# Lecture 9 and 10

## (17 and 19 Feb)

"Synopsis"

$$\sigma_N(t) = \frac{t}{2} + \sum_1^N \frac{\sin nt}{n} \qquad \text{look at this on } 0 < t \lesssim 2\pi - \delta$$

$$\sigma_N(t) = \int_0^t \frac{\sin[(N+\frac{1}{2})v]}{2\sin(v/2)}\,dv \qquad \boxed{\text{Lec 8 p. 16}} \text{ etc}$$

but $\dfrac{1}{2\sin\frac{v}{2}} = \dfrac{1}{v} + h(v)$, $h(v)$ $C^\infty$ on $[0, 2\pi - \delta]$ and analytic near $v = 0$

$$\Rightarrow \sigma_N(t) = \int_0^{(N+\frac{1}{2})t} \frac{\sin q}{q}\,dq + \int_0^t h(v)\sin[(N+\tfrac{1}{2})v]\,dv$$

Freeze $t$. Let $N \to \infty$. Use $\mathrm{Im}\left[\int_0^t h(v)e^{i\lambda v}dv\right] = O(\frac{1}{\lambda})$. Get:

$$\frac{t}{2} + \sum_1^\infty \frac{\sin nt}{n} = A + 0, \qquad A \equiv \int_0^\infty \frac{\sin q}{q}\,dq$$

$$\Downarrow$$

$$\sum_1^\infty \frac{\sin nt}{n} = A - \frac{t}{2}, \qquad \text{all } 0 < t < 2\pi.$$

Plug in $t = \pi$. Get $A = \frac{\pi}{2}$.

Write $t = 2\pi q$, $0 < q < 1$, get

$$\sum_1^\infty \frac{\sin(2\pi n q)}{n} = \frac{\pi}{2} - \pi q \Rightarrow \sum_1^\infty \frac{\sin(2\pi n q)}{-\pi n} = q - \frac{1}{2}.$$

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By the way, regarding this, note that this method of proof immediately yields uniform convergence for $\delta \leq t \leq 2\pi - \delta$.

Lec 8, p. 23

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$$\frac{t}{e^t - 1} \equiv \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \quad , \quad |t| < 2\pi, \quad t \in \mathbb{C}$$

def of Bernoulli numbers

| $k$ | $B_k$ |
| --- | --- |
| 0 | 1 |
| 1 | $-1/2$ |
| 2 | $1/6$ |
| $\vdots$ | $\vdots$ |

Easy Lemma

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2} \operatorname{ctnh}\left(\frac{t}{2}\right) = \text{even}$$

hence $B_3 = B_5 = B_7 = \cdots = 0$.

$$1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} t^n$$

Put $t = 2iz$. Get:

$$iz \operatorname{ctnh}(iz) = 1 + \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{B_k}{k!} (2iz)^k$$

$$\Downarrow$$

Thus

$$z \operatorname{ctn}(z) = 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (-1)^n (2z)^{2n} \quad , \quad |z| < \pi$$

and

$$\pi \operatorname{ctn}(\pi w) = \frac{1}{w}\left[ \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} (-1)^n (2\pi w)^{2n} \right] ; \quad |w| < 1 .$$

The function $\pi \operatorname{ctn}(\pi z)$ has familiar properties in complex analysis.

(A) periodic $z \to z+1$

(B) simple poles at $z = n$, $n \in \mathbb{Z}$

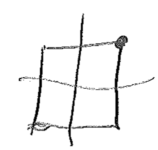
(C) residue always 1

(D) $\operatorname{ctn} \pi(x+iy) = -i + O(e^{-2\pi y}) \quad y \to +\infty$

(E) $\operatorname{ctn} \pi(x+iy) = i + O(e^{-2\pi |y|}) \quad y \to -\infty$

Standard Cauchy residue theorem set-up with

$$\frac{1}{2\pi i} \oint_{C_N} f(z)\, \pi \operatorname{ctn} \pi z \, dz$$



$C_N$ = square with vertices $\left(\pm(N+\frac{1}{2}), \pm(N+\frac{1}{2})\right)$.

EG $f(z) = z^{-2m}$, $m \geq 1$. Use Thm on ②.

Take $N \to \infty$. Get:

$$\boxed{\zeta(2m) = 2^{2m-1} \pi^{2m} \frac{(-1)^{m+1} B_{2m}}{(2m)!}.} \quad \text{(Euler)}$$

In particular, note that $B_{2m} = (-1)^{m+1} |B_{2m}|$ since $\zeta(2m) > 0$.

Another interesting $f(z)$ is $f = \dfrac{1}{\xi - z}$.

Think $\mathcal{D} = \mathbb{C} \sim \mathbb{Z}$ and $\xi \in [\mathcal{D}$ compacta$]$.

By CRT,

$$-\pi \operatorname{ctn} \pi\xi + \sum_{|n| \leq N} \frac{1}{\xi - n} = \frac{1}{2\pi i} \int_{\mathcal{C}_N} \frac{\pi \operatorname{ctn} \pi z}{\xi - z} dz .$$

Write:

$$f(z) = -\frac{1}{z} + \left[ \frac{1}{\xi - z} + \frac{1}{z} \right] = -\frac{1}{z} + \frac{\xi}{(\xi - z) z}$$

$$= -\frac{1}{z} + \underline{r(z)} , \quad r(z) = O(z^{-2}) .$$

Notice that:

$$\int_{\mathcal{C}_N} h(z) dz = \underline{0} \quad \text{for } \underline{\text{ANY}} \text{ even + continuous } h$$

$\{$ uses symmetry of $\mathcal{C}_N$ and $w = -z = z e^{i\pi}$ $\}$.

So, we still get:

$$\int_{\mathcal{C}_N} \frac{\pi \operatorname{ctn} \pi z}{\xi - z} dz = O\left(\frac{1}{N}\right) .$$

So,

$$\lim_{N \to \infty} \sum_{|n| \leq N} \frac{1}{\xi - n} = \pi \operatorname{ctn} \pi \xi , \quad \xi \in \mathcal{D}.$$

By reviewing the proof, we see the limit is uniform for $z \in [\mathcal{D}$ compacta].

Tautology:

$$\frac{1}{z} + \sum_{\substack{-N \\ n \neq 0}}^{N} \left( \frac{1}{z-n} + \frac{1}{n} \right) = \sum_{-N}^{N} \frac{1}{z-n}.$$

THM $z \in \mathcal{D}$.

(i) $\pi \operatorname{ctn} \pi z = \lim_{N \to \infty} \sum_{-N}^{N} \frac{1}{z-n}$

(ii) $\pi \operatorname{ctn} \pi z = \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right)$.

$\uparrow$ unif conv on $\mathcal{D}$ compacta

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Use thm on (2). Get:

$$\pi w \operatorname{ctn}(\pi w) = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} (2\pi)^{2n} (-1)^n w^{2n}, \quad |w| < 1.$$

But, now use the THM above:

$$\pi \operatorname{ctn}(\pi z) = \frac{1}{z} + \sum_{m \neq 0} \left( \frac{1}{z+m} - \frac{1}{m} \right) \qquad \begin{array}{l}\text{Keep} \\ 0 < |z| < 1\end{array}$$

$$\frac{1}{m+z} = \frac{1}{m(1+\frac{z}{m})} = \frac{1}{m}\left[1 - \frac{z}{m} + \frac{z^2}{m^2} \pm \cdots\right]$$

$$\pi \operatorname{ctn} \pi z = \frac{1}{z} + \sum_{m \neq 0} \left[ -\frac{z}{m^2} + \frac{z^2}{m^3} - \frac{z^3}{m^4} \pm \cdots \right]$$

$$\pi \operatorname{ctn} \pi z = \frac{1}{z} - 2z\,\zeta(2) - 2z^3\,\zeta(4) - \cdots$$

$$\pi z \operatorname{ctn} \pi z = 1 - 2z^2\,\zeta(2) - 2z^4\,\zeta(4) - \cdots, \quad |z|<1$$

$$\Downarrow$$

$$-2\,\zeta(2n) = \frac{B_{2n}}{(2n)!}(2\pi)^{2n}(-1)^{n}$$

$$\Downarrow$$

$$\zeta(2n) = \frac{B_{2n}}{(2n)!}(-1)^{n+1}\,2^{2n-1}\,\pi^{2n}$$

= 2nd proof of Euler's formula.

NICE!

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Thm

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

unif conv on $\mathcal{D}$ compacta

Pf

Differentiate THM on 5 by Weierstrass conv thm.

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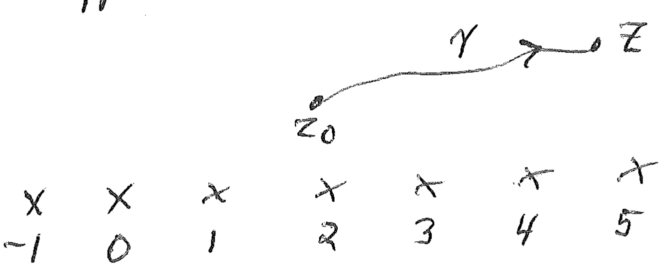
Another nice trick. Take $z_0$ and $z$ in upper half-plane. Connect via $\gamma$.




$\gamma$ = compact subset

$$\frac{d}{dz}\log(\sin\pi z) = \pi \operatorname{ctn} \pi z$$

$$\Downarrow$$

(with some branches)

$$\log[\sin\pi z] - \log[\sin\pi z_0] = \int_\gamma \pi \operatorname{ctn} \pi w \, dw$$

$$\frac{\sin\pi z}{\sin\pi z_0} = \exp\left[\int_\gamma \pi \operatorname{ctn} \pi w \, dw\right]$$

<u>substitute</u> THMS 5 (i)

$$= \exp\left[o(1) + \sum_{-N}^{N}\left(\log(z-n) - \log(z_0-n)\right)\right]$$

$o(1)$ = <u>little</u> "oh"

$$= [1+o(1)] \prod_{-N}^{N} \frac{z-n}{z_0-n}$$

$$= [1+o(1)] \frac{z}{z_0} \frac{\prod_1^N (k^2-z^2)}{\prod_1^N (k^2-z_0^2)}$$

$$= [1+o(1)] \frac{z}{z_0} \frac{\prod_1^N \left(1-\frac{z^2}{k^2}\right)}{\prod_1^N \left(1-\frac{z_0^2}{k^2}\right)}$$

$$\left\{\begin{array}{l} z\in H,\ z_0\in H \quad \text{think nonzero} \\ \text{infinite products etc etc} \end{array}\right\}$$

$$\frac{\sin \pi z}{\pi z} = \left( \frac{\sin \pi z_0}{\pi z_0} \right) \frac{1}{\prod_{1}^{\infty} \left(1 - \frac{z_0^2}{k^2}\right)} \left( \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \right)$$

$$\frac{\sin \pi z}{\pi z} = c \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

first for $Im(z) > 0$, then for ALL $z \in \mathbb{C}$ by analyticity of both sides

Let $z \to 0$. Get $c = 1$.

<u>THM</u>

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \quad z \in \mathbb{C}.$$

Known to Euler

## Baby Fact

Let $\beta(x) = x - [\![x]\!] - \frac{1}{2}$ for $x \in \mathbb{R}$.

Let $F$ be Riemann integrable on $[a,b]$.

Then:

$$\int_a^b F(x)\beta(x)\,dx = \sum_{n=1}^{\infty} \int_a^b F(x)\frac{\sin 2\pi n x}{-\pi n}\,dx.$$

## Proof

$$S_N(x) = -\sum_1^N \frac{\sin 2\pi n x}{\pi n}.$$

Know $S_N(x)$ is unif bdd all $x, N$. Also, $S_N(x) \rightrightarrows \beta(x)$ on compact subsets of $\mathbb{R}$ away from $\mathbb{Z}$.

Think $a < 1 < b$ (say) and $|F| \leq M$, say

$$\left|\int_a^b F(x)(\beta - S_N)\,dx\right| \leq \int_a^{1-\delta} |F||\beta - S_N|\,dx$$
$$+ \int_{1-\delta}^{1+\delta} |F||\beta - S_N|\,dx$$
$$+ \int_{1+\delta}^{b} |F||\beta - S_N|\,dx$$

$\Rightarrow$ done! ∎

Take, e.g., $f \in C^{2R+1}[0,1]$. Know

$$\tfrac{1}{2}f(0) + \tfrac{1}{2}f(1) = \int_0^1 f\,dx + \int_0^1 f' \beta(x)\,dx$$

by E-M version I

$$= \int_0^1 f\,dx + \int_0^1 f'\Big(-\sum_1^\infty \frac{\sin 2\pi n x}{\pi n}\Big)dx$$

$$= \int_0^1 f\,dx + \sum_1^\infty \Big(-\frac{1}{\pi n}\Big)\int_0^1 f' \sin 2\pi n x\,dx$$

by Baby Fact.

Now just look at EACH integral

$$\int_0^1 \underline{f'} \sin(2\pi n x)\,dx$$

and repeatedly integrate by parts.

Step 1

$$\int_0^1 f'\,d\Big(\frac{\cos 2\pi n x}{-2\pi n}\Big)$$

$$= f' \frac{\cos 2\pi n x}{-2\pi n}\Big]_0^1 + \int_0^1 \frac{\cos 2\pi n x}{2\pi n} f''\,dx$$

$$= \frac{f'(1) - f'(0)}{-2\pi n} + \frac{1}{2\pi n}\int_0^1 \cos 2\pi n x \cdot f''\,dx.$$

Step 2

$$\frac{f'(1)-f'(0)}{-2\pi n} + \frac{1}{2\pi n}\int_0^1 f'' \, d\left[\frac{\sin 2\pi n x}{2\pi n}\right]$$

$$= \frac{f'(1)-f'(0)}{-2\pi n} + 0 - 0 - \frac{1}{(2\pi n)^2}\int_0^1 \sin 2\pi n x \cdot f''' \, dx$$

$$= \frac{f'(1)-f'(0)}{-2\pi n} - \frac{1}{(2\pi n)^2}\int_0^1 f'''(x)\sin(2\pi n x)\,dx \,.$$

Clearly a recursion has begun!

$$\int_0^1 f' \sin(2\pi n x)\,dx = \sum_{k=1}^{R} \frac{(-1)^k}{(2\pi n)^{2k-1}}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

$$+ \frac{(-1)^R}{(2\pi n)^{2R}}\int_0^1 f^{(2R+1)}(x)\sin(2\pi n x)\,dx \,.$$

So, on 10 line 5,

$$-\frac{2}{2\pi n}\int_0^1 f'(x)\sin(2\pi n x)\,dx$$

$$= \sum_{k=1}^{R} \frac{2(-1)^{k+1}}{(2\pi n)^{2k}}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

$$+ \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}}\int_0^1 f^{(2R+1)}(x)\sin(2\pi n x)\,dx \,.$$

<u>Thm</u> (E–M version #2, prelim form)

$f \in C^{2R+1}[0, N]$. Then:

$$\sum_{n=0}^{N} f(n) = \frac{1}{2} f(0) + \frac{1}{2} f(N)$$

$$+ \int_0^N f\,dx$$

$$+ \sum_{k=1}^{R} \frac{2(-1)^{k+1}}{(2\pi)^{2k}} \underline{\zeta(2k)} \left[ f^{(2k-1)}(N) - f^{(2k-1)}(0) \right]$$

$$+ 2(-1)^{R+1} \int_0^N f^{(2R+1)}(x) \left[ \sum_1^{\infty} \frac{\sin 2\pi n x}{(2\pi n)^{2R+1}} \right] dx$$

and (in the above) we actually have

$$\frac{2(-1)^{k+1}}{(2\pi)^{2k}} \zeta(2k) \equiv \frac{B_{2k}}{(2k)!} .$$

<u>PF</u>

Write

$$\frac{1}{2} f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2} f(N) = \sum_{j=0}^{N-1} \frac{1}{2}[f(j) + f(j+1)],$$

then use 10 (top) + 11 (bottom). The final observation about $B_{2k}$ follows from Euler's formula for $\zeta(2k)$. ▥

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$\boxed{\text{Lec 10}}$ begins here.

Think about $\displaystyle\int_{C_N} f(z)\,\frac{e^{iqz}}{1-e^{2\pi i z}}\,dz$. Entertain yourself.

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$\zeta(2k) \to 1$ as $k \to \infty$. So: $\displaystyle 1 \sim \frac{(2\pi)^{2k}}{2}\frac{|B_{2k}|}{(2k)!}$ ;

IE $\displaystyle |B_{2k}| \sim \frac{2(2k)!}{(2\pi)^{2k}}$.

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$B_2 = \frac{1}{6}$ $\qquad B_{10} = \frac{5}{66}$ $\qquad$ there are long tables!

$B_4 = -\frac{1}{30}$ $\qquad B_{12} = -\frac{691}{2730}$

$B_6 = \frac{1}{42}$ $\qquad B_{14} = \frac{7}{6}$

$B_8 = -\frac{1}{30}$ $\qquad B_{16} = -\frac{3617}{510}$ $\qquad \left(B_1 = -\frac{1}{2}\right)$ p. 2

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Bernoulli polynomials on $0 < x < 1$ ?

$B_n(x) =$ degree $n$, leading coefficient 1

$B_n(0) = B_n$

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \qquad |t| < 2\pi$$

$B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$ ; ...

Let $\tilde{B}_n(x)$ be the period 1 periodic extension of $B_n(x)$ to $\mathbb{R} - \mathbb{Z}$.

Not hard to check:

$$\frac{\widetilde{B}_{2R}(x)}{(2R)!} = 2(-1)^{R+1} \sum_{1}^{\infty} \frac{\cos(2\pi n x)}{(2\pi n)^{2R}}$$

$$\frac{\widetilde{B}_{2R+1}(x)}{(2R+1)!} = 2(-1)^{R+1} \sum_{1}^{\infty} \frac{\sin(2\pi n x)}{(2\pi n)^{2R+1}} \longleftarrow \text{compare (12) line 6}$$

Note that $B_1(x)$ à la $R=0$ certainly fits!

<u>THEOREM</u> (E-M version 2)

$f \in C^{2R+1}[0,N]$, $f$ complex OK. Then:

$$\sum_{j=0}^{N} f(j) = \tfrac{1}{2} f(0) + \tfrac{1}{2} f(N) + \int_0^N f(x)\,dx$$
$$+ \sum_{1}^{R} \frac{B_{2k}}{(2k)!} \left[ f^{(2k-1)}(N) - f^{(2k-1)}(0) \right]$$
$$+ \text{Remainder}_R, \quad \text{where}$$

$$\left\{ \begin{aligned} \text{Rem}_R &= \int_0^N f^{(2R+1)}(x) \frac{\widetilde{B}_{2R+1}(x)}{(2R+1)!}\,dx \\ &= -\int_0^N f^{(2R)}(x) \frac{\widetilde{B}_{2R}(x)}{(2R)!}\,dx, \quad \text{via easy integ by parts} \end{aligned} \right\}$$

<u>Pf</u>

Use (12) THM + the formulae at top of this page. ∎

## Cor 1

$$|\mathrm{Rem}_R| \leq \frac{|B_{2R}|}{(2R)!} \int_0^N |f^{(2R)}(x)|\, dx .$$

**Pf**

Plug into 2nd form of $\mathrm{Rem}_R$ on (14).

$$|\mathrm{Rem}_R| \leq \int_0^N |f^{(2R)}| \cdot 2 \sum_1^\infty \frac{1}{(2\pi n)^{2R}}\, dx$$

$$= \frac{2}{(2\pi)^{2R}} \zeta(2R) \int_0^N |f^{(2R)}|\, dx$$

$$= \frac{|B_{2R}|}{(2R)!} \int_0^N |f^{(2R)}|\, dx \qquad \text{by (3) box.} \;\blacksquare$$

## Cor 2 (very useful in numerical work)

$$\mathrm{Rem}_R = 2(-1)^{R+2} \int_0^N f^{(2R+2)}(x) \sum_{n=1}^\infty \overbrace{\frac{1-\cos 2\pi n x}{(2\pi n)^{2R+2}}}^{\text{non-negative}}\, dx$$

**Pf**

$$\mathrm{Rem}_R = \int_0^N f^{(2R+1)}(x) \left[ 2(-1)^{R+1} \sum_1^\infty \frac{\sin 2\pi n x}{(2\pi n)^{2R+1}} \right] dx$$

$$= \sum_n \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}} \int_0^N f^{(2R+1)}\, d\left( \frac{1-\cos 2\pi n x}{2\pi n} \right)$$

$$= \sum_n \frac{2(-1)^{R+1}}{(2\pi n)^{2R+1}} \left[ 0 - 0 - \int_0^N \frac{1-\cos 2\pi n x}{2\pi n} f^{(2R+2)}(x)\, dx \right]$$

$$= \sum_n \frac{2(-1)^{R+2}}{(2\pi n)^{2R+2}} \int_0^N (1-\cos 2\pi n x)\, f^{(2R+2)}(x)\,dx$$

$$= 2(-1)^{R+2} \int_0^N \sum_{n=1}^{\infty} \frac{1-\cos 2\pi n x}{(2\pi n)^{2R+2}}\, f^{(2R+2)}(x)\,dx \,. \qquad \blacksquare$$

Cor 3 (very commonly used)

$f$ complex. We always have:

$$|Rem_R| \leqq 2 \frac{|B_{2R+2}|}{(2R+2)!} \int_0^N |f^{(2R+2)}|\,dx \,.$$

Pf

Use Cor 2.

$$|Rem_R| \leqq 2 \int_0^N |f^{(2R+2)}| \cdot \sum_{n=1}^{\infty} \frac{2}{(2\pi n)^{2R+2}}\, dx$$

$$= 2 \cdot \frac{2}{(2\pi)^{2(R+1)}} \zeta(2R+2) \int_0^N |f^{(2R+2)}|\,dx$$

{apply 3 box}

$$= 2 \cdot \frac{|B_{2R+2}|}{(2R+2)!} \int_0^N |f^{(2R+2)}|\,dx \qquad \blacksquare$$

For the sake of clarity, notice too that

$$\mathrm{Rem}_R \equiv \frac{B_{2R+2}}{(2R+2)!}\left[f^{(2R+1)}(N) - f^{(2R+1)}(0)\right]$$

$$+ \ \mathrm{Rem}_{\underline{R+1}} .$$

Cor 3 can thus be obtained <u>equally</u> <u>well</u> by Cor 1; indeed,

$$|\mathrm{Rem}_R| \leqq \frac{|B_{2R+2}|}{(2R+2)!}\left|f^{(2R+1)}(N) - f^{(2R+1)}(0)\right|$$

$$+ \ \frac{|B_{2R+2}|}{(2R+2)!}\int_0^N |f^{(2R+2)}(x)|\,dx \leftarrow \text{Cor } 1$$

$$\leqq 2\frac{|B_{2R+2}|}{(2R+2)!}\int_0^N |f^{(2R+2)}(x)|\,dx .$$

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**THM** (Corollary of E–M à la Euler)

$\zeta(z)$ is analytic on each half-plane $\{Re(z) > -\Delta\}$ except for a simple pole of residue 1 at $z = 1$. Hence,

$$\zeta(z) - \frac{1}{z-1} \text{ is analytic on } \mathbb{C}.$$

**Pf**

E–M $\quad f(t) = (1+t)^{-z} \quad$ keep $Re(z) > 1$ at first

$$f^{(j)}(t) = (-1)^j z(z+1)\cdots(z+j-1)(1+t)^{-z-j}$$

$$\Downarrow$$

$$\sum_{k=1}^{N+1} k^{-z} = \frac{1}{2} + \frac{1}{2}(1+N)^{-z} + \int_0^N (1+t)^{-z}\,dt$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!}\Big[(-1)\, z(z+1)\cdots(z+2k-2)(1+N)^{-z-(2k-1)}$$
$$+ z(z+1)\cdots(z+2k-2)\cdot 1\Big]$$

$$+ \int_0^N \frac{\tilde{B}_{2R+1}(t)}{(2R+1)!}(-1)\, z(z+1)\cdots(z+2R)(1+t)^{-z-2R-1}\,dt$$

Let $N \to \infty$.

$$\zeta(z) = \frac{1}{2} + \int_0^\infty (1+t)^{-z}\, dt$$

$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!} z(z+1)\cdots(z+2k-2)\cdot 1$$

$$+ \int_0^\infty (-1) \frac{z(z+1)\cdots(z+2R)}{(1+t)^{z+2R+1}} \frac{\tilde{B}_{2R+1}(t)}{(2R+1)!}\, dt$$

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1}$$

$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)!} z(z+1)\cdots(z+2k-2)$$

$$+ (-1) z(z+1)\cdots(z+2R) \int_0^\infty \frac{1}{(1+t)^{z+2R+1}} \frac{\tilde{B}_{2R+1}(t)}{(2R+1)!}\, dt$$

$Rem_R$ with $R \geq 1$

note $R = 0$ is OK too;
recall Lec 5 p. 9 line 3
and $r(t) = \frac{1}{2} + \beta(t)$

The integral containing $\tilde{B}_{2R+1}(t)$ is clearly nicely convergent, hence analytic, for compact subsets of $\{Re(z) > -2R\}$.

At once,

$$\zeta(z) - \frac{1}{z-1}$$

is analytic on each $\{Re(z) > -2R\}$ and we are done! $\blacksquare$

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Examples (Numerology!)

Recall $B_k$ on p. (13).

$R = 3$ say $\Rightarrow$ get

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} + \frac{B_2}{2!} z + \frac{B_4}{4!} z(z+1)(z+2)$$

$$+ \frac{B_6}{6!} z(z+1)\cdots(z+4)$$

$$+ (-1) z(z+1)\cdots(z+6) \int_0^\infty \frac{1}{(1+t)^{z+7}} \frac{\tilde{B}_7(t)}{7!} dt$$

$$Re(z) > -6$$

$$\zeta(0) = -\frac{1}{2}$$

$$\zeta(-1) = \frac{1}{2} - \frac{1}{2} + \frac{B_2}{2!}(-1) + 0 = -\frac{1}{12}$$

$$\zeta(-2) = \frac{1}{2} - \frac{1}{3} + \frac{B_2}{2!}(-2) + 0 = \frac{1}{6} + \frac{1}{12}(-2) = \underline{\underline{0}}$$

$$\zeta(-4) = \frac{1}{2} - \frac{1}{5} + \frac{B_2}{2!}(-4) + \frac{B_4}{4!}(-4)(-3)(-2)$$

$$+\, 0$$

$$= \frac{1}{2} - \frac{1}{5} + \frac{1}{12}(-4) + \left(-\frac{1}{30}\right)\frac{(-1)4!}{4!} + 0$$

$$= \frac{3}{10} - \frac{1}{3} + \frac{1}{30} = \frac{9-10+1}{30} = \underline{\underline{0}}$$

One conjectures $\zeta(-2\ell) = 0$, $\ell \geqq 1$.

Euler proved this (playing with the $B_{2k}$).

---

of course he used only $\zeta(x)$
and his "natural formulae".

$\Gamma(z)$, more accurately $\Gamma(\frac{z}{2})$, is part of the "modern Riemann zeta fcn". It's the Archimedean part.

$$\pi^{-z/2}\,\Gamma(\tfrac{z}{2})\,\zeta(z)$$

---

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t}\,dt\,, \quad Re(z) > 0$$

GAMMA FCN.

$$\Gamma(k) = (k-1)! \quad k \geq 1$$

By Weierstrass M-test, the improper integral is unif conv on $\{Re(z) > 0\}$-compacta.

$$|t^{z-1}| = t^{x-1}$$

Hence $\Gamma(z)$ analytic on $Re(z) > 0$.

---

Easy integ by parts:

$$Re(z) > 0 \Rightarrow \Gamma(z+1) = z\,\Gamma(z).$$

---

Hence

$$\Gamma(z+R) = z(z+1)\cdots(z+R-1)\,\Gamma(z)\,, \quad R \geq 1.$$

$$\Rightarrow \boxed{\Gamma(z) = \frac{\Gamma(z+R)}{z(z+1)\cdots(z+R-1)}}$$

But, RHS is analytic for $\{Re(z) > -R\}$, except at $0, -1, -2, \ldots, -(R-1)$.

Hence, $\Gamma(z)$ is analytic on

$$\mathbb{C} - \{0, -1, -2, \ldots\}.$$

Easy to check:

$z = -k$ is a <u>simple</u> pole

$$\text{Res} = \frac{(-1)^k}{k!}.$$

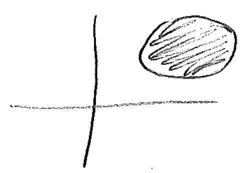
<u>Thm</u> (Euler)

$$\Gamma(z) = \lim_{N \to \infty} \frac{N!\, N^z}{z(z+1)\cdots(z+N)}$$

with <u>unif</u> conv on $\{\text{Re}(z) > 0\}$ compacta.

<u>Pf</u>

Keep $z \in K$ say.



$$e^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n, \quad t > 0 \qquad \text{(baby calc)}$$

We hope to <u>approximate</u> $\Gamma(z)$ by

$$\int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n dt.$$

$\exp[(z-1)\ln t]$

Baby calculus $\Rightarrow$

previous integral $= n^z \int_0^1 v^{z-1}(1-v)^n \, dv$

{ integrate by parts repeatedly }

NOT HARD

$$= n^z \frac{n!}{z(z+1)\cdots(z+n-1)} \left(\frac{1}{z+n}\right)$$

$$\int_0^1 v^{z+n-1}(1-v)^0 \, dv$$

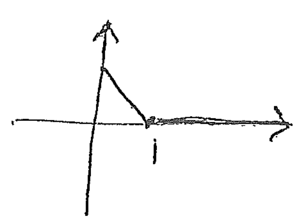
$$= \frac{n^z \cdot n!}{z(z+1)\cdots(z+n)} .$$

Must now prove

$$\lim_{N\to\infty} \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt = \int_0^\infty t^{z-1} e^{-t} \, dt$$

<u>unif</u> on $K$.

Let $(1-u)_* = \begin{cases} 0, & u > 1 \\ 1-u, & 0 \le u \le 1 \end{cases}$.



We will look at

$$\int_0^\infty t^{z-1}\left(1-\frac{t}{N}\right)_*^N dt .$$

which equals $\int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt$

For one moment

Take $t > 0$ and keep $0 < t < n$.

$$\left(1-\frac{t}{n}\right)^n = e^{n \log\left(1-\frac{t}{n}\right)}$$
$$= e^{n\left[-\frac{t}{n} - \frac{1}{2}\frac{t^2}{n^2} - \cdots\right]}$$
$$= e^{-t} e^{-\frac{1}{2}\frac{t^2}{n}} e^{-\frac{1}{3}\frac{t^3}{n^2}} \cdots$$

$$\Downarrow$$

$$0 < \left(1-\frac{t}{n}\right)^n < \left(1-\frac{t}{n+1}\right)^{n+1} < \cdots < e^{-t}.$$

We thus see that

$$0 \leqq \left(1-\frac{t}{N}\right)_*^N \nearrow \text{ to } e^{-t}$$

as $N \to \infty$. In addition, by the expansion above, we have uniform convergence as $N \to \infty$ on any $0 \leqq t \leqq \Delta$, $\Delta$ big.

The issue with $\int_0^\infty t^{z-1}\left(1-\frac{t}{N}\right)_*^N dt$ is now a simple manipulation with the dominated convergence thm for $z \in K$.

Just write

$$\int_0^\infty t^{z-1}\left(1-\frac{t}{N}\right)_*^N dt$$

$$= \int_0^\delta + \int_\delta^T t^{z-1}\left(1-\frac{t}{N}\right)_*^N dt + \int_T^\infty$$

$$\left\{ \begin{array}{l} \text{then adjust } \delta \text{ and } T > 1 \\ \text{appropriately} \\ \hline \alpha = \inf \mathrm{Re}(K), \quad \beta = \sup \mathrm{Re}(K) \end{array} \right\}$$

$$\left| \int_0^\delta \right| \leq \int_0^\delta t^{\alpha-1} e^{-t}\, dt < \frac{\varepsilon}{100}$$

$$\left| \int_T^\infty \right| \leq \int_T^\infty t^{\beta-1} e^{-t}\, dt < \frac{\varepsilon}{100}$$

etc etc. ∎

Thm

$\Gamma(z) \neq 0$ for $\mathrm{Re}(z) > 0$.

Pf

$\Gamma(x) > 0 \Rightarrow \Gamma(z) \not\equiv 0$.

Notice that $\dfrac{N!\, N^z}{z(z+1)\cdots(z+N)} \neq 0$ on $\mathrm{Re}(z) > 0$.

By Hurwitz's thm in analytic fcns, the limit is either $\equiv 0$ or <u>never</u> zero. So, we are done thanks to THM 23. ∎

<u>Cor</u>

$\Gamma(z) \neq 0$ for $z \in \mathbb{C}$.

<u>Pf</u>

$z = -k$ is no problem! $(k \geq 0)$

Suppose $\Gamma(z_0) = 0$, some $Re(z_0) \leq 0$. $z_0 \notin \mathbb{Z}$.

But, then, 22 box $\Rightarrow$

$$\Gamma(z_0+m) = z_0(z_0+1)\cdots(z_0+m-1)\,\Gamma(z_0) = 0$$

for all $m$ large. Contradicts Thm 26. ∎

---

Hence, $\frac{1}{\Gamma(z)}$ is analytic <u>on $\mathbb{C}$</u> with zeros at $\{0, -1, -2, \ldots\}$, each of multiplicity 1.

---

Let $K$ be any compact subset of

$$\mathbb{C} - \{0, -1, -2, \ldots\}.$$

Choose $m$ so big that

$$m + \inf \operatorname{Re}(K) \geq 1.$$

The relation

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}$$

holds first for $\operatorname{Re}(z) > 0$, then ALL $z$ by analytic continuation.

One can apply Thm 23 to $\Gamma(z+m)$ for $z \in K$.

Thm

$$\Gamma(z) = \lim_{N \to \infty} \frac{N!\, N^z}{z(z+1)\cdots(z+N)}$$

for $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$ with unif conv on compacta.

PF

Exercise — using the procedure suggested. ▮

**Lemma**

Let $Q_n(z)$ be analytic on $|z-z_0| < 2h$.
Let $Q_n(z)$ conv unif on, say, $|z-z_0| = h$.
Then: $Q_n(z)$ conv unif on $|z-z_0| \leq h$ too!

**Pf**

Apply max mod principle to $Q_m(z) - Q_n(z)$, $m > n$.
Get unif Cauchy condition for $|z-z_0| \leq h$. ∎

**Thm**

$$\frac{1}{\Gamma(z)} = \lim_{N\to\infty} \frac{z(z+1)\cdots(z+N)}{N!\, N^z} \quad \underline{\text{on } \mathbb{C}}$$

with unif conv on compacta.

**Pf**

Combine Thm 28 with Lemma. ∎

**Thm** (all well-known)

Let $\underline{D} = \mathbb{C} \sim \{0, -1, -2, \ldots\}$.

(a) $\Gamma(z) \neq 0$ on $\mathbb{C}$, $\frac{1}{\Gamma(z)}$ entire, simple zeros at $z = 0, -1, -2, \ldots$

(b) $$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{k=1}^{\infty} \left( \frac{1}{z+k} - \frac{1}{k} \right)$$

with unif conv on $D$ compacta

(c) $$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

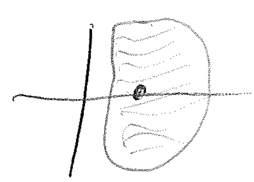
(d) $$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + \tfrac{1}{2})$$

(e) $$\gamma = \lim_{N \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \ln N \right) = -\Gamma'(1)$$

Pf

Exercise!

Some pointers. In (b), by analytic continuation, it suffices to work on a compact subset $\underline{\text{of } Re(z) > 0}$ containing 1. Use thm 28.



$$f_n(z) \rightrightarrows \Gamma(z)$$

so

$$\frac{f_n'(z)}{f_n(z)} \rightrightarrows \frac{\Gamma'(z)}{\Gamma(z)}, \quad \frac{f_n'(1)}{f_n(1)} \to \frac{\Gamma'(1)}{\Gamma(1)}$$

Study

$$\frac{f_n'(z)}{f_n(z)} - \frac{f_n'(1)}{f_n(1)} \quad \text{as} \quad n \to \infty .$$

Get (b) and (e).

For (c), take $\operatorname{Im}(z) > 0$ w/log.* Use thm 28 and recall

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) .$$

In (d), use thm 28 again with obvious factorings to get

$$\Gamma(2z) = A\, \Gamma(z)\, \Gamma(z + \tfrac{1}{2})\, 2^{2z} .$$

Evaluate $A$ via the knowledge that $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$.

∎

* must prove

$$\Gamma(z)\Gamma(-z) = \frac{\pi}{-z \sin(\pi z)}$$

Thm (very well-known)

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k}$$

with unif conv on $\mathbb{C}$ - compacta.

Pf

Work on $K = \{|z| \leq R\}$, $R$ large.

$$\frac{1}{\Gamma(z)} = \lim_{N\to\infty} \frac{z \prod_{k=1}^{N} (z+k)}{N^z \prod_{k=1}^{N} k} \quad (29) \quad \text{uniformly}$$

$$= \lim_{N\to\infty} \frac{z \prod_{k=1}^{N} \left(1+\frac{z}{k}\right)}{e^{z \ln N}}$$

$$\left\{ \begin{aligned} &\sum_{1}^{N} \frac{1}{k} = \ln N + \gamma + \varepsilon_N, \quad \varepsilon_N \to 0 \\ &\ln N = \sum_{k=1}^{N} \frac{1}{k} - \gamma - \varepsilon_N \end{aligned} \right\}$$

$$= \lim_{N\to\infty} \frac{z \prod_{k=1}^{N} \left(1+\frac{z}{k}\right)}{e^{z\left(\sum \frac{1}{k} - \gamma - \varepsilon_N\right)}}$$

$$= \lim_{N\to\infty} e^{\gamma z} e^{\varepsilon_N z} z \prod_{k=1}^{N} \left(1+\frac{z}{k}\right) e^{-z/k}$$

$$= z e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k}$$

Note:

for $k \gtrsim 100R$, say, get $\left|\frac{z}{k}\right| \lesssim \frac{1}{100}$

but $(1+w)e^{-w} = 1 + O(w^2)$ for $|w| \lesssim \frac{1}{100}$

$$\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} = 1 + O\left(\frac{R^2}{k^2}\right)$$

$$\Downarrow$$

Weierstrass M-test for product is fine. ∎

---

We now return to 28 and wish to apply E-M version 2 to $\log \Gamma(z)$. We wish to do this for $z \in D = \mathbb{C} - \{0,-1,-2,\ldots\}$, keeping $z$ on some compact subset of $D$ initially.

Clearly, we need:

$$\ln N! + z \ln N - \sum_{j=0}^{N} \log(z+j),$$

for some branch $\log \Gamma(z)$

and for the logs, we should try to use Log.

For ease of calculation, focus on

$$T_N(z) \equiv \sum_{j=0}^{N} \mathrm{Log}(z+j) - \sum_{k=1}^{N} \ln k - z \ln N$$

Can use E–M

Can use E–M

$$\sum_{j=0}^{N-1} \mathrm{Log}(1+j)$$

The calculation is easy, but boring. We just give a few steps.

$f(t) = \mathrm{Log}(t+z)$ $\quad z =$ held fixed

$f'(t) = (t+z)^{-1}$

$f''(t) = (-1)(t+z)^{-2}$

$f^{(j)}(t) = (-1)^{j-1}(j-1)!\,(t+z)^{-j}$

$\Downarrow$

$$\sum_{j=0}^{N} \mathrm{Log}(z+j) = \frac{1}{2}\mathrm{Log}\, z + \frac{1}{2}\mathrm{Log}(z+N) \qquad \text{see } 14$$
$$+ \int_0^N \mathrm{Log}(z+t)\,dt$$
$$+ \sum_1^R \frac{B_{2k}}{(2k)!}\left[(2k-2)!\,(z+N)^{-(2k-1)} - (2k-2)!\,(z)^{-(2k-1)}\right]$$
$$+ \int_0^N \frac{\tilde{B}_{2R+1}(t)}{(2R+1)!}\,(2R)!\,(t+z)^{-2R-1}\,dt$$

$$\left\{ \int_0^N \mathrm{Log}(z+t)\,dt = (z+N)\mathrm{Log}(z+N) - z\,\mathrm{Log}\, z - N \right\}$$

$$\sum_{j=0}^{N} \mathrm{Log}(z+j) = \frac{1}{2}\mathrm{Log}\, z + \left(z+N+\frac{1}{2}\right)\mathrm{Log}(z+N)$$
$$- z\,\mathrm{Log}\, z - N$$
$$+ \sum_1^R \frac{B_{2k}}{(2k)(2k-1)}\left[(N+z)^{-2k+1} - z^{-2k+1}\right]$$
$$+ \int_0^N \frac{\tilde{B}_{2R+1}(t)}{2R+1}\,(t+z)^{-2R-1}\,dt$$

---

to get $\sum_{k=1}^{N} \ln k$ just take $z=1$ and $N \hookrightarrow N-1$

$$\Downarrow$$

$$\sum_{k=1}^{N} \ln k = (N+\tfrac{1}{2}) \operatorname{Log} N - (N-1) + \sum_{1}^{R} \frac{B_{2k}}{(2k)(2k-1)} \left[ N^{-2k+1} - 1 \right] + \int_0^{N-1} \frac{\tilde{B}_{2R+1}(t)}{2R+1} (t+1)^{-2R-1} dt$$

---

So, by flipping all signs in the above, get:

$$T_N(z) \simeq (z+N+\tfrac{1}{2}) \left\{ \operatorname{Log}(z+N) - \operatorname{Log}(N) \right\} - (z-\tfrac{1}{2}) \operatorname{Log} z - 1$$

$$+ \sum_{1}^{R} \frac{B_{2k}}{(2k)(2k-1)} \left[ 1 - z^{-2k+1} + (N+z)^{-2k+1} - N^{-2k+1} \right]$$

$$+ \frac{1}{2R+1} \int_0^{N} \tilde{B}_{2R+1}(t)(t+z)^{-2R-1} dt$$

$$- \frac{1}{2R+1} \int_0^{N-1} \tilde{B}_{2R+1}(t)(t+1)^{-2R-1} dt$$

$$\left\{ \text{use } \underset{z \in \mathbb{D}}{\operatorname{Log}(z+N) - \operatorname{Log}(N) = \operatorname{Log}\left(\tfrac{z}{N}+1\right)} \right\}$$

$$T_N(z) \approx \left(z+N+\tfrac{1}{2}\right)\operatorname{Log}\left(1+\frac{z}{N}\right)$$

$$-\left(z-\tfrac{1}{2}\right)\operatorname{Log} z \;-1$$

$$+\sum_{1}^{R}\frac{B_{2k}}{(2k)(2k-1)}\left[\cdots\right]$$

$$+\frac{1}{2R+1}\int_0^N \tilde{B}_{2R+1}(t)\,(t+z)^{-2R-1}\,dt$$

$$-\frac{1}{2R+1}\int_0^{N-1}\tilde{B}_{2R+1}(t)\,(t+1)^{-2R-1}\,dt \,.$$

Remember that $|\tilde{B}_{2R+1}(t)| \leq$ some $\mathcal{B}_R$, all $t \in \mathbb{R}$.

Now let $N \to \infty$.

$$\left(z+\tfrac{1}{2}\right)\operatorname{Log}\left(1+\frac{z}{N}\right) \rightrightarrows 0$$

$$N\operatorname{Log}\left(1+\frac{z}{N}\right) \rightrightarrows z$$

Conclude:

$$T_N(z) \rightrightarrows z-\left(z-\tfrac{1}{2}\right)\operatorname{Log} z \;-1$$

$$+\sum_{1}^{R}\frac{B_{2k}}{(2k)(2k-1)}\left[1-z^{-2k+1}\right]$$

$$+\frac{1}{2R+1}\int_0^{\infty}\frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\,dt-\frac{1}{2R+1}\int_0^{\infty}\frac{\tilde{B}_{2R+1}(t)}{(t+1)^{2R+1}}\,dt \,.$$

Recall 33 bottom. Deduce:

$$-\log\Gamma(z) = z - \left(z-\tfrac{1}{2}\right)\operatorname{Log} z$$

$\uparrow$
some branch

$$-\sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$

$$+\frac{1}{2R+1}\int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\,dt$$

$$+ C_R,$$

where $C_R$ = some appropriate real constant.

Thus, on $D$, $D = \mathbb{C} - \{0,-1,-2,\ldots\}$,

$$\log\Gamma(z) = \left(z-\tfrac{1}{2}\right)\operatorname{Log} z - z$$

$$+\sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$

$$-\frac{1}{2R+1}\int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\,dt$$

$$+ E_R,$$

$E_R$ = suitable real constant.

The preceding relation is an identity.

Note that RHS is real if $z = x > 0$.

Hence the branch of $\log \Gamma(z)$ under discussion is the one that reduces to $\ln \Gamma(x)$ for $z = x > 0$.

Also, note:

$$\left| \frac{1}{2R+1} \int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+x)^{2R+1}}\, dt \right|$$

$$\lesssim \frac{1}{2R+1} \int_0^\infty \frac{\mathcal{B}_R}{(t+x)^{2R+1}}\, dt$$

$$= \frac{1}{(2R+1)(2R)} \mathcal{B}_R\, x^{-2R}$$

$$= O\left(\frac{1}{x^{2R}}\right), \quad \text{each } R.$$

So,

$$\ln \Gamma(x) = \left(x - \tfrac{1}{2}\right) \ln x - x + \sum_1^R \frac{B_{2k}}{(2k)(2k-1)} x^{-2k+1}$$

$$+ E_R + O(x^{-2R}) \qquad \text{as } x \to +\infty.$$

At once, by comparing for different $R$'s,

$$E_1 = E_2 = E_3 = \cdots = E_R = \cdots .$$

Call the common value $E$.

An easy substitution into

$$\Gamma(2x) = 2^{2x-1}\pi^{-1/2}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) \qquad (30)\,d$$

$$\Downarrow$$

$$\ln\Gamma(2x) = (2x-1)\ln 2 - \tfrac{1}{2}\ln\pi$$
$$+ \ln\Gamma(x) + \ln\Gamma(x+\tfrac{1}{2})$$

gives $(R=1)$

$$(2x-\tfrac{1}{2})\ln(2x) - 2x + O(\tfrac{1}{x}) + E$$

$$= (2x-1)\ln 2 - \tfrac{1}{2}\ln\pi$$
$$+ (x-\tfrac{1}{2})\ln x - x + O(\tfrac{1}{x}) + E$$
$$+ (x)\ln(x+\tfrac{1}{2}) - (x+\tfrac{1}{2}) + O(\tfrac{1}{x}) + E$$

$$\Downarrow$$

$$-\frac{1}{2}\ln 2 + O\left(\frac{1}{x}\right) + E$$

$$= -\ln 2 - \frac{1}{2}\ln \pi + 2E + O\left(\frac{1}{x}\right)$$

$$\frac{1}{2}\ln 2 + \frac{1}{2}\ln \pi + O\left(\frac{1}{x}\right) = E$$

$$\Rightarrow E = \frac{1}{2}\ln(2\pi).$$

On 38 bottom, we therefore get:

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\operatorname{Log} z - z + \frac{1}{2}\ln(2\pi)$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$
$$- \frac{1}{2R+1}\int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\,dt$$

AS AN IDENTITY ON $\mathcal{D}$.

**Theorem** (Stirling – corollary of E–M)

Keep $z \in D$. $D = \mathbb{C} - \{0, -1, -2, \dots\}$.

We have

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right) \operatorname{Log} z - z + \tfrac{1}{2} \ln(2\pi)$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1}$$
$$- \frac{1}{2R+1} \int_0^{\infty} \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}} \, dt ,$$

where the branch of $\log \Gamma(z)$ reduces to $\ln \Gamma(x)$ when $z = x > 0$.

Pf

As above. ∎

FAMOUS

**Corollary** (Stirling's asymptotic formula)

Fix any $R \geqq 1$ and $\delta > 0$. Then:

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right) \operatorname{Log} z - z + \tfrac{1}{2} \ln(2\pi)$$
$$+ \sum_{k=1}^{R} \frac{B_{2k}}{(2k)(2k-1)} z^{-2k+1} + O_{R\delta}\left(\frac{1}{|z|^{2R+1}}\right)$$

as $z \to \infty$ in $|\operatorname{Arg}(z)| \leqq \pi - \delta$.

Pf

A simple absolute value estimate of

$$-\frac{1}{2R+1}\int_0^\infty \frac{\tilde{B}_{2R+1}(t)}{(t+z)^{2R+1}}\,dt$$

compare (39) middle

does <u>not</u> work. One uses a minor trick instead. Namely:

$$\mathrm{Rem}_R = \frac{B_{2R+2}}{(2R+2)(2R+1)} z^{-2R-1}$$

$$-\frac{1}{2R+3}\int_0^\infty \frac{\tilde{B}_{2R+3}(t)}{(t+z)^{2R+3}}\,dt \quad .$$

Notice that:

$$|\text{last term}| \leqq \frac{\mathcal{B}_{R+1}}{2R+3}\int_0^\infty \frac{dt}{|t+z|^{2R+3}}$$

$$\left\{\begin{array}{c} z = \rho e^{i\omega}, \quad |\omega| \leqq \pi - \delta, \\ \rho \text{ large} \\ \text{put } t = \rho v \end{array}\right\}$$

$$= \frac{\beta_{R+1}}{2R+3} \int_0^\infty \frac{\rho \, dv}{\rho^{2R+3} |v + e^{i\omega}|^{2R+3}}$$

$$= \frac{\beta_{R+1}}{2R+3} \rho^{-2R-2} \int_0^\infty \frac{dv}{|v+e^{i\omega}|^{2R+3}}$$

These integrals are $O_{R\delta}(1)$ for $|\omega| \leq \pi - \delta$

$$\Downarrow$$

$$|\mathrm{Rem}_R| \lessapprox O_{R\delta}(z^{-2R-1}) + O_{R\delta}(1)|z|^{-2R-2}$$

$$= O_{R\delta}(1)|z|^{-2R-1} .$$

$\blacksquare$

## Remark 1

Refer to 14 line 4 and 14 Thm (E-M vers 2). Though it may be tempting to allow $R=0$ on page 42, note that

$$\int_0^\infty \frac{\tilde{B}_1(t)}{t+z}\,dt$$

is NOT absolutely convergent. For this reason, allowing $R=0$ in Thm 42 is usually avoided. In the Corollary, it is of course OK, since

$$\frac{B_2}{2\cdot 1} z^{-1} + O_\delta\left(\frac{1}{|z|^3}\right) = O_\delta\left(\frac{1}{|z|}\right).$$

## Remark 2 (classical Stirling) $R=0$ OK HERE.

By use of 15 Cor 2 (nontrivial), it is possible to show

$$\log N! = (N+\tfrac{1}{2})\ln N - N + \tfrac{1}{2}\ln(2\pi) + \sum_1^R \frac{B_{2k}}{(2k)(2k-1)} N^{-2k+1} + REM_R,$$

$$REM_R = \lambda \frac{B_{2R+2}}{(2R+2)(2R+1)} N^{-2R-1} \quad \text{with } 0<\lambda<1.$$