5/31/16

Addendum A

[concerning Lecture 30]

An alternate ending for Lec 30 ("skip Turing, stay with Euler").*

After developing ③ ~ ⑭ in Lec 30, go back to Euler's *Opera Omnia* (e.g. vol. 8, p. 291) and say:

We now adapt the idea of Newman's general thm on page ⑨ of Lec 28 — motivated by Newman's *Amer Math Monthly* article cited on p. ①.

Take $\mathrm{Re}(z) > 0$ initially and define

$$g_N(z) = \sum_{n=1}^{N} \frac{\mu(n)}{n^{1+z}}$$

$$g(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+z}} = \frac{1}{\zeta(1+z)} .$$

$\zeta(1+iy) \neq 0$

Form analytic continuations. Keep N and R large. Use exactly the same path as on p. ⑩ of Lec 28 (it depends on R). Get:

$$g(0) - g_N(0) = \frac{1}{2\pi i} \oint_C (g - g_N) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

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* This is the comment referred to on p. ㉔ of Lec 28.

$$= \frac{1}{2\pi i}\int_{C_+} (g-g_N)\, N^z \left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$- \frac{1}{2\pi i}\int_{C_-} g_N(z)\, N^z \left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$+ \frac{1}{2\pi i}\int_{C_-} g(z)\, N^z \left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} .$$

On C_+, observe that:

$$|g(z)-g_N(z)| = \left|\sum_{N+1}^{\infty} \frac{\mu(n)}{n^{1+z}}\right| \leq \sum_{N+1}^{\infty} \frac{1}{n^{1+x}}$$

$$\leq \int_N^{\infty} u^{-1-x}\,du = \frac{N^{-x}}{x} \quad ;$$

$$|N^z| = N^x \quad ;$$

$$\left|1+\frac{z^2}{R^2}\right| = \left|1+\frac{z}{\bar z}\right| = \frac{2|x|}{R} \quad ;$$

$$\left|\frac{1}{z}\right| = \frac{1}{R} \quad ;$$

$$|dz| = R\,d\theta \quad (z = Re^{i\theta}) .$$

Hence,

$$\left| \frac{1}{2\pi i} \int_{C_+} (g - g_N) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right|$$

$$\lessapprox \frac{1}{2\pi} \int_{C_+} \frac{N^{-x}}{x} N^x \frac{2x}{R} \frac{R\,d\theta}{R}$$

$$\approx \frac{1}{R} .$$

Exactly as on p. 13 of Lec 28, we have

$$\frac{1}{2\pi i} \int_{C_-} g_N(z) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int_{\left(\substack{\text{left half} \\ \text{of } |z| = R}\right)} [\cdots] \frac{dz}{z} .$$

Along $\{|z| = R,\ x < 0\}$, we have:

$$|g_N(z)| = \left| \sum_1^N \frac{\mu(n)}{n^{1+z}} \right|$$

$$\le \sum_1^N \frac{1}{n^{1+x}} = \sum_1^N n^{|x|-1}$$

$$\le \begin{Bmatrix} 1, & |x|<1 \\ \frac{N^{|x|}}{N}, & |x|\ge 1 \end{Bmatrix} + \int_1^N u^{|x|-1}\, du$$

$$\le \frac{N^{|x|}}{N} + \frac{N^{|x|}}{|x|} \quad ;$$

$$|N^z| = N^x = N^{-|x|} \quad ;$$

$$\left|1+\frac{z^2}{R^2}\right| = \left|1+\frac{z^2}{z\bar{z}}\right| = \frac{2|x|}{R} \quad ;$$

$$\left|\frac{1}{z}\right| = \frac{1}{R} \quad ;$$

$$|dz| = R\, d\theta \quad .$$

So,

$$\left| \frac{1}{2\pi i} \int_{C_-} g_N(z) N^z \left(1+\frac{z^2}{R^2}\right) \frac{dz}{z} \right|$$

$$\le \frac{1}{2\pi} \int_{(\text{semicircle})} N^{|x|} \left(\frac{1}{N} + \frac{1}{|x|}\right) N^{-|x|} \frac{2|x|}{R} \frac{R\, d\theta}{R}$$

$$= \frac{1}{\pi} \int_{(\text{semicircle})} \left(\frac{1}{N} + \frac{1}{|x|}\right) \frac{|x|}{R}\, d\theta$$

$$\lesssim \frac{1}{\pi} \int_{(\text{semicircle})} \left(\frac{1}{N} + \frac{1}{R}\right) d\theta$$

$$= \frac{1}{N} + \frac{1}{R} \,.$$

Finally, imitate Lec 28 pages 15 + 16. Remember that R and δ are held fixed. Get:

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{C_-} g(z) N^z \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} = 0 \,.$$

Conclude that:

$$\limsup_{N\to\infty} |g(0) - g_N(0)| \lesssim \frac{1}{R} + 0 + \underbrace{\frac{1}{R}} + 0 \,.$$

Since R is arbitrary, deduce that

$$\limsup_{N\to\infty} |g(0) - g_N(0)| = 0 \,.$$

IE,
$$\lim_{N\to\infty} g_N(0) = g(0) \,.$$

But $g(0) = 0$ since $\zeta(1+z)$ has a simple pole at $z = 0$. Hence,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$$

and Euler was right!!

Thanks to part I (i.e. ① - ⑭) of Lec 30, this proves the PNT!

"VERY NICE INDEED!"

NOTE: clearly, the same argument utilized above adapts to show that

$$\sum_{1}^{\infty} \mu(n) n^{-1-i\tau} = \frac{1}{\zeta(1+i\tau)}$$

for $\tau \neq 0$. (Just declare $a_n = \mu(n) n^{-i\tau}$.)