PARTIAL DIARY ENTRY for

Lecture 5

(3 Feb 2016)

We first went over a number of elementary facts and properties. The goal today was to begin the Riemann zeta fcn in earnest.

Topic I

About Riemann-Stieltjes integrals.

Showed that even if $\alpha(x)$ is right continuous and $\nearrow$ on $[0,1]$, taking f to be piecewise continuous can lead to

$$\overline{\int_0^1} f(x)\,d\alpha(x) = 1 \,, \quad \underline{\int_0^1} f(x)\,dx = 0 \,.$$

Discouraging!! So, best to use R-S for continuous f when possible.

Showed:

$f \in C[1,N] \Rightarrow$

$$\int_1^N f(x)\, d[\![x]\!] = f(2) + \cdots + f(N)$$

$\boxed{\text{note carefully}}$

$g \in C[\beta, N] \quad (0 < \beta < 1) \Rightarrow$

$$\int_\beta^N g(x)\, d[\![x]\!] = g(1) + \cdots + g(N).$$

$\boxed{\text{similarly for } g \in C[\beta, N+\beta]}$

Hence, R-S has natural connection with sums!

Topic II

Abel's Lemma for power series.

Given $\sum_{n=0}^{\infty} a_n z^n$ which converges at $z_1 \neq 0$.

Then: $|a_n| \leq \dfrac{\mathcal{M}}{|z_1|^n}$ for some $\mathcal{M}$ and all $n \geq 0$.

Hence, the orig power series conv uniformly and absolutely on each closed disk $\{|z| \leq |z_1| - \delta\}$.

$\boxed{\text{And Weierstrass Conv Thm applies!! on } |z| < |z_1|}$

Pf Trivial. ▥

Topic III

Another well-known result of Abel.

Thm (Abel)

$S(z)$

Let $\sum_{n=0}^{\infty} a_n z^n$ converge at, say, $z = 1$ (to S).

Then:

$$\sum_{n=0}^{\infty} a_n x^n$$

conv. uniformly on $[0,1]$. Hence $\lim_{x \to 1^-} S(x) = S$.

(similarly along $z = re^{i\theta}$.)

Pf

Uniform Cauchy estimate + Abel summation.

Must prove

$$|S_N(x) - S_M(x)| < \varepsilon \quad , \quad \text{all } N > M \geq \mathcal{N}_\varepsilon .$$

We know, of course,

$$|a_{M+1} + \cdots + a_N| < \varepsilon \quad \text{for } N > M \geq N_\varepsilon .$$

Claim that we can take $\mathcal{N}_\varepsilon = N_\varepsilon$. Put

$$T_k = a_{M+1} + \cdots + a_k \quad , \quad k \geq M+1 .$$

Get:

$$a_{M+1} x^{M+1} + \cdots + a_N x^N$$

$$= T_{M+1} x^{M+1} + (T_{M+2} - T_{M+1}) x^{M+2} + \cdots + (T_N - T_{N-1}) x^N$$

$$= T_{M+1} (x^{M+1} - x^{M+2}) + \cdots + T_{N-1}(x^{N-1} - x^N) + T_N x^N .$$

Know $|T_k| < \varepsilon$, $k \geq M+1$. Get:

$$\text{ABS VALUE} < \varepsilon(x^{M+1} - x^{M+2}) + \cdots + \varepsilon(x^{N-1} - x^N) + \varepsilon x^N$$

$$\{0 \leq x \leq 1\} \quad x^j - x^{j+1} \geq 0$$

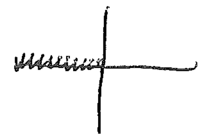
$$= \varepsilon x^{M+1} \leq \varepsilon .$$

Hence all is OK. ▥

This proof can clearly be generalized to work in <u>many</u> other settings!

Topic IV

Traditional to define principal value of $\arg(w)$ by declaring $-\pi < \text{Arg}(w) < \pi$ and keeping w off the negative real axis $(-\infty, 0]$.



$$\text{Log}(w) = \ln|w| + i\,\text{Arg}(w)$$

Nice analytic fcn for $\mathbb{C} \sim (-\infty, 0]$.

$$\frac{d}{dw}\text{Log}\, w = \frac{1}{w} \qquad \left\{ \begin{array}{l} \text{local inverses} \\ \text{are analytic, etc} \end{array} \right\}$$

So, $f(z) = \text{Log}(1+z)$ is analytic for $|z| < 1$.

Cauchy - Taylor $\Rightarrow$

$$f(z) = \text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \pm \cdots, \quad |z| < 1.$$

get unif + abs conv for $|z| \leq 1 - \delta$

Thm 1 (basic def of $\zeta(z)$)

↑ RIEMANN ZETA FCN.

We write

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \qquad \{ n^{-z} \equiv \exp(-z \ln n) \}$$

for $Re(z) > 1$. The series conv unif and absolutely in every half-plane $\{Re(z) \geq 1+\delta\}$. Hence, $\zeta(z)$ is nicely analytic on $\{Re(z) > 1\}$.

Pf

Weierstrass M-test with $M_n = n^{-1-\delta}$. ▥

Also Weierstrass Conv Thm!

Thm 2

There exists a function $F(z)$ which is analytic on $\{Re(z) > 0\} - \{1\}$ such that $F(z) = \zeta(z)$ whenever $Re(z) > 1$. The fcn F is unique (numerically). We call it the analytic continuation of $\zeta(z)$. One can see that, near $z=1$,

$$F(z) = \frac{1}{z-1} + [\text{something analytic}].$$

in, say, $|z-1| < \frac{1}{2}$

Pf

Suppose there were two: F_1 and F_2.
The fcn $F_1 - F_2$ is analytic on $\{Re(z) > 0\} - \{1\}$ but $\equiv 0$ for $Re(z) > 1$. By properties of analytic fcns, get $F_1 - F_2 \equiv 0$ everywhere.

Hence F must be unique.

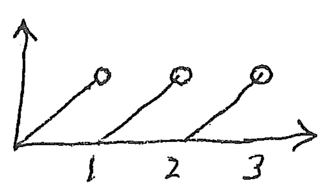
Must now find one F.

Take $R = N + \varepsilon$ for some tiny $\varepsilon > 0$. Keep $Re(z) > 1$.

$$1 + \int_1^R t^{-z} \, d[\![t]\!] = 1 + 2^{-z} + 3^{-z} + \cdots + N^{-z} .$$

Notice (see 2) that nothing is lost if we simply take $\varepsilon = 0$ rather than let $\varepsilon \to 0$. (Make sure you understand this; this type of trick is used a lot!)

Write $t = [\![t]\!] + r(t)$.



right continuous

diff of two increasing right continuous fcns

$0 \leq r(t) < 1$

Get:

$$\sum_{n=1}^{N} n^{-z} = 1 + \int_1^N t^{-z}\, d[\![t]\!]$$

$$= 1 + \int_1^N t^{-z}\, d(t - r(t))$$

$$= 1 + \int_1^N t^{-z}\, dt - \int_1^N t^{-z}\, dr(t)$$

R-S integral and t^{-z} nicely C^1 wrt t

$$\boxed{\begin{gathered}\frac{d}{dv} e^{cv} = c e^{cv} \\ \text{for } v \in \mathbb{R}\end{gathered}} \Rightarrow \left\{ \begin{aligned} \frac{d}{dt} t^c &= \frac{1}{t} \frac{d}{d(\ln t)} t^c \\ &= \frac{1}{t} \frac{d}{d(\ln t)} e^{c(\ln t)} \\ &= \frac{1}{t} e^{c(\ln t)} \cdot c = \underline{c t^{c-1}} \\ &\text{for } c \in \mathbb{C} \text{ and } t > 0 \end{aligned} \right\}$$

$$= 1 + \frac{N^{1-z} - 1}{1 - z} - \left[t^{-z} r(t)\right]_1^N + \int_1^N r(t)(-z) t^{-z-1}\, dt$$

parts

$$= 1 + \frac{1 - N^{1-z}}{z - 1} - 0 + 0 - z \int_1^N \frac{r(t)}{t^{z+1}}\, dt$$

$r(1) = r(N) = 0$

So, $Re(z) > 1$ gives (by taking $N \to \infty$)

$$\sum_{n=1}^{N} n^{-z} = 1 + \frac{1 - N^{1-z}}{z-1} - z \int_1^N \frac{r(t)}{t^{z+1}} \, dt$$

$$\zeta(z) = 1 + \frac{1}{z-1} - z \int_1^\infty \frac{r(t)}{t^{z+1}} \, dt$$

$$\left\{ \begin{aligned} |t^c| &= \left| e^{(\alpha + i\beta)\ln t} \right| = e^{\alpha \ln t} \\ &= t^\alpha = t^{Re(c)} \, , \quad t > 0 \end{aligned} \right\} .$$

In the formulae above, note that:

formula #1 holds for <u>any</u> $z \in \mathbb{C} \sim \{1\}$;

formula #2 holds for $Re(z) > 1$;

the <u>integral</u> in formula #2 is nicely absolutely + uniformly convergent so long as $x \geqq \delta > 0$ (!!!)

$$\boxed{\text{HENCE analytic à la Weierstrass conv thm on } Re(z) > 0}$$

Note: there *is* a Weierstrass M-test for improper integrals $\int_1^\infty$; one should review it. (Cf. any adv calc book.)

We can thus put

$$F(z) \equiv 1 + \frac{1}{z-1} - z\int_1^\infty \frac{r(t)}{t^{z+1}}\,dt$$

for all $\mathrm{Re}(z) > 0$ except $z = 1$. This choice *works* in Thm 2 on page 6. $\blacksquare$

$\boxed{z = x + iy}$

Thm 3

We have $|\zeta(z)| \leq \zeta(x)$ for all $\mathrm{Re}(z) > 1$.

We also have

$$|\zeta(z) - 1| < 2^{-x}\left(1 + \frac{2}{x-1}\right)$$

for all $\mathrm{Re}(z) > 1$. {Note that RHS is $< 3 \cdot 2^{-x}$ whenever $x > 2$.}

<u>Pf</u>

$$\left|\sum_{n=1}^{\infty} n^{-z}\right| \leq \sum_{n=1}^{\infty} |n^{-z}| = \sum_{n=1}^{\infty} n^{-x} \qquad (\text{see } 9)$$

$$= \zeta(x)$$

Also:

$$\left|\sum_{n=2}^{\infty} n^{-z}\right| \leq \sum_{n=2}^{\infty} n^{-x} < 2^{-x} + \int_2^{\infty} u^{-x}\,du \qquad (x>1)$$

$$\{\text{by baby areas}\}$$

$$= 2^{-x} + \left[\frac{u^{1-x}}{1-x}\right]_2^{\infty}$$

$$= 2^{-x} + \frac{2^{1-x}}{x-1}$$

$$= 2^{-x}\left\{1 + \frac{2}{x-1}\right\}.$$

∎

By Thm 2,

$$\zeta(x) = \frac{1}{x-1} + O(1)$$

as $x \to 1^{+}$.

Thm 4 (very crude)

Keep $|z-1| \geqq \frac{1}{3}$ say. Take any $0<\delta<1$. We then have

$$|\zeta(x+iy)| = O(1)\frac{1}{\delta}(1+|y|)$$

whenever $\delta \leqq x \leqq 1+\delta$. For $x \geqq 1+\delta$, we have

$$|\zeta(x+iy)| = O(1)\frac{1}{\delta}.$$

{In $O(1)$, the implied constant is absolute.}

Pf

Use (10) line 5. Keep $\delta \leqq x \leqq 1+\delta$. Get:

$$|\zeta(x+iy)| \leqq 1+3+|z|\int_1^\infty \frac{1}{t^{x+1}}\,dt$$

$$\leqq 4 + (|x|+|y|)\int_1^\infty \frac{1}{t^{x+1}}\,dt$$

$$\leqq 4 + (2+|y|)\frac{1}{x} \qquad (x>0)$$

$$\leqq 4(1+|y|) + 2(1+|y|)\frac{1}{x}$$

$$= (1+|y|)\left(4+\frac{2}{x}\right)$$

$$\leqq (1+|y|)\left(\frac{4}{\delta}+\frac{2}{\delta}\right) = \frac{6}{\delta}(1+|y|).$$

For $x \geqq 1+\delta$, simply use $|\zeta(z)| \leqq \zeta(x)$ (thm 3) and $\zeta(1+\delta) = \frac{1}{\delta} + O(1)$. ∎

We then paused to discuss infinite products, a topic which seems to have disappeared from UM's undergrad math curriculum!

We do not give a treatise; AND we will deal with products of COMPLEX numbers.

using only REAL is much easier!!!

Like with $\sum_{n=1}^{\infty} a_n$ vis à vis conv/div, matters should focus on the tail end of the series (or product), NOT on the first 10^{100} terms.

Unless $a_n \to 0$, $\sum_{1}^{\infty} a_n$ is div.

Unless $a_n \to 1$, $\prod_{1}^{\infty} a_n$ is (said to be) div!

We therefore focus on products with the first 10^{100} terms erased and presuppose that $a_n = 1 + b_n$, with $|b_n| \lesssim \lambda < 1$ for some λ.

Def

Given $a_n = 1 + b_n$ with $|b_n| \leqq \lambda < 1$.

We say

$$\prod_{n=1}^{\infty} a_n \quad \textbf{conv} \quad \text{to} \quad P$$

if

(a) $P \neq 0$

and

(b) $\dfrac{P_N}{P} \to 1$ as $N \to \infty$.

"Multiplicative style"

Here $P_N = a_1 \cdots a_N$.

If $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leqq \lambda < 1$, $z \in E$,

we say

$$\prod_{n=1}^{\infty} a_n(z) \quad \textbf{conv unif} \quad \text{to} \quad P(z)$$

if

$$P(z) \neq 0 \quad \text{and} \quad \frac{P_N(z)}{P(z)} \to 1 \quad \text{uniformly as } N \to \infty.$$

Def

Given $a_n = 1 + b_n$ as above. We say

$$\prod_{n=1}^{\infty} a_n \quad \text{conv absolutely}$$

when

$$\prod_{n=1}^{\infty} (1 + |b_n|) \quad \text{converges.}$$

NOT $|a_n|$

N.B. see 24 below!

as above 14

Lemma

Suppose $\prod_{n=1}^{\infty} a_n(z)$ conv unif to $P(z)$ on E.

Then, there exist $c_j > 0$ so that

$$c_1 < |P(z)| < c_2 \quad \text{on } E.$$

Pf

Choose M so big that

$$\left| \frac{P_N(z)}{P(z)} - 1 \right| < 10^{-6} \qquad (z \in E)$$

for all $N \geq M$.

Get:

$$\left|\frac{P_M}{P}-1\right| < 10^{-6} \qquad \left|\frac{P}{P_M}-1\right| < 10^{-5}$$

$$\Downarrow$$

$$\frac{3}{4} < \left|\frac{P}{P_M}\right| < \frac{5}{4} \qquad \text{certainly}$$

$$\Downarrow$$

$$\frac{3}{4}|P_M| < |P| < \frac{5}{4}|P_M|$$

$$\Downarrow$$

$$\frac{3}{4}(1-\lambda)^M < |P| < \frac{5}{4}(1+\lambda)^M . \qquad \blacksquare$$

Corollary (important)

Notation as above. Suppose $\prod_{n=1}^{\infty} a_n(z)$ conv unif to $P(z)$ on E. We then **also** have

$$P_N(z) \rightrightarrows P(z) \quad \text{on } E.$$

recall that this means UNIF CONV

Pf

Obvious because of Lemma on 15. ▨

Lemma

Recall $Log(w)$ and $Arg(w)$ on 5.

Suppose that $|w_1 - 1| < 1$ and $|w_2 - 1| < 1$.

Then:

$$w_1 w_2 \notin (-\infty, 0]$$

and

$$Arg(w_1 w_2) = Arg(w_1) + Arg(w_2).$$

Pf

Write $w_j = R_j e^{i\theta_j}$. Clearly $-\frac{\pi}{2} < \theta_j < \frac{\pi}{2}$ and $R_j > 0$. Hence

$$w_1 w_2 = R_1 R_2 e^{i(\theta_1 + \theta_2)}$$

and

$$-\pi < \theta_1 + \theta_2 < \pi.$$

Done! ▨

For $|w_j - 1| < \frac{1}{100}$, baby trig

$\Rightarrow Arg(w_1 \cdots w_{100}) = \sum_1^{100} Arg(w_j)$.

General Thm

Let E be some set which might possibly be just one point. Given $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leq \lambda < 1$, $z \in E$, as above.

We then have:

$\prod_{n=1}^{\infty} a_n(z)$ conv unif to some $P(z)$ on E

if and only if

$\sum_{n=1}^{\infty} \mathrm{Log}(1 + b_n(z))$ conv unif to some $S(z)$ on E.

And, if so,

$$P(z) = \exp\{S(z)\}.$$

Pf

This thm is not just hand-waving trivia by "passing to logs". It is NOT true that

$$\mathrm{Log}(w_1 \cdots w_N) = \sum_1^N \mathrm{Log}(w_j)$$

in general, even if $w_j \approx 1$.

think, eg, $w_j = e^{2\pi i/N}$

Suppose first that $S_N(z) \rightrightarrows S(z)$, where

$$S_N = \sum_{n=1}^{N} \mathrm{Log}(1+b_n(z)).$$

From unif conv (and λ), automatically (get) $|S(z)| < \mathfrak{M}$ for some $\mathfrak{M}$.

↖ just imitate p. 15 Lemma

$$\Downarrow$$

We can now exponentiate freely.

$$P_N = \exp(S_N)$$

$$P = \exp(S)$$ <u>makes</u> <u>sense</u> on E

$$\frac{P_N}{P} \rightrightarrows 1 \quad \text{as} \quad N \to \infty, \quad z \in E$$

Hence: $\prod_{1}^{\infty} a_n(z)$ conv unif on E and $P = \exp(S)$. <THIS MUCH IS <u>TRIVIAL</u>.>

The <u>problem</u> is with the converse!

Suppose now that $\prod_{n=1}^{\infty} a_n(z)$ conv unif to $P(z)$ on E.

<u>Choose</u> M so large that

$$\left| \frac{P_N}{P} - 1 \right| < 10^{-6}$$

for all $N \geq M$, $z \in E$. Do some baby algebra.

Get

$$\left| \frac{P_{N_2}}{P_{N_1}} - 1 \right| < 10^{-5}, \quad \left| \frac{P}{P_{N_1}} - 1 \right| < 10^{-5}$$

for all $N_2 \geq N_1 \geq M$. See (16) top.

Let:

$$\operatorname{Arg}\Big[\underbrace{(1+b_{M+1})\cdots(1+b_N)}_{\frac{P_N}{P_M}}\Big] = \sum_{j=M+1}^{N} \operatorname{Arg}(1+b_j) + 2\pi i t_N$$

$t_N \in \mathbb{Z}$

for $N \geq M+1$.

<u>CLAIM</u>: $t_N = 0$ for <u>all</u> $N \geq M+1$.

<u>Pf of claim</u>

Take $N_2 = N_1 + 1$ and $N_1 \geq M$. Get: $|a_{N_2} - 1| < 10^{-5}$.

IE $|a_L - 1| < 10^{-5}$ for all $L \geq M+1$.

IE b_L

Clearly $t_{M+1} = 0$ by def.

Use induction. Suppose $0 = t_{M+1} = \cdots = t_N$.

<u>Must</u> <u>prove</u> $t_{N+1} = 0$.

Use <u>Lemma</u> on 17. Take:

$$w_1 = (1+b_{M+1})\cdots(1+b_N) \leftarrow \frac{P_N}{P_M} \qquad 10^{-5} \text{ etc}$$

$$w_2 = 1+b_{N+1} \leftarrow \frac{P_{N+1}}{P_N} \qquad 10^{-5} \text{ etc}$$

Get:

$$\mathrm{Arg}(w_1 w_2) = \mathrm{Arg}(w_1) + \mathrm{Arg}(w_2) \quad \text{by } |w_j - 1| < 10^{-5}$$

OR

$t_N = 0$

$$\mathrm{Arg}[(1+b_{M+1})\cdots(1+b_{N+1})] = \sum_{j=M+1}^{N} \mathrm{Arg}(1+b_j) + 2\pi i(0)$$

$$+ \mathrm{Arg}(1+b_{N+1})$$

hence $t_{N+1} = 0$.

OK

We have just proved Claim for our given M.
But the same reasoning works with

$N_2 > N_1 \geqq M$ and N_1 in place of M.

IE

$$\operatorname{Arg}[(1+b_{N_1+1})\cdots(1+b_{N_2})] = \sum_{j=N_1+1}^{N_2} \operatorname{Arg}[1+b_j] + 0$$

hence

$$\boxed{\operatorname{Log}\left[\frac{P_{N_2}(z)}{P_{N_1}(z)}\right] = \sum_{j=N_1+1}^{N_2} \operatorname{Log}[1+b_j(z)] \quad \text{so long as } N_2 > N_1 \geqq M}$$

This is the key equation! Since $\frac{P_N(z)}{P(z)} \rightrightarrows 1$ on E and $c_1 < |P(z)| < c_2$ (15), we get a multiplicative Cauchy condition

$$\left|\frac{P_{N_2}(z)}{P_{N_1}(z)} - 1\right| < \varepsilon \text{ anytime } N_2 > N_1 \geqq \mathcal{N}_\varepsilon$$

(and, wlog, $\mathcal{N}_\varepsilon \geqq M$).

This shows that there is a uniform Cauchy condition for

$$\sum_{j=N_1+1}^{N_2} \operatorname{Log}(1+b_j(z)) \quad , \quad \text{i.e. } S_{N_2}(z)-S_{N_1}(z),$$

for $z \in E$. HENCE: $\sum_{j=1}^{\infty} \operatorname{Log}(1+b_j(z))$ conv uniformly on E to some $S(z)$.

By referring to 19, we again have $P = \exp(S)$.

Done! ∎

Important Remark.

> If you know $P(z)$,
> note that you do NOT
> in general know $S(z)$ without
> further playing around with
> $\sum_{1}^{\infty} \operatorname{Log}(1+b_j(z))$. Indeed,
>
> $n \in \mathbb{Z} \rightsquigarrow \exp[S(z)+2\pi i n] = P(z)$ too.
>
> I.E. which "branch" of $\log P(z)$ applies? YOU DO NOT KNOW THIS IN GENERAL, even if $E = \{\text{one point}\}$.

Thm (Yes, this IS a theorem!!)

Given $a_n(z) = 1 + b_n(z)$, $z \in E$, $|b_n(z)| \leq \lambda < 1$ as usual.

If $\prod_{n=1}^{\infty} (1 + b_n(z))$ converges absolutely on E,

then
$$\prod_{n=1}^{\infty} (1 + b_n(z)) \quad \text{converges} \quad \text{on } E.$$

[Remember E could be one point.]

Pf

By hypothesis, we know $\prod_{n=1}^{\infty} (1 + |b_n(z)|)$ conv at each $z \in E$.

Apply 18. Get $\sum_{n=1}^{\infty} \ln(1 + |b_n|)$ conv on E.

But, baby calculus $\Rightarrow$
$$\frac{1}{2} t \leq \ln(1+t) \leq t \quad \text{for} \quad 0 \leq t \leq 1.$$

Hence:
$$\sum_{n=1}^{\infty} |b_n(z)| \quad \text{conv}, \quad \text{each } z \in E.$$

But, baby analytic functions $\Rightarrow$
$$a_\lambda |w| \leq |\mathrm{Log}(1+w)| \leq b_\lambda |w| \quad \text{for} \quad |w| \leq \lambda < 1.$$

$$0 < a_\lambda < b_\lambda < \infty$$

EG use Taylor series or else

$$\operatorname{Log}(1+w) = \underset{\text{line}}{\int_0^w} \frac{d\xi}{1+\xi}.$$

Accordingly,

$$\sum_{n=1}^{\infty} \left| \operatorname{Log}(1+b_n(z)) \right| \text{ converges, each } z \in E.$$

This absolute conv $\Rightarrow$ ordinary conv. of $\sum_1^\infty \operatorname{Log}(1+b_n)$.

Now just apply 18 at each single point of E.

$\blacksquare$

Thm (Weierstrass M-test for products)

Given $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leqq \lambda < 1$, $z \in E$.

Assume that

$$|b_n(z)| \leqq M_n \quad \text{and} \quad \sum_1^\infty M_n < \infty \quad \text{on } E.$$

Then:

$$\prod_1^\infty (1 + b_n(z)) \quad \text{conv} \quad \text{unif} \quad \text{on} \quad E.$$

(In fact, so does $\prod_1^\infty (1 + |b_n(z)|)$.)

Pf

Apply (18). Must show that $\sum_1^\infty \mathrm{Log}(1 + b_n(z))$ conv unif on E. Recall (24) last line! Get:

$$|\mathrm{Log}(1 + b_n(z))| \leqq b_\lambda |b_n(z)| \leqq b_\lambda M_n .$$

A standard Weierstrass M-test now applies to $\sum_1^\infty \mathrm{Log}(1 + b_n(z))$. Hence $\sum_1^\infty \mathrm{Log}(1 + b_n(z))$ conv uniformly as needed. OK

For the "in fact", just replace $b_n(z)$ by $|b_n(z)|$. The same M_n still work. ▮

Simple Exercise

Given $a_n(z) = 1 + b_n(z)$, $|b_n(z)| \leqq \lambda < 1$, $z \in E$.

(a) $\prod_1^\infty (1 + |b_n|)$ conv pointwise on $E \Leftrightarrow \sum_1^\infty |b_n|$ does;

(b) $\prod_1^\infty (1 + |b_n|)$ conv uniformly on $E \Leftrightarrow \sum_1^\infty |b_n|$ does;

(c) $\prod_1^\infty (1 + |b_n|)$ conv unif on $E \Rightarrow \prod_1^\infty (1 + b_n)$ does too.

See 18. Note $\text{Log}(1 + |w|) = \ln(1 + |w|) \sim |w|$ as $w \to 0$.
Also recall uniform Cauchy condition for unif conv.

Mind-Twister Exercise $\left(\begin{array}{c} \text{otherwise known as} \\ \text{Log}(1+w) \neq w \end{array} \right)$

Let E = one point. Keep $|b_n| \leqq \lambda < 1$. Put $a_n = 1 + b_n$.

(a) Find $\{b_n\}_{n=1}^\infty$ so that $\sum_1^\infty b_n$ conv, but $\prod_1^\infty a_n$ div!

(b) Find $\{b_n\}_{n=1}^\infty$ so that $\prod_1^\infty a_n$ conv, but $\sum_1^\infty b_n$ div!

Eye-opening Exercise

This exercise really goes with Lecture #6, but is placed here for convenience.

(A) Prove that $\prod_{n=1}^{\infty} \cos\left(\frac{z}{2^n}\right)$ is unif and abs conv on every closed disk $\{|z| \leq R\}$, hence its value $P(z)$ is some analytic fcn on $\mathbb{C}$.

(B) [MAIN PROBLEM!] Evaluate $P(z)$ in simple terms.

Note too:
$\sin(z) = 0 \Leftrightarrow$
$z = n\pi$, etc.

Recall that:

$$\cos(z) \equiv \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) \equiv \frac{e^{iz} - e^{-iz}}{2i}$$

when $z \in \mathbb{C}$. These trig fcns are analytic on $\mathbb{C}$. Standard identities are therefore true; eg $\sin^2 z + \cos^2 z = 1$.