PARTIAL DIARY ENTRY for Lecture 6 (5 Feb 2016)

(I) More was discussed on infinite products, especially Weierstrass M-test. E.g., for

$$\prod_{n=1}^{\infty} \cos\left(\frac{z}{n}\right).$$

(II) $\prod_{n=1}^{\infty} (1+b_n(z))$ with $|b_n(z)| \leq \lambda < 1$ on E. We stressed that when <u>unif</u> conv holds, you get both

$$\frac{P_N(z)}{P(z)} \rightrightarrows 1 \qquad \text{AND} \qquad P_N(z) \rightrightarrows P(z)$$

since $c_1 < |P(z)| < c_2$ on E. Hence:

$b_n(z)$ continuous $\Rightarrow$ $P(z)$ continuous on E

$b_n(z)$ analytic $\Rightarrow$ $P(z)$ analytic (in the usual "on compacta" sense associated with Weierstrass convergence thm)

(III) Cauchy products. I remarked that

$$A = \sum_{n=0}^{\infty} a_n \text{ , abs conv} \qquad (a_n \in \mathbb{C})$$

$$B = \sum_{n=0}^{\infty} b_n \text{ , abs conv} \qquad (b_n \in \mathbb{C})$$

$$\Downarrow$$

$$AB = \sum_{n=0}^{\infty} \Big(\underbrace{\sum_{j+k=n} a_j b_k}_{} \Big) = \sum_{n=0}^{\infty} c_n \text{ , abs conv .}$$

(Same proof as in $\mathbb{R}$.)

(IV) $\mathrm{Re}(z) > 1$ say. Use (III). Take $T \geq 2$.

$$\prod_{p \leq T} \frac{1}{1-p^{-z}} = \prod_{p \leq T} \left\{ 1 + p^{-z} + p^{-2z} + \cdots \right\}$$

$$= \sum_{\substack{n \geq 1 \\ n \text{ is factorizable} \\ \text{into primes which} \\ \text{are all } \leq T}} n^{-z} \text{ .}$$

includes all $n \leq T$ obviously

Notice too:

$$\left| p^{-z} + p^{-2z} + \cdots \right| \leq p^{-x} + p^{-2x} + \cdots = \frac{p^{-x}}{1-p^{-x}} = \frac{1}{p^{x}-1}$$

$$\Downarrow$$

$$|p^{-z} + p^{-2z} + \cdots| \leqq \frac{1}{2^x - 1} \quad \text{for all } p .$$

Hence, for $x \geqq 1 + \delta$, we have a good "λ" of

$$\frac{1}{2^{1+\delta} - 1} < 1 .$$

All of our earlier thms about infinite products apply — when we opt to let $T \to \infty$. All is well.

We clearly get: (Euler)

$$\boxed{\prod_p \frac{1}{1 - p^{-z}} = \zeta(z) = \sum_{n=1}^{\infty} n^{-z}}$$

with uniform + absolute convergence on each closed half-plane $\{x \geqq 1 + \delta\}$.

In particular, since LHS is nonzero (by def of conv infinite product), we get:

$$\zeta(z) \neq 0 \quad \text{for } \operatorname{Re}(z) > 1 .$$

IV I drew attention to <u>Euler's</u> <u>identity</u>

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \{1 + f(p) + f(p^2) + f(p^3) + \cdots\}$$

in Ingham p. 16 — under the assumption that

$\boxed{1}$ $$\sum_{1}^{\infty} |f(n)| < \infty$$

and f is multiplicative

$\boxed{2}$ $$\left\{ \begin{array}{l} f(1) = 1 \\ f(mn) = f(m) f(n) \quad \text{if } (m, n) = 1 \end{array} \right\}.$$

(Read proof there!)

V Defined a natural branch of $\log \zeta(z)$ on $\mathrm{Re}(z) > 1$ by writing ← used 3 BOX

$$\mathrm{Log}\ \zeta(z) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-z}.$$

This is <u>NOT</u> in general $\mathrm{Log}\ \zeta(z)$!

For $z = x > 1$, however, one readily checks $\mathrm{Log}\ \zeta(z) = \ln \zeta(x)$. RECALL $\zeta(x) = \sum_{1}^{\infty} n^{-x} > 1$.

Clarification: (regarding $\mathcal{L}og\, \zeta(z)$)

Recall 2 last line and 3 lines 5~7. "All is well" because we are using Weierstrass M~test with

$$M_p = \frac{1}{p^{1+\delta} - 1} \quad (\text{for } x \gtrsim 1+\delta).$$

Our $\sum_p \mathrm{Log}(1+b_p(z))$ for the "infinite product equivalence thm" in Lec #5 is $-\sum_p \mathrm{Log}(1-p^{-z})$.

This infinite series converges to some $S(z)$.

The series is just

$$\sum_p \left\{ p^{-z} + \frac{1}{2} p^{-2z} + \frac{1}{3} p^{-3z} + \cdots \right\}$$

$$\equiv \sum_n \frac{\Lambda(n)}{\ln n} n^{-z} \quad \longleftarrow \text{nice analytic fcn}$$

(with good abs conv). As in Lec 5, we always have:

$$P(z) = \exp\{S(z)\}.$$

So, here, 3 BOX,

$$\zeta(z) = P(z) = \exp\{S(z)\}.$$

Ie, there is no question $S(z) =$ some branch of $\log \zeta(z)$ on $\{\mathrm{Re}(z) > 1\}$.

Clearly, by inspection, $S(x) > 0$ for $x > 1$.

Hence, we do have: $S(x) \equiv \mathcal{L}og\, \zeta(x) = \mathrm{Log}\, \zeta(x)$.
$(x > 1)$

VI Clearly, in V,

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_n \frac{\Lambda(n)}{n^z}, \quad \mathrm{Re}(z) > 1.$$

(by Weierstrass' conv thm
for analytic fcns)

Thm (Hadamard)

For $x > 1$, $y \neq 0$

$$|\zeta(x)|^3 \, |\zeta(x+iy)|^4 \, |\zeta(x+2iy)| \geq 1.$$

Pf

Take ln. We Want:

$$3\ln|\zeta(x)| + 4\ln|\zeta(x+iy)| + \ln|\zeta(x+2iy)| \geq 0.$$

But,

$$\ln|\zeta(z)| = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} \, n^{-x} \cos(y \ln n)$$

$\mathrm{Re}\{n^{-x} e^{-iy\ln n}\}$

by (4) $\{x > 1\}$.

Since, for any $\theta \in \mathbb{R}$,

$$
\begin{aligned}
3 + 4\cos\theta + \cos(2\theta) &= 3 + 4\cos\theta + 2\cos^2\theta - 1 \\
&= 2 + 4\cos\theta + 2\cos^2\theta \\
&= 2(1+\cos\theta)^2 \geqq 0,
\end{aligned}
$$

and $\frac{\Lambda(n)}{\ln n} \geqq 0$, a trivial substitution now gives what we claimed. ▩

<u>Corollary</u> (Hadamard ← famous result)

$\zeta(1+iy) \neq 0$ if $y \neq 0$.

<u>Pf</u>

Suppose we had $\zeta(1+iy) = 0$ <u>at</u> some $y \neq 0$.

$$\Downarrow$$

$$(x-1)^3 |\zeta(x)|^3 \frac{|\zeta(x+iy)|^4}{(x-1)^4} |\zeta(x+2iy)| \geq \frac{1}{x-1}$$

$$\{(x-1)|\zeta(x)|\}^3 \left| \frac{\zeta(x+iy)-\zeta(1+iy)}{x-1} \right|^4 |\zeta(x+2iy)| \geq \frac{1}{x-1}$$

let $x \to 1^+$

$$1^3 \cdot |\zeta'(1+iy)|^4 \cdot |\zeta(1+2iy)| \geq \infty \Rightarrow$$

a contradiction {since $\zeta(z)$ is nicely analytic for $\mathrm{Re}(z) > 0$ except at $z=1$} !! ∎

Theorem (essentially like Ingham p. 27)

Let $0 < \delta < 1$. We then have:

(A) $|\zeta(x+iy)| \lesssim A \ln|y|$ for $x \geq 1$, $|y| \geq 2$

(B) $|\zeta'(x+iy)| \lesssim B \ln^2|y|$ for $x \geq 1$, $|y| \geq 2$

(C) $|\zeta(x+iy)| \lesssim \dfrac{C}{\delta(1-\delta)} |y|^{1-\delta}$ for $x \geq \delta$, $|y| \geq 2$.

Here A, B, C are certain absolute constants.

Pf

$$\sum_{n=1}^{N} n^{-z} = 1 + \frac{1 - N^{1-z}}{z-1} - z \int_1^N \frac{r(t)}{t^{z+1}}\, dt$$

$(z \neq 1)$ Lec 5, p. 8 + 9

$$\zeta(z) = 1 + \frac{1}{z-1} - z \int_1^\infty \frac{r(t)}{t^{z+1}}\, dt$$

$(\mathrm{Re}(z) > 1)$ Lec 5 p. 9

Then, we used this last formula to define $\zeta(z)$ for $\mathrm{Re}(z) > 0$. Lec 5 p. 9

By subtraction,

$$\zeta(z) - \sum_{n=1}^{N} n^{-z} = \frac{N^{1-z}}{z-1} - z\int_N^\infty \frac{r(t)}{t^{z+1}}\,dt\ .$$

This is a very useful TRICK!!

$$\zeta(z) = \sum_{n=1}^{N} n^{-z} + \frac{N^{1-z}}{z-1} - z\int_N^\infty \frac{r(t)}{t^{z+1}}\,dt\ ,$$

$\mathrm{Re}(z) > \underline{0}$. $(z \neq 1)$

We propose to begin with (C) [even though it looks to be the most complicated].

To prove (C), notice that it suffices to prove it for, say, $|y| \gtrsim 100$.

In fact, for $2 \lesssim |y| \lesssim 100$, we can just use our old VERY CRUDE Theorem 4 from Lec 5, page 12.

In this connection, recall too that

$$|\zeta(z)| \lesssim \frac{1}{\delta} + O(1) \lesssim \frac{\text{const}}{\delta}$$

for all $\mathrm{Re}(z) \gtrsim 1+\delta$. (Also on p. 12.)

Use p. 10 line 4 above. $\boxed{|y| \gtrapprox 100\,,\ x \gtrapprox \delta}$

$$|\zeta(x+iy)| \lessapprox \sum_{n=1}^{N} n^{-x} + \frac{N^{1-x}}{|z-1|} + |z| \int_N^\infty \frac{1}{t^{x+1}}\,dt$$

$$|\zeta(x+iy)| \lessapprox \sum_{n=1}^{N} n^{-\delta} + \frac{N^{1-\delta}}{|y|} + (x+|y|) \int_N^\infty \frac{dt}{t^{1+\delta}}$$

$$\sum_{n=1}^{N} n^{-\delta} < 1 + \int_1^N u^{-\delta}\,du \qquad \left\{\begin{matrix}\text{by}\\ \text{areas}\end{matrix}\right\}$$

$$\sum_{n=1}^{N} n^{-\delta} < 1 + \left.\frac{u^{1-\delta}}{1-\delta}\right]_1^N$$

$$\sum_{n=1}^{N} n^{-\delta} < 1 + \frac{N^{1-\delta}}{1-\delta} < 2\frac{N^{1-\delta}}{1-\delta}$$

$$|\zeta(x+iy)| \lessapprox 2\frac{N^{1-\delta}}{1-\delta} + \frac{N^{1-\delta}}{100} + (x+|y|)\frac{N^{-\delta}}{\delta}$$

For $x \gtrapprox 1+\delta$, we already know $|\zeta(x+iy)| \lessapprox \dfrac{\text{const}}{\delta}$, hence (C) is certainly OK here [if c is taken sufficiently big].

For this reason, there is no harm in proceeding under the assumption

$$|y| \gtrsim 100, \quad \delta \lesssim x \lesssim 1+\delta .$$

Get:

$$|\zeta(x+iy)| \lesssim 2\frac{N^{1-\delta}}{1-\delta} + \frac{N^{1-\delta}}{100} + 2|y|\frac{N^{-\delta}}{\delta}$$

$$\lesssim 3\frac{N^{1-\delta}}{1-\delta} + 2\frac{|y|}{\delta}N^{-\delta}$$

$$\lesssim 3N^{-\delta}\left[\frac{N}{1-\delta} + \frac{|y|}{\delta}\right] .$$

This estimate can admittedly be improved. But, a sloppy one is sufficient.

Also, recall $\zeta(x-iy) = \overline{\zeta(x+iy)}$. Hence, wlog, $y \gtrsim 100$.

Let's try $N = G\frac{y}{\delta}$ where $1 \lesssim G \lesssim 10$, say, and we always adjust it to make $N \in \mathbb{Z}$.

(Note $\frac{y}{\delta} \gtrsim \frac{100}{\delta} \gtrsim 100$.)

Get:

$$|\zeta(x+iy)| \leq 3\left(G\frac{y}{\delta}\right)^{-\delta}\left[N+\frac{y}{\delta}\right]\frac{1}{1-\delta} \quad \text{by } (12)$$

$$= 3\left(G\frac{y}{\delta}\right)^{-\delta}\left[G\frac{y}{\delta}+\frac{y}{\delta}\right]\frac{1}{1-\delta}$$

$$= \frac{3}{1-\delta}G^{-\delta}y^{-\delta}\delta^{\delta}(G+1)\frac{y}{\delta}$$

$$\boxed{\begin{array}{l} \delta^{\delta} = e^{\delta \ln \delta} \\ \text{is bdd away from } 0 \\ \text{and } \infty \text{ for } 0 \leq \delta \leq 1 \end{array}}$$

$$\leq \frac{c_1}{1-\delta}G^{-\delta}y^{1-\delta}\cdot\frac{2G}{\delta} \qquad \{1 \leq G \leq 10\}$$

$$\leq \frac{c_2}{(1-\delta)\delta}y^{1-\delta}, \text{ AS REQUIRED.}$$

This proves (C).

It is important to note $\delta \in (0,1)$ is arbitrary. It could even be taken as a fcn of y.

(A) is now a <u>trivial</u> consequence of (C).

Indeed, since $\zeta(z)$ is a nice analytic fcn for $\mathrm{Re}(z) \geqq 1$, $|y| \gtrsim 2$, there is nothing to do for $\{1 \lesssim x \lesssim 2,\ 2 \lesssim |y| \lesssim 100\}$. For $\{x \gtrsim 2,\ 2 \lesssim |y| \lesssim 100\}$ just use (10) last 3 lines; again nothing to do.

So, wlog, we can assume $|y| \gtrsim 100$. Also $y \gtrsim 100$.

Put $\delta \approx 1 - \dfrac{1}{\ln y}$ in (C). Note $\ln 100 \approx 4.605^{+}$.
Hence $.75 < \delta < 1$. By (C), get (see (9)):

$$|\zeta(x+iy)| \lesssim \frac{2C}{1-\delta}\, y^{1-\delta}, \quad x \gtrsim 1 - \frac{1}{\ln y}$$

$$|\zeta(x+iy)| \lesssim 2C(\ln y)\, y^{\frac{1}{\ln y}}$$

$$|\zeta(x+iy)| \lesssim 2Ce(\ln y) \lesssim 6C(\ln y).$$

Now just specialize to $x \gtrsim 1$. Done!

(B) is "almost" as trivial <u>once</u> we recall Cauchy's inequality for $|f'(z_0)|$. IE

$$f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\, dz$$

$$\Downarrow$$

$$|f'(z_0)| \lesssim \frac{1}{2\pi} \frac{M(R)}{R^2} (2\pi R)$$

where $M(R) \equiv \max_{|z-z_0|=R} |f|$

$$\Downarrow$$

$$\boxed{|f'(z_0)| \lesssim \frac{M(R)}{R}}\ .$$

Here are the details for (B).

First, since $f(z)$ is a nice analytic fcn for $Re(z) \gtrsim 1$, $|y| \lesssim 2$, there is nothing to do for $\{1 \lesssim x \lesssim 2,\ 2 \lesssim |y| \lesssim 150\}$. For $\{x \gtrsim 2,\ 2 \lesssim |y| \lesssim 150\}$, apply Cauchy's inequality with $R = 1/2$ and (10) last 3 lines. Again nothing to do. ⊛

So, wlog, take $|y| \gtrsim 100$. We can also assume $y \gtrsim 100$. We will use (C) with a δ similar to $1 - \frac{1}{\ln y}$.

⊛ I do want 150 here, i.e. a slight overshoot over 100.

Take $\delta = 1 - \frac{\lambda}{\ln y}$, where $0 < \lambda \le 1$. We'll choose λ in a few moments. Note that we have $.75 < \delta < 1$ by (14) line 8. Apply (C) on page (9).

Get:

$$|\zeta(x+iy)| \le \frac{2e}{1-\delta}\, y^{\frac{\lambda}{\ln y}}, \quad \text{all } x \ge 1 - \frac{\lambda}{\ln y},\ y \ge 100$$

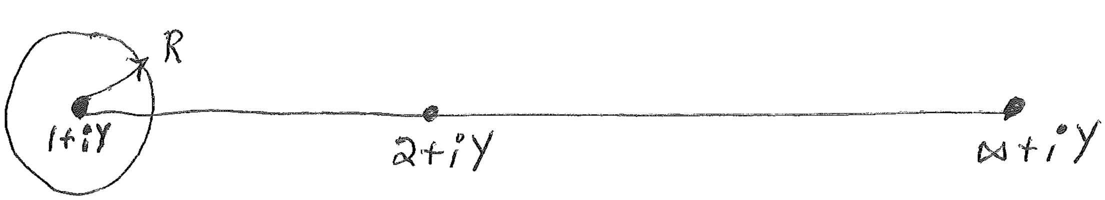
$$|\zeta(x+iy)| \le \frac{2e}{\lambda}(\ln y)e^{\lambda}$$

$$|\zeta(x+iy)| \le \frac{6e}{\lambda}(\ln y), \quad \text{for all } x \ge 1 - \frac{\lambda}{\ln y},\ y \ge 100.$$

We want to RIG THINGS so we can select $Y \ge 110$, then use $R = \frac{1}{10}\frac{\lambda}{\ln Y}$ (say) in Cauchy's inequality for a center z_0 along the segment $[1+iY, \infty+iY)$.

Note that $R \le \frac{1}{10}\frac{1}{\ln Y} \le \frac{1}{46} < \frac{1}{2}$.

$\boxed{\ln 100 = 4.605^{+}}$



As the circle slides along, its y-values clearly stay between $Y-1$ and $Y+1$.

Hence, obviously, $y \gtrapprox 100$.

But we <u>must</u> make certain that no matter what happens, we have $x \gtrapprox 1 - \frac{2}{\ln y}$ at <u>all</u> times on the circle.

<u>Baby Calculus Lemma</u>

Given $T \gtrapprox 110$. Keep $y \in [T-1, T+1]$.

Then:

$$\frac{4}{5} \lessapprox \frac{\ln y}{\ln T} \lessapprox \frac{5}{4}.$$

<u>Pf</u>

$$\frac{\ln(T-1)}{\ln T} \lessapprox \frac{\ln y}{\ln T} \lessapprox \frac{\ln(T+1)}{\ln T}$$

But, by theorem of the mean,

$$\ln(T+1) - \ln(T) \lessapprox \frac{1}{T}(1)$$

$$\ln(T) - \ln(T-1) \lessapprox \frac{1}{T-1}(1).$$

So,

$$\ln(T+1) \lesssim \ln(T) + \frac{1}{T}$$

$$\ln(T-1) \gtrsim \ln(T) - \frac{1}{T-1}$$

$$\Downarrow$$

$$\ln(T+1) \lesssim \ln T + \frac{1}{110} < \ln T + \frac{1}{100}$$

$$\ln(T-1) \gtrsim \ln T - \frac{1}{109} > \ln T - \frac{1}{100}$$

$$\Downarrow$$

$$\frac{\ln T - \frac{1}{100}}{\ln T} \lesssim \frac{\ln y}{\ln T} \lesssim \frac{\ln T + \frac{1}{100}}{\ln T}$$

$$\{\text{but } \ln T \gtrsim \ln 100 = 4.605^+\}$$

$$.99 \lesssim \frac{\ln y}{\ln T} \lesssim 1.01 \qquad \text{OK} \qquad \blacksquare$$

In the moving circle on 16 bottom, obviously

$$x \gtrsim 1 - \frac{1}{10}\frac{\lambda}{\ln Y}. \qquad \text{1-R}$$

But $\ln Y = \omega \ln y$ with $\frac{4}{5} \lesssim \omega \lesssim \frac{5}{4}$ by Calc Lemma.

So,

$$x \gtrsim 1 - \frac{1}{10}\frac{\lambda}{\omega \ln y} \qquad \text{on circle.}$$

Must make sure

$$\frac{\lambda}{\ln y} \gtrsim \frac{\lambda}{10\omega}\frac{1}{\ln y}$$

i.e.,

$$1 \gtrsim \frac{1}{10\omega} .$$

but $8 \lesssim 10\omega \lesssim 12.5$

Thus things are OK and (actually) λ is irrelevant.

So: just put $\lambda = 1$.

$$\boxed{\text{Get: } R = \frac{1}{10}\frac{1}{\ln Y} .}$$

By Cauchy's inequality 15, and 16 (middle), we find that:

$$|\zeta'(x+iY)| \lesssim \frac{6e[\ln(Y+1)]}{R} \lesssim \frac{12e\ln Y}{R}$$

$$\lesssim 120e(\ln Y)^2$$

for any $x \in [1, \infty)$. Recalling 15 lines 5–9, we have thus proved (B). ∎

Two Remarks

1. Take $\lambda \in [11, 12]$ say. Note $\ln 10^6 = 13.815^+$.
By mimicking 14 – 19, one easily sees that

$$|\zeta(x+iy)| = O(\ln y) \quad \text{for } x \gtrsim 1 - \frac{5}{\ln y}, \; y \gtrsim 10^6$$

$$|\zeta'(x+iy)| = O(\ln^2 y) \quad \text{for same } (x, y).$$

One can take $R = \dfrac{\lambda}{2\ln y}$. Key necessity is

$$\frac{\lambda}{\ln y} \gtrsim \frac{5}{\ln y} + \frac{\lambda}{2\ln y}$$

or

$$\frac{\lambda}{\ln y} \gtrsim \frac{5}{\omega \ln y} + \frac{\lambda}{2\omega \ln y}$$

or

$$\lambda\left(1 - \frac{1}{2\omega}\right) \gtrsim \frac{5}{\omega}.$$

But, by 18, $\omega = 1 \pm [.01]$. OK

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2. Why do we use $\delta = 1 - \dfrac{\lambda}{\ln y}$?
Answer: go to 13 5 lines from bottom.

We wonder: if δ is close to 1 and y is very large, what is the <u>smallest</u> that

$$\frac{y^{1-\delta}}{1-\delta}$$

can be? This is a trivial calc problem.

$\delta = 1-u \Rightarrow$ look at $\frac{y^u}{u} \Rightarrow$ look at

$$f(u) = u \ln y - \ln u, \quad 0 \leq u \leq 1$$

$$f' = \ln y - \frac{1}{u}$$

get $f' > 0 \iff u > \frac{1}{\ln y}$.

So key δ is $1 - \frac{1}{\ln y}$, which gives $e^1 \ln y$.

The insertion of λ allows us to "move around" a bit.