PARTIAL DIARY ENTRY FOR
Lectures 3 and 4
(27 Jan and 29 Jan)

+ SOME NEW STUFF

In lecture #3 and part of #4, we reviewed some key points in complex analysis (a subject regarded by many mathematicians as the <u>most beautiful</u> in mathematics, not only aesthetically but also vis à vis logical unity / coherence).

List of Some Definitions and Thms

$C^\infty(D) \leftarrow$ the usual

$A^\infty(D) =$ the subset of $C^\infty(D)$ consisting of those complex-valued $f = u + iv$ for which we have a complex derivative

"A" for analytic!

$$f'(z_0) \equiv \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at each $z_0 \in D$

$A(D) =$ the set of complex-valued f for which we have a complex derivative $f'(z_0)$ at each $z_0 \in D$ (no other assumptions about $f = u + iv$)

Proved using $\Delta x = \frac{\Delta z + \overline{\Delta z}}{2}$, $\Delta y = \frac{\Delta z - \overline{\Delta z}}{2i}$ that

$$A^{\infty}(D) = \{u+iv \in C^{\infty}(D) : \underbrace{u_x = v_y,\ u_y = -v_x}_{\text{C-R eqs.}}\}.$$

Noted: $A^{\infty}(D)$ is a subring of $C^{\infty}(D)$; also that quotients f_1/f_2, composites $G[f(z)]$, and local inverses $\{$for $w = f(z)$, $f'(z_0) \neq 0\}$ have good properties.

Also noted: $f \in A^{\infty}(D) \Rightarrow f'(z) = u_x + iv_x = \frac{\partial f}{\partial x}$.
And that $f' \in A^{\infty}(D)$ as well.

C-R equations are <u>equivalent</u> to

$$\frac{\partial f}{\partial \bar{z}} \equiv \frac{1}{2}(f_x + i f_y) = 0 \quad \text{on } D.$$

Showed standard examples of $f \in A^{\infty}(\text{suitable } D)$.

z^n, rational fcns of z,
$\exp(z) \equiv e^x(\cos y + i \sin y)$
$\log z = \ln|z| + i\arg(z)$ locally near some $z_1 \neq 0$

In Lec #4, did standard hand-waving about approximating "sensible" $f \in C^{\infty}(\text{suitable } D)$ by polynomials in $(z, \bar{z})$. Hence being able to <u>explain</u> $A^{\infty}(D)$ to "man in the street".

Showed that

$$f \in A^{\infty}(D) \Rightarrow \quad f(z)dz = (u+iv)(dx+idy)$$
$$\equiv (udx-vdy)+i(vdx+udy)$$

is locally exact (i.e., closed). Accordingly, Green's thm can be brought to bear for suitable independence of path results for $\int_{\gamma} f(z)dz$.

no need to belabor

Showed that

$$F \in A^{\infty}(D) \Rightarrow \quad F'(z)dz = d\mathcal{U} + i d\mathcal{V} \quad (F \equiv \mathcal{U} + i\mathcal{V})$$

in the sense of standard differentials on RHS.

This produced

$$\int_{\gamma} F'(z)dz = F(B)-F(A)$$

as the "fundamental thm of complex integral calculus".

If $H(z)$ is continuous on γ, explained $\int_{\gamma} H(z)dz$ and why

$$\left|\int_{\gamma} H dz\right| \leq \int_{\gamma} |H(z)| ds .$$

Proved the standard Cauchy – Goursat thm for $f \in A(D)$, $D \approx$ domain straddling closed rectangle R. Got:

$$\oint_{\partial R} f(z)\,dz = 0 . \qquad (\approx 1900)$$

Used bisection and nested interval/box thm.

Immediately went further to get the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z - z_0}\,dz$$

for $z_0 \in \mathrm{int}(R)$. Here $f \in A(D)$.

Used Leibnitz's rule from adv calc to establish the fund thm that

$$A(\mathcal{D}) = A^{\infty}(\mathcal{D})$$

on any domain $\mathcal{D}$.

Turned quickly to a host of CLASSICAL theorems (in the "Cauchy theory").

(1) Cauchy Integral Thm
(2) Cauchy Integral Formula for $f(z_0)$

(3) Cauchy Integral Formula for $f^{(n)}(z_0)/n!$

(4) Max. Modulus Thm for $|f(z_0)|$, $z_0 \in D$ (gave the slick proof with CIF).

(5) Proved standard Cauchy-Taylor development for $f \in A(D)$, $D = \{|z-z_0| < R\}$: $f = \sum_0^\infty \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$.

(6) Given uniformly conv. $S(z) = \sum_{n=1}^\infty g_n(z)$ with g_n continuous; remarked about $S(z)$ being continuous, and

$$\int_\gamma p(z)\, S(z)\, dz = \sum_{n=1}^\infty \int_\gamma p(z)\, g_n(z)\, dz$$

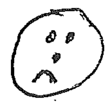
for sensible $p(z)$.

(7) Weierstrass M-test for uniform conv.

(8) Used pathwise connectedness and a "marble" to prove that, when $f \in A(D)$, having $f \equiv 0$ on $\{|z-z_0| < \varepsilon\} \Rightarrow f \equiv 0$ on all D.

(9) Stated Laurent series for $f \in A(D)$, $D = \{R_1 < |z-z_0| < R_2\}$.

(10) Mentioned in one microsecond the idea of isolated singularities. Never said the words remov singularity, pole of order N, essential singularity.



(11) WANTED TO also stress the availability of Abel's lemma for an <u>arbitrary</u> power series $\sum_{n=0}^{\infty} a_n z^n$ which converges at $z_1 \neq 0$.

(12) Likewise, WANTED TO develop the Cauchy residue theorem (CRT)

$$\oint_{\Gamma} f(z)\,dz = 2\pi i \sum_{j=1}^{m} Res(f; z_j)$$

after defining "Residue". Also wanted to do 2 quick examples. *

(13) Very quickly outlined the Weierstrass Convergence Theorem for $\sum_{n=1}^{\infty} f_n(z)$, $f_n \in A(D)$, and its proof (see 9 below).

(Time and Fortitude ran out on (10), (11), (12).)

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* Also missed the argument principle

$$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f'(z)}{f(z)}\,dz = N_0(f) = \frac{1}{2\pi}\Delta_{\Gamma} \arg f \quad ; \quad f \neq 0 \text{ on } \Gamma.$$

Other topics reviewed:

Used nested interval/box thm, bisection, and a Cantor diagonal to prove the Bolzano-Weierstrass thm in $\mathbb{R}^2$. Similarly $\mathbb{R}^k$. Noted that same proof shows that any <u>bdd + closed</u> $E \subseteq \mathbb{R}^k$ is sequentially compact.

Discussed the Riemann integrability criterion for

$$\int_a^b \underbrace{f(x)}_{\text{bdd}}\, d\underbrace{\alpha(x)}_{\text{monotonic increasing on } [a,b]}$$

using $U(P,f,\alpha)$, $L(P,f,\alpha)$ $\{P = \text{partition}\}$ and

$$\underline{\int_a^b} f(x)\,d\alpha(x) \;,\quad \overline{\int_a^b} f(x)\,d\alpha(x).$$

Using uniform continuity of $f \in C[a,b]$, proved that we actually have

$$\int_a^b f(x)\,d\alpha(x) = \lim_{\|P\| \to 0} \sum_{j=1}^{N} f(c_j)\,\Delta\alpha_j$$

whenever $f \in C[a,b]$. $\{\|P\| = \text{largest } \Delta x_j\}$

For $\int_a^b f(x)\,\underline{\underline{dx}}$ à la Riemann, remarked that

$$f \text{ monotonic} \Rightarrow f \in \mathcal{R}[a,b] \qquad \text{(easy)}$$

$$f, g \in \mathcal{R}[a,b] \Rightarrow fg \in \mathcal{R}[a,b].$$

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \qquad a<c<b$$

in a sensible way

Used summation by parts to derive the <u>integration by parts</u> formula:

$$\int_a^b f(x)\,d\alpha(x) = \left[f(x)\alpha(x)\right]_a^b - \int_a^b \alpha(x) f'(x)\,\underline{\underline{dx}}$$

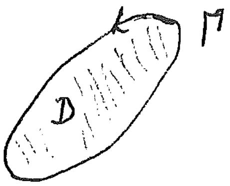
valid whenever $f \in C^1[a,b]$ and $\alpha(x) \nearrow$ on $[a,b]$.

This Riemann integral DOES exist!

One gets an excellent review of the power of the structural properties of analytic functions by studying (developing) several results closely tied to the Weierstrass Convergence Thm.

It is always striking when one appears to be getting something for nothing.

THEOREM (the standard Weierstrass conv thm; similarly for multiply-connected D)



Let D be a simply-connected domain bounded by a piecewise smooth Jordan curve Γ. Let $\{s_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on D which converges to a limit function $s(z)$ for each $z \in D$.

ASSUME THAT the convergence is UNIFORM on every closed subset K of D. Then:

(a) $s(z)$ must be analytic on D;

(b) we automatically have $s_n'(z) \to s'(z)$ uniformly on every closed $K \subsetneqq D$;

(c) similarly for $s_n^{(j)}(z) \to s^{(j)}(z)$, $j \geqq 2$.

$s_n(z)$ could, for instance, be $\sum_{k=1}^{n} f_k(z)$ with $f_k \in A(D)$

Weierstrass' Theorem is very well-known (and important). I sketched the proof of it in Lec #4. See 20¼ – 20¾ below for the essential details.

In the pages that follow, we give a review of some techniques which, taken together, permit one to formulate 2 less well-known [and much more impressive] variants of Weierstrass' theorem.

Lemma (stated in $\mathbb{R}^3$, but valid for $\mathbb{R}^k$, $k \geq 2$)

Let R be the box $[A_1, B_1] \times [A_2, B_2] \times [A_3, B_3]$. Let f be continuous on R. Then:

$$\mathcal{I}(x,y) \equiv \int_{A_3}^{B_3} f(x,y,z)\, dz$$

is continuous on $\mathcal{R} = [A_1, B_1] \times [A_2, B_2]$, and

$$\iiint_R f(x,y,z)\, dV = \iint_{\mathcal{R}} \mathcal{I}(x,y)\, dA$$

as in multi-variable calc.

Pf

As indicated, this is just a form of Fubini's thm from elem calc. The continuity of $\mathcal{I}(x,y)$ is a familiar fact which is part of that theorem (or should be!) and is a simple consequence of the uniform continuity of f on R. ∎

Lemma (stated in $\mathbb{R}^3$ but valid in $\mathbb{R}^k$, $k \geq 2$)

Let R be the box $[A_1, B_1] \times [A_2, B_2] \times [A_3, B_3]$. Let $f(x, y; z)$ be continuous on R. In addition, let all partial derivatives of f wrt x & y also be continuous fcns of $(x, y; z)$ on R. $\{f_x, f_y, f_{xx}, f_{xy}, f_{yy}, \text{etc}\}$ Let

$$I(x,y) = \int_{A_3}^{B_3} f(x, y; z)\, dz .$$

Then $I(x,y)$ is C^∞ on $[A_1, B_1] \times [A_2, B_2]$ and we have

$$\frac{\partial I}{\partial x} = \int_{A_3}^{B_3} \frac{\partial f}{\partial x}(x, y; z)\, dz$$

$$\frac{\partial^2 I}{\partial x \partial y} = \int_{A_3}^{B_3} \frac{\partial^2 f}{\partial x \partial y}(x, y; z)\, dz$$

etc etc.

Pf

This is Leibnitz's rule from advanced calc stated in iterated form — and relying on the foregoing lemma with a general $g(x, y; z)$. The proof is standard adv calc. ▮

Lemma

Let $F = \mathcal{U} + i\mathcal{V}$ (real) be a C^∞ function on, say, the open neighborhood $N = \{|z - z_0| < \delta\}$. The Cauchy-Riemann equations for $\mathcal{U}$ and $\mathcal{V}$ on N are equivalent to the relation

$$F_x + iF_y \equiv 0$$

on N. Hence $F_x + iF_y \equiv 0$ is the condition for F to belong to $A^\infty(N)$ (i.e., C^∞ + analytic).

Pf

Trivial calculation gives

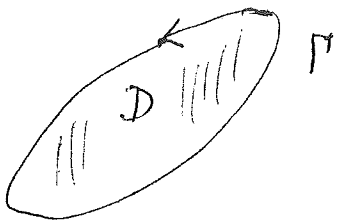
$$F_x + iF_y = \mathcal{U}_x + i\mathcal{V}_x + i(\mathcal{U}_y + i\mathcal{V}_y)$$
$$= (\mathcal{U}_x - \mathcal{V}_y) + i(\mathcal{U}_y + \mathcal{V}_x).$$

Note:

$$\frac{\partial F}{\partial \bar{z}} \equiv \frac{1}{2}(F_x + iF_y)$$ is the standard definition.

Observe: $$\frac{\partial(\bar{z})}{\partial \bar{z}} = 1 \quad \text{and} \quad \frac{\partial(z)}{\partial \bar{z}} = 0.$$

LEMMA (about Cauchy-type integrals; similarly for multiply-connected D)



Let D be a simply-connected domain bounded by a piecewise smooth Jordan curve Γ. (See fig.) Let $\sigma(w)$ be a piecewise continuous complex-valued function on Γ.

Let

$$F(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\sigma(w)}{w-z}\, dw \qquad \text{(DEF.)}$$

for $z \in D$. We then have:

(a) $F(z) \in C^{\infty}(D)$

(b) $F(z) \in A^{\infty}(D)$ (i.e., C^{∞} + analytic)

(c)
$$F^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{\sigma(w)}{(w-z)^{n+1}}\, dw, \quad n \geq 1,\ z \in D.$$

Proof

We can assume WLOG that Γ is smooth and that $\sigma(w)$ is continuous on Γ. Otherwise, in what follows, just split $F(z)$ into a SUM of several natural "chunk integrals".

Convert Γ to a parametric equation $w = w(t)$, $a \leqq t \leqq b$, $t \nearrow$. Get:

$$F(z) = \frac{1}{2\pi i}\int_a^b \frac{\sigma[w(t)]\,w'(t)}{w(t) - z}\,dt$$

Write $z = x + iy$. The integrand, as a fcn. of (x, y, t), satisfies the hypotheses of the Leibnitz Lemma on page 11 so long as $\alpha_1 \leqq x \leqq \beta_1$, $\alpha_2 \leqq y \leqq \beta_2$ is some small rectangle <u>inside</u> D.

Note that the numerator can be "pushed aside" since it has <u>no</u> dependence on z. Also, the partial derivatives of any

$$\frac{1}{W - x - iy} \qquad W \in \Gamma$$

wrt x and y are 100% trivial calc. Of course $\{ (W - z)^{-1}$ lying in $\underline{A}^{\infty} \}$:

$$\frac{\partial}{\partial x}\left(\frac{1}{W - x - iy}\right) + i\,\frac{\partial}{\partial y}\left(\frac{1}{W - x - iy}\right)$$

$$= \frac{1}{(W - x - iy)^2} + i\,\frac{i}{(W - x - iy)^2} = 0\,!$$

(a) and (b) are now immediate on each $(\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$, hence on all D by the Lemmas on 10 + 11 + 12.

To get (c), since we know $F \in A^{\infty}(D)$, just use

$$F^{(n)}(z) = \left(\frac{\partial}{\partial x}\right)^n F(z)$$

and the trivial fact that

$$\left(\frac{\partial}{\partial x}\right)^n (w-x-iy)^{-1} = n!\,(w-x-iy)^{-n-1}$$

[in addition to Leibnitz' Lemma on 11]. ∎

OK

Lemma (stated for $\mathbb{R}^2$, readily adapted to $\mathbb{R}^k$, $k \geq 1$)

Let $s_n(z) = s_n(x+iy)$ be a sequence of [complex-valued] functions on the closed rectangle $R: [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ OR, if you prefer, closed disk R. Assume that $s_n(z) \to$ some function $s(z)$ pointwise for $z \in R$. Assume further that, for some $\mathcal{M} > 0$, we have

$$|s_n(z_1) - s_n(z_2)| \leq \mathcal{M} |z_1 - z_2| \quad *$$

for all $n \geq 1$ and $z_j \in R$. (Uniform Lipschitz condition!) THEN: the convergence of $s_n(z)$ to $s(z)$ is automatically uniform on R.

Pf

The procedure for this is standard. Choose any tiny $\varepsilon > 0$. Look at R and select a finite grid of points $\{P_1, \ldots, P_L\} \subseteq R$ so that every point $z \in R$ is located within $\frac{\varepsilon}{5\mathcal{M}}$ units of some P_ℓ. THIS IS CERTAINLY POSSIBLE!

Since L is finite, we can select N_ε so big

* The word "equicontinuity" may come to mind here.

$$|s_n(P_j^\circ) - s(P_j^\circ)| < \frac{\varepsilon}{10}$$

for all $n \geqq N_\varepsilon$ and all $j \in [1, h]$. Get:

$$|s_n(P_j^\circ) - s_m(P_j^\circ)| < \frac{\varepsilon}{5}$$

for all $n \geqq m \geqq N_\varepsilon$, $j \in [1, h]$.

Take __any__ $z \in R$. Select P_ℓ so that

$$|z - P_\ell| \leqq \frac{\varepsilon}{5\mathcal{M}} .$$

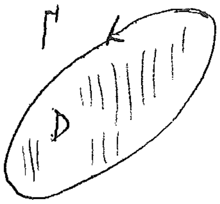
For $n \geqq m \geqq N_\varepsilon$, notice that

$$\begin{aligned} |s_n(z) - s_m(z)| &\leqq |s_n(z) - s_n(P_\ell)| + |s_n(P_\ell) - s_m(P_\ell)| \\ &\quad + |s_m(P_\ell) - s_m(z)| \\ &\leqq \mathcal{M}\left(\frac{\varepsilon}{5\mathcal{M}}\right) + \frac{\varepsilon}{5} + \mathcal{M}\left(\frac{\varepsilon}{5\mathcal{M}}\right) \\ &= \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} < \varepsilon . \end{aligned}$$

This is the standard __uniform__ __Cauchy__ condition for __uniform__ conv of $\{s_i(z)\}_{i=1}^{\infty}$ over R. Indeed, by letting $n \to \infty$, we get

$$|s(z) - s_m(z)| \leqq \varepsilon \quad \text{anytime} \begin{Bmatrix} m \geqq N_\varepsilon \\ z \in R \end{Bmatrix} .$$

Theorem (strengthened Weierstrass conv. thm; similarly for multiply-connected D)



Let D be a simply-connected domain bounded by a piecewise smooth Jordan curve Γ. Let $\{s_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on D which converges pointwise to some [not necessarily analytic] function $s(z)$ for each $z \in D$.

Assume that, for each closed subset K of D, there exists a constant $M(K)$ so that

$$|s_n(z)| \leqq M(K)$$

whenever $z \in K$ and $n \geqq 1$. THEN:

(a) $s(z)$ must be analytic on D;

(b) $s_n(z)$ converges UNIFORMLY to $s(z)$ on every closed subset of D;

(c) we have $s_n'(z) \to s'(z)$ UNIFORMLY on every closed subset of D;

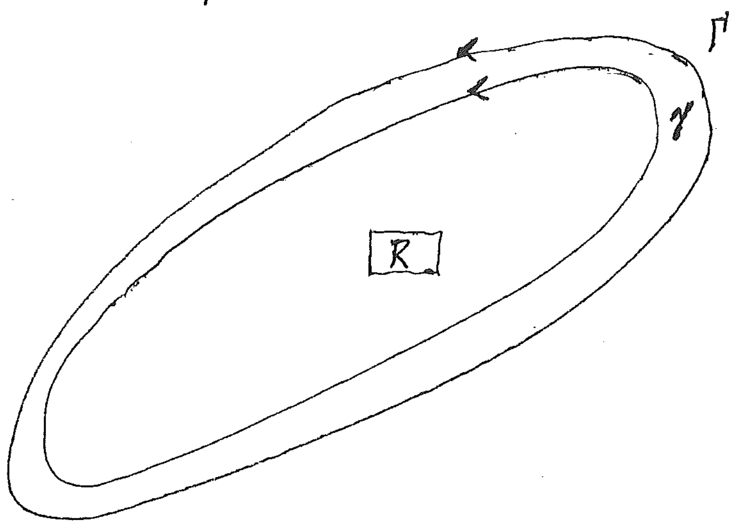
(d) similarly for $s_n^{(j)}(z) \to s^{(j)}(z)$, $j \geqq 2$.

Key Issue: where did all the UNIFORM conv. come from?

Pf

Not surprisingly, we rely on page 16 Lemma.

Every closed set $K \subseteq D$ is automatically bounded. Hence K is bdd + closed; hence, sequentially compact, etc. Any such K will lie a positive distance from Γ. Likewise from a piecewise smooth Jordan curve γ in D "paralleling Γ" extremely closely!



Note that γ is itself a closed subset of D.

Let $|s_n(z)| \leq M_0$ for $z \in \gamma$, $n \geq 1$.

Consider now any small closed rectangle R situated "inside" γ. Let $h = \text{dist}(R, \gamma)$.

For $\xi \in R$,

$$s_n'(\xi) = \frac{1}{2\pi i} \oint_\gamma \frac{s_n(z)}{(z-\xi)^2}\, dz \quad \text{by CIF for deriv.}$$

$$\Rightarrow \quad |s_n'(\xi)| \leq \frac{1}{2\pi} \int_\gamma \frac{M_0}{h^2}\, ds = \underbrace{\frac{1}{2\pi} \frac{M_0}{h^2}\, \ell(\gamma)}_{\text{call this } \tilde{M}}.$$

Connect any two points ξ_1 and ξ_2 of R by a line segment to get

$$s_n(\xi_2) - s_n(\xi_1) = \underset{\text{linear}}{\int_{\xi_1}^{\xi_2}} s_n'(z)\,dz$$

$$|s_n(\xi_2) - s_n(\xi_1)| \leq \int_{\xi_1}^{\xi_2} \tilde{\mathfrak{M}}\,ds = \tilde{\mathfrak{M}}\,|\xi_2 - \xi_1|.$$

Because of this, Lemma on page 16 applies. Hence: $s_n(z) \to s(z)$ UNIFORMLY on R !!

Since R can be reduced in size and slid around, and γ can always be pushed closer to Γ, we get that $s_n(z) \to s(z)$ UNIFORMLY on each closed subset K of D. (It's best to fix K first, then select the Jordan curve γ.) This proves (b).

At this point, the standard Weierstrass convergence thm applies and, so, we get (a) + (c) + (d).

OK ∎

A person wishing to keep matters completely self-contained would reason alternatively as follows. (here)

Let $\rightrightarrows$ signify uniform conv.

Refer to the picture on (19). Let R_0 be a closed rectangle just slightly bigger than R. 
Since we could have just as well used R_0 instead of R, we know that $s_n(z) \rightrightarrows s(z)$ on R_0.

By unif conv, $s(z)$ is continuous on R_0.

For $z_1 \in R$,

$$s_n(z_1) = \frac{1}{2\pi i} \oint_{\partial R_0} \frac{s_n(w)}{w - z_1}\, dw \qquad (n \geq 1).$$

By unif conv of $s_n(w)$ on ∂R_0 and use of test function $p(w) = \frac{1}{w - z_1}$, we get (via $n \to \infty$)

$$s(z_1) = \frac{1}{2\pi i} \oint_{\partial R_0} \frac{s(w)}{w - z_1}\, dw, \quad \text{each } z_1 \in R.$$

By the FACT on p. (13), $s(z)$ must belong to A^∞ on the interior of R.

Now just take the whole "picture frame" $\{R, R_0\}$ and slide it around slightly within the interior of γ. See picture on 19.

We get $s(z) =$ analytic on an open set containing R_0. We already know $s_n(z) \rightrightarrows s(z)$ on R_0.

Let $h = \text{dist}(R, \partial R_0)$. For $z \in R$,

$$s_n'(z) - s'(z) = \frac{1}{2\pi i} \oint_{\partial R_0} \frac{s_n(w) - s(w)}{(w-z)^2}\, dw$$

$\{$ by CIF for deriv. $\}$

$$|s_n'(z) - s'(z)| \leqq \frac{1}{2\pi} \oint_{\partial R_0} \frac{|s_n(w) - s(w)|}{h^2} |dw|$$

$$|s_n'(z) - s'(z)| \leqq \frac{1}{2\pi h^2} \ell(\partial R_0) \max_{\partial R_0} |s_n(w) - s(w)| .$$

But, choose N_ε so $|s_n(w) - s(w)| < \varepsilon$ for all $n \geqq N_\varepsilon$ and $w \in R_0$. Hence, $n \geqq N_\varepsilon \Rightarrow$

$$|s_n'(z) - s'(z)| < \frac{\ell(\partial R_0)}{2\pi h^2} \varepsilon \quad , \ z \in R .$$

This proves $s_n'(z) \rightrightarrows s'(z)$ on R. Similarly for $s_n^{(j)}(z)$.

Since R can be reduced in size and slid around, and γ can always be pushed closer to Γ, we immediately get (a), then (b)(c)(d). $\blacksquare$ $\blacksquare$

variant

The following version of the Weierstrass conv. theorem is **still** stronger and was proved by Vitali.

It shows how the **structure** of analytic functions can be exploited to induce a kind of "global rigidity".

THEOREM (Vitali). $\boxed{\text{Similarly for multiply-connected } D.}$

Let D and Γ be as on 18.

Let $\{s_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on D which satisfies the $M(K)$ hypothesis on 18 for each closed $K \subseteq D$.

ASSUME THAT $\lim_{n\to\infty} s_n(\xi_j)$ exists for each $j \geq 1$, where the ξ_j are distinct points of D tending to a point (say, ξ_∞) of D.

THEN:

(a) $\{s_n(z)\}_{n=1}^{\infty}$ automatically converges to a limit function $s(z)$ at each point of D;

(b) the convergence in (a) is UNIFORM on each closed $K \subseteq D$.

Pf

We first recall a simple fact about complex numbers.

Lemma

Given sequence $\{w_n\}_{n=1}^{\infty}$ in $\mathbb{C}$. This sequence converges to $\underline{L}$ as $n \to \infty$ if and only if every subsequence $\{w_{n_j} : n_1 < n_2 < n_3 < \cdots\}$ admits a further subsequence which converges to $\underline{L}$.

Pf of lemma

The "only if" is obvious. For the "if", we use contradiction. Hence there must exist some bad $\varepsilon_0 > 0$ with no "N_{ε_0}". IE we can find arbitrarily big $\tilde{n}$ for which $|w_{\tilde{n}} - L| \geq \varepsilon_0$. Make a recursion construction to get $n_{j+1} > n_j \geq 1$ satisfying

$$|w_{n_j} - L| \geq \varepsilon_0 \,, \quad j \geq 1.$$

By hypothesis, there exists a (increasing) subseq $\mathcal{S}$ of $\{n_j\}_{j=1}^{\infty}$ for which

$$\{w_m : m \in \mathcal{S}\} \to L \,.$$

But $\mathcal{S} \subseteq \{n_j\} \Rightarrow |w_k - L| \geq \varepsilon_0$ for each $k \in \mathcal{S}$.

Contradiction! ∎

We now turn to the proof of the THM.

The reasoning that follows is closely related to the Arzela–Ascoli theorem in real analysis (or point-set topology).

> See, e.g., Rudin, Principles of Math Analysis, 3rd ed, Theorem 7.25. Also 7.23.

Choose any $\delta > 0$. By taking a grid of points on D, we can clearly find a finite set $E_\delta \subseteq D$ such that every point of D lies within δ units of some point of E_δ. The set

$$\mathcal{C} = \bigcup_{k=1}^{\infty} E_{1/k}$$

is then countable and <u>dense</u> in D.

Let $Q_j = \lim_{n\to\infty} s_n(\xi_j)$ for each $j \geq 1$.

Also let $\mathcal{S}$ be any increasing subsequence of $\{n\}_{n=1}^{\infty}$.

By combining hypothesis $\mathcal{M}(K)$, the Bolzano–Weierstrass thm, and the Cantor diagonal process, we can construct an increasing subsequence $\mathcal{S}_1$ of $\mathcal{S}$ such that

$$\lim_{n\to\infty} \{ s_n(P) : n \in \mathcal{S}_1 \} \text{ exists}$$

for EACH $P \in \mathcal{C}$.

At this juncture, we go back into the <u>proof</u> of p. 18 THM.

The key initial observation is this. Let R be any closed rectangle situated within γ. See picture on 19. Since $\mathcal{C}$ is dense in D, there exists a finite set of points $\{P_1, \dots, P_L\} \subseteq \underline{R \cap \mathcal{C}}$ satisfying the $\varepsilon/5M$-unit condition on 16 bottom. This assertion <u>requires</u> just a bit of care in handling points near ∂R; see also the very important page 20 (top 4 lines).

Keeping $n \in \mathcal{S}_1$, notice that lines 3–13 on 17 can now be re-used [[since $P_j \in \mathcal{C}$ and $\lim_n \{s_n(P_j) : n \in \mathcal{S}_1\}$ <u>exists</u> !!!]].

We conclude that $\{s_n : n \in \mathcal{S}_{\underline{1}}\}$ is <u>uniformly</u> <u>Cauchy</u> on R.

Hence $\{s_n : n \in \mathcal{S}_1\} \rightrightarrows$ some $s(z)$ on each R.

By sliding R as in the proof of p. 18 THM (see especially 20), we get that $\{s_n : n \in \mathcal{S}_{\underline{1}}\} \rightrightarrows s(z)$ on every <u>closed</u> $K \subseteq D$.

Note that the points $\{\xi_j^\circ\}_{j=1}^{\infty}$ will lie in some fixed closed $K \subseteq D$.

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We can now apply either the traditional or p. 18 strengthened Weierstrass conv theorem to $\{s_n : n \in \mathcal{S}_1\}$. The fcn $s(z)$ is thus analytic on D. Moreover, by substituting $z = \xi_j^\circ$, we find that

$$s(\xi_j^\circ) = Q_j^\circ \,, \quad j \geq 1 \,.$$

For Q_j°, recall 23.

Let $\tilde{\mathcal{S}}$ be any other increasing subseq of $\{n\}_{n=1}^{\infty}$. See 23. Form $\tilde{\mathcal{S}}_1$ by analogy with $\mathcal{S}_1$. The limit function for $\{s_n : n \in \tilde{\mathcal{S}}_1\}$ will be $\tilde{s}(z)$. The function $\tilde{s}(z)$ is again analytic on D, and satisfies $\tilde{s}(\xi_j^\circ) = Q_j^\circ$.

The function $H(z) \equiv s(z) - \tilde{s}(z)$ is analytic on D and vanishes at each ξ_j°. Hence also at ξ_∞. If $H(z) \not\equiv 0$ on D, any zero at ξ_∞ would need to be isolated. [[This follows from the local Taylor expansion.]] Since $\xi_j^\circ \to \xi_\infty$, we get an immediate violation. Hence: $H(z) \equiv 0$ and $s(z) \equiv \tilde{s}(z)$ on D.

CLAIM:

For each $\tau \in D$, $\{s_n(\tau) : n \geq 1\} \to s(\tau)$.

Pf of Claim

Just use the Lemma on 22. We need only show that any increasing subseq $\tilde{\mathcal{S}}$ of $\{n\}_{n=1}^{\infty}$ admits a subsequence $\tilde{\mathcal{S}}_0$ such that

$$\{s_n(\tau) : n \in \tilde{\mathcal{S}}_0\} \longrightarrow s(\tau).$$

But, $\tilde{\mathcal{S}}_1$ works in the role of $\tilde{\mathcal{S}}_0$ (since we just proved that $\tilde{s}(z) \equiv s(z)$). OK on the Claim. ∎

One is now exactly in the situation of p. 18 THM — and so we are done. ∎