frac{\sin^2 \lambda (y-u)}{\lambda (y-u)^2}\, du$$

$$= \int_{-\infty}^{\lambda y} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv - \int_{-\infty}^{\lambda y} \frac{\sin^2 v}{v^2}\, dv.$$

Bei $y \to \infty$ strebt das Integral links als FOURIERkonstante gegen 0, das letzte Integral gegen π.

Beweis des Satzes: 1) Mit $a = \sqrt{\lambda}$ ist

$$\pi \geqq \overline{\lim_{y=\infty}} \int_{-a}^{a} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\, dv \geqq \overline{\lim_{y=\infty}} \int_{-a}^{a} H\left(y - \frac{a}{\lambda}\right) e^{-\frac{2a}{\lambda}} \frac{\sin^2 v}{v^2}\, dv,$$

$$\overline{\lim_{y=\infty}}\, H(y) \leqq \pi e^{\frac{2a}{\lambda}} : \int_{-a}^{a} \frac{\sin^2 v}{v^2}\, dv = P_1(\lambda) \to 1.$$

3) Gebraucht wird nur $\pi > 0$, nicht π = LUDOLPHsche Zahl.

2) Nach 1) ist $H(y)$ beschränkt und zuletzt $< 2\,P_1(\lambda)$. Mit $b = \frac{4}{\pi}\,P_1(\lambda) + \sqrt{\lambda}$ ist also

$$\pi = \lim_{y=\infty} \int_{-\infty}^{\sqrt{y}} H\left(y - \frac{v}{\lambda}\right) \frac{\sin^2 v}{v^2}\,dv$$

$$\leqq 2\,P_1(\lambda) \int_{-\infty}^{-b} \frac{dv}{v^2} + \lim_{y=\infty} \int_{-b}^{b} H\left(y + \frac{b}{\lambda}\right) e^{\frac{2b}{\lambda}} \frac{\sin^2 v}{v^2}\,dv + 2\,P_1(\lambda) \int_{b}^{\infty} \frac{dv}{v^2},$$

$$\underline{\lim}_{y=\infty} H(y) \geqq e^{-\frac{2b}{\lambda}} \left(1 - \frac{4\,P_1(\lambda)}{\pi b}\right) = P_2(\lambda) \rightarrow 1.$$