5/31/16

Addendum B

[regarding Lecture 28]

Newman's proof (in Lec 28) is clearly very slick. One seeks to understand the GENERAL THM on page (9) better — especially its genesis.

In regard to the genesis issue, one is inevitably forced to keep things a bit speculative.

With this in mind, it is helpful to back up and recall several very basic facts from Fourier analysis (on a fundamentally formal level).

Given appropriately decaying f on $[0,\infty)$. Let

$$\mathcal{L}(s) = \int_0^\infty e^{-sx} f(x)\,dx = \text{Laplace transform}.$$

To get the inversion formula, note that

$$\mathcal{L}(k+it) = \int_0^\infty [e^{-kx} f(x)]\, e^{-itx}\,dx$$

$$= \int_{-\infty}^\infty e^{-kx} \begin{bmatrix} f(x) \\ 0 \end{bmatrix} e^{-itx}\,dx$$

$$e^{-kx} \begin{bmatrix} f(x) \\ 0 \end{bmatrix} = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{L}(k+it)\, e^{itx}\,dt \qquad (x \neq 0)$$

$$\begin{bmatrix} f(x) \\ 0 \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{L}(k+it)e^{(k+it)x}\,dt$$

$$\begin{bmatrix} f(x) \\ 0 \end{bmatrix} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathcal{L}(s)e^{sx}\,ds$$

$$\boxed{f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathcal{L}(s)e^{sx}\,ds, \quad x>0}\,.$$

This is the familiar formula; typically $k>0$. For suitable f, one can reason just as well with $k=0$. In both cases, the s-integral is technically a principal value with $[-R, R]$ and $R\to\infty$. See Lec 25, p. 5 middle.

or continuous + piecewise C^1

Let $h(x)$ be C^1 + appropriately decaying. Note that

$$\int_0^\infty h'(x)e^{-sx}\,dx = \int_0^\infty e^{-sx}\,dh(x)$$

$$= 0 - h(0) + s\int_0^\infty h(x)e^{-sx}\,dx$$

$$\Downarrow$$

$$\boxed{\mathcal{L}_h(s) = \frac{h(0)+\mathcal{L}_{h'}(s)}{s}}\,.$$

Assume now that our original f satisfies

$$(*) \qquad \int_0^\infty f(u)\,du = 0 .$$

Putting

$$h(x) = \int_0^x f(u)\,du$$

and observing that

$$h(x) - \int_\infty^x f(u)\,du = \int_0^x + \int_x^\infty = 0 ,$$

we deduce that

$$h(0) = 0 , \quad h'(x) = f(x) \quad [a.e.]$$

$$\mathcal{L}_h(s) = \frac{1}{s}\mathcal{L}_f(s)$$

$$h(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathcal{L}_h(s)\, e^{sx}\, ds$$

$$h(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{1}{s}\mathcal{L}_f(s)\, e^{sx}\, ds$$

$$\int_0^x f(u)\,du = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{1}{s}\mathcal{L}_f(s)\, e^{sx}\, ds$$

(the RHS being a $[-R, R]$ principal value) .

For suitable f, we can reason just as well with $k=0$, thus getting

$$\int_0^x f(u)\,du = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\mathcal{L}(s)}{s} e^{sx}\,ds$$

under (*).

In alternate notation,

$$\int_0^T f(u)\,du = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\mathcal{L}(s)}{s} e^{sT}\,ds$$

$$\int_0^T f(u)\,du = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\mathcal{L}(z)}{z} e^{zT}\,dz$$

$$\Downarrow$$

$$\int_0^T f(u)\,du = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{-iR}^{iR} \frac{\mathcal{L}(z)}{z} e^{zT}\,dz$$

$$\boxed{\int_0^T f(u)\,du = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\mathcal{L}(z)}{z} e^{Tz}\,\omega\!\left(\frac{y}{R}\right)dz \quad \text{wherein } \omega(u) = \chi_{[-1,1]}(u),\ u\in\mathbb{R}}$$

The foregoing format [in the box] is highly ~~suggestive~~ given that it is a time-honored trick in Fourier transform theory to replace the earlier ⟦even⟧ ω with other "more interesting" choices.

The case of Fejér summability corresponds, for instance, to taking $\omega(y) = \max\{0, 1-\lvert y\rvert\}$; see Lec 25 p. 9.

The essential point in these things is that keeping

$$\omega(0) = 1 \qquad \text{and} \qquad |\omega| = O(1)$$

guarantees that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} F(y)\,\omega\left(\frac{y}{R}\right) dy = \int_{-\infty}^{\infty} F(y)\,dy$$

for every $F \in L_1(\mathbb{R})$. Indeed, let $|\omega| \leq M$ and $|\omega(u) - 1| < \varepsilon$ for $|u| < \delta$. Then:

$$\left| \int_{-\infty}^{\infty} F(y)\,dy - \int_{-\infty}^{\infty} F(y)\,\omega\left(\frac{y}{R}\right) dy \right|$$

$$\leqq \int_{|y| \geqq R\delta} |F(y)|\,(1+\mathfrak{M})\,dy$$

$$+ \int_{|y| < R\delta} |F(y)|\,\varepsilon\,dy$$

$$\leqq (1+\mathfrak{M}) \int_{|y| \geqq R\delta} |F(y)|\,dy + \varepsilon \int_{-\infty}^{\infty} |F(y)|\,dy .$$

Inspired by the Fejér case, it is more-or-less <u>mandatory</u> to observe that [for any sensible ω] we have:

$$\int_{-\infty}^{\infty} f_1(y)\,\overline{f_2(y)}\,dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_1(u)\,\overline{\tilde{f}_2(u)}\,du$$

{see Lec 25 pp. 5 8}

$$\int_{-\infty}^{\infty} L(y)\,\omega\left(\frac{y}{R}\right) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{L}(u)\,R\,\tilde{\omega}(-Ru)\,du$$

{but ω is even}

$$\int_{-\infty}^{\infty} L(y)\,\omega\left(\frac{y}{R}\right) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{L}(u)\,R\,\tilde{\omega}(Ru)\,du$$

$$\boxed{\int_{-\infty}^{\infty} L(y)\,\omega\!\left(\frac{y}{R}\right)dy \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{L}\!\left(\frac{v}{R}\right)\tilde{\omega}(v)\,dv},$$

whereupon the expression $\int_{\mathbb{R}} L(y)\,\omega\!\left(\frac{y}{R}\right)dy$ {with given $L \in L_1(\mathbb{R})$} is <u>again</u> seen to converge (as $R \longrightarrow \infty$) to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{L}(0)\,\tilde{\omega}(v)\,dv = \tilde{L}(0)\,\omega(0) = \int_{-\infty}^{\infty} L(y)\,dy\,.$$

We've switched (here) to L (in lieu of F) to be generally suggestive of B4 box. Issues with 2π are ignored.

think $L \leftrightarrow \frac{\mathcal{Z}(z)}{z}e^{Tz}$

For $\max\{0, 1-|y|\}$, one knows that $\tilde{\omega}(v) = \left(\frac{\sin\frac{v}{2}}{\frac{v}{2}}\right)^2$ by Lec 25 9 line 4.

The Fourier transform of $\max\{0, 1-y^2\}$ is slightly more complicated; viz.,

$$\frac{4}{v^2}\left(\frac{\sin v}{v} - \cos v\right) \equiv \frac{4}{v^3}\int_0^v (\xi \sin \xi)\,d\xi\,.$$

We stress that ALL of the foregoing is just rudimentary / completely classical Fourier transform theory! B8

In the Landau paper from 1932 attached to Lec 28, it is nearly self-evident that a Fejér type $w(u)$ is lurking in the background; cf. B7 (box), B5 (lines 1+2), and Landau 526 lines 6+11.

Being aware of this, and guided by a concomitant desire to exploit the Cauchy integral theorem in the counterpart of B4 (box), it stands to reason that an ω which is analytic prior to "turning off" is best.

One can guess that this thought motivated Newman's choice of $\omega(u) = \max\{0, 1-u^2\}$.

BOTTOM LINE: the fcn $e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}$ is thus completely natural in B4 (box).

It is worthwhile at this juncture to quickly record the counterpart of all this for

$$\mathfrak{M}(s) = \int_1^\infty x^{-s}\, dA(x),$$

where $A(1) = 0$ and, for instance, $A(x) = \int_1^x \phi(v)\,dv$.

One gets:

$$\mathfrak{M}(s) = s\int_1^\infty \frac{A(x)}{x^{s+1}}\,dx \qquad (\mathrm{re}(s) > 1)$$

$$\frac{\mathfrak{M}(s)}{s} = \int_0^\infty \frac{A(e^u)}{e^{us}}\,du$$

$$\frac{\mathfrak{M}(k+it)}{k+it} = \int_0^\infty e^{-ku} A(e^u)\, e^{-itu}\,du$$

$$\frac{\mathfrak{M}(k+it)}{k+it} = \int_{-\infty}^\infty e^{-ku}\begin{bmatrix} A(e^u) \\ 0 \end{bmatrix} e^{-itu}\,du$$

$$e^{-ku}\begin{bmatrix} A(e^u) \\ 0 \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\mathfrak{M}(k+it)}{k+it}\, e^{itu}\,dt$$

$\{ [-R, R]$ principal value $\}$

$$\begin{bmatrix} A(e^u) \\ 0 \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\mathfrak{M}(k+it)}{k+it}\, e^{(k+it)u}\,dt$$

$$\begin{bmatrix} A(e^u) \\ 0 \end{bmatrix} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\mathfrak{M}(s)}{s}\, e^{su}\,ds$$

$$A(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\mathfrak{M}(s)}{s} x^s\, ds\,, \quad x > 1$$

(the RHS being a $[-R, R]$ principal value)

This is a very familiar formula (typically having $k > 1$). Indeed: recall PERRON'S FORMULA in Lec 19, pp. 12 (bot) ~ 13 (top). (Also Lec 17, 10 top.)

Again, under certain hypotheses, it will be possible to proceed with $k = 1$. The situation is clearly analogous to B4 (box). One is thus led to the expression

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{\mathfrak{M}(z)}{z} \gamma^z\, \omega\left(\frac{y}{R}\right) dz\,, \quad \gamma > 1\,,$$

with the possible need therein to have "engineered matters" so as to have a removable singularity at, say, $z = 1$. Compare: Landau 526 line 11.

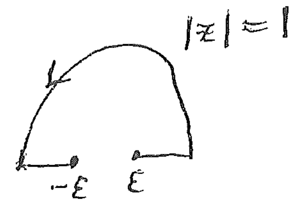
During Lec 28, I mentioned that Newman's General Thm could be strengthened rather easily. This was already indicated on pp. 17 ~ 19 (top) of Lec 28.

To bring matters into a still better form, two elementary lemmas are required.

Let $D = \{|z| < 1,\ y > 0\}$, $K = \{|z| \leq 1,\ y \geq 0\}$.

Let $\Gamma = \partial D$ (counterclockwise) and

$$\Gamma_\varepsilon = \Gamma \sim (-\varepsilon, \varepsilon).$$



[Here $0 < \varepsilon < 1$.]

Let $\operatorname{Log} w = \ln|w| + i\operatorname{Arg}(w)$, with $-\pi < \operatorname{Arg}(w) \leq \pi$.

LEMMA 1

Let $F(z)$ be continuous on K and analytic on D. We then have

$$\frac{1}{2}F(0) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{F(z)}{z}\,dz.$$

P.f

Notice that

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{F(0)}{z}\, dz$$

$$= \lim_{\varepsilon \to 0} \frac{F(0)}{2\pi i} \left[\operatorname{Log}(-\varepsilon) - \operatorname{Log}(\varepsilon) \right]$$

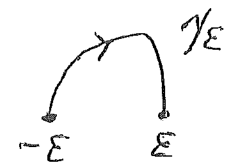
$$= \lim_{\varepsilon \to 0} \frac{F(0)}{2\pi i} [\pi i] = \frac{F(0)}{2} .$$

Our task is to check that

$$0 = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{F(z) - F(0)}{z}\, dz .$$

As such, we might as well simply hypothesize $F(0) = 0$ at the outset. We do so. By the extended Cauchy integral theorem,

$$\int_{\Gamma_\varepsilon} \frac{F(z)}{z}\, dz + \int_{\gamma_\varepsilon} \frac{F(z)}{z}\, dz = 0 .$$



But,

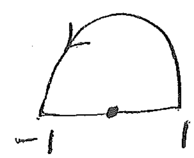
$$\left| \int_{\gamma_\varepsilon} \frac{F(z)}{z}\, dz \right| \lessapprox \max_{\gamma_\varepsilon} |F| \cdot \int_{\gamma_\varepsilon} \frac{|dz|}{\varepsilon} = \pi \max_{\gamma_\varepsilon} |F| .$$

Hence,

$$\left| \int_{\Gamma'_\varepsilon} \frac{F(z)}{z}\,dz \right| \lesssim \pi \max_{\gamma_\varepsilon} |F| .$$

Since $F(0) = 0$, we have $\max_{\gamma_\varepsilon} |F| \to 0$ as $\varepsilon \to 0$.

∎



LEMMA 2

Let $F(z)$ be continuous on K and analytic on D. Assume that $F(0) = 0$ and that

$$\int_{-1}^{1} \left| \frac{F(x)}{x} \right| dx < \infty .$$

We then have

$$0 = \frac{1}{2\pi i} \int_{\Gamma'} \frac{F(z)}{z}\,dz$$

in a [natural] Lebesgue integral sense.

Pf

That $\frac{1}{2\pi i} \int_{\Gamma'} \frac{F(z)}{z}\,dz$ exists as a Lebesgue integral is obvious. But, then, by a standard specialization,

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{F(z)}{z}\,dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{n}}} \frac{F(z)}{z}\,dz = 0,$$

thanks to Lemma 1. ∎

IMPROVED THEOREM.

Newman's General Theorem on p. 9 of Lec 28 is actually valid anytime the function $g(s)$ on $\{Re(s)>0\}$ admits a continuous extension to $\{Re(s)\geqq 0\}$ having the additional property that

$$\int_{-1}^{1}\left|\frac{g(it)-g(0)}{t}\right|dt < \infty.$$

Pf

We first claim that one can take $g(0)=0$, wlog. Suppose, e.g., that $v=1$ is a point of continuity of f. Let $A=g(0)\neq 0$ and define

$$f_1(v) = \begin{cases} f(v)-A, & 0\leqq v<1 \\ f(v), & v\geqq 1 \end{cases}.$$

The fcn f_1 is still bounded + piecewise continuous on $[0,\infty)$. We get: {for $re(s)>0$ initially}

$$g_1(s) = g(s) - A\int_0^1 e^{-sv}\,dv$$
$$= g(s) - A\,E(s)\,, \quad E(s) \equiv \frac{1-e^{-s}}{s}\,.$$

The fcn E is entire and equals $1+O(s)$ near $s=0$. Clearly, $g_1(0)=0$ and g_1 is continuous for all $\{\operatorname{Re}(s) \geqq 0\}$. For $|t| \leqq 1$, notice that

$$|g(it) - g(0) - g_1(it)| = |A|\,|E(it)-1|$$
$$= O(|t|)\,.$$

Hence,

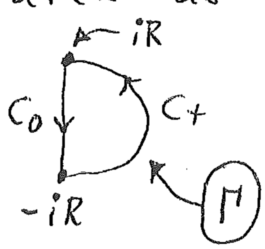
$$\int_{-1}^{1}\left|\frac{g_1(it)}{t}\right| dt < \infty \iff \int_{-1}^{1}\left|\frac{g(it)-g(0)}{t}\right| dt < \infty\,.$$

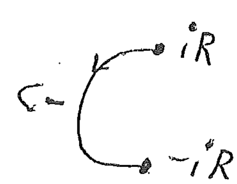
Switching to $\{g_1, f_1\}$ and establishing $0 = \int_0^\infty f_1(v)\,dv$ will thus lead to $0 = -A + \int_0^\infty f(v)\,dv$, which is exactly what we want.

From this point on, we assume $g(0)=0$.

Fix any large R. Let Γ be the (counterclockwise) path $\{|z|=R,\ x>0\} \cup \{z=iy,\ -R \leqq y \leqq R\}$. Introduce arcs as shown:

$$\boxed{\Gamma = C_+ \cup C_0}$$





Keep T big. Write

$$g_T(z) = \int_0^T e^{-zv} f(v)\,dv$$

as usual. By a trivial variant of Lemma 2, we have:

$$(**) \qquad 0 = \oint_{\Gamma} \frac{g(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)}{z}\,dz \,.$$

(as a Lebesgue integral)

The fcn $g_T(z)$ is entire; by the Cauchy integral formula, we know:

$$g_T(0) = \frac{1}{2\pi i}\int_{C_+} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$+ \frac{1}{2\pi i}\int_{C_-} g_T(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} \,.$$

We propose to subtract (**) from this equation for $g_T(0)$.

Get:

$$g_T(0) = \frac{1}{2\pi i}\int_{C_+} [g_T(z) - g(z)]\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$+ \frac{1}{2\pi i}\int_{-iR}^{iR} \frac{g(z)}{z}\, e^{Tz}\left(1+\frac{z^2}{R^2}\right) dz$$

this arc is $-C_0$

$$+ \frac{1}{2\pi i}\int_{C_-} g_T(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$\equiv J_1 + J_2 + J_3 \,, \quad \text{say.}$$

The estimations for $|J_1|$ and $|J_3|$ go exactly like before.

$|J_1| \leqq \frac{B}{R}$ (11) bot ~ (12) in Lec 28

$|J_3| \leqq \frac{B}{R}$ (13) (middle) ~ (14) in Lec 28

To estimate $|J_2|$, we write

$$|J_2| = \frac{1}{2\pi} \left| \int_{-R}^{R} \frac{g(iy)}{y} e^{iyT} \left(1 - \frac{y^2}{R^2}\right) dy \right| ,$$

note the Lebesgue integrability of $g(iy)/y$, and then apply the standard Riemann-Lebesgue lemma. Get: see Lec 7 p. 22

$$\lim_{T \to \infty} |J_2| = 0 , \quad \text{each } R.$$

IE $J_2 = o(1)$, akin to Lec 28 p. 19 (line 2).

It follows that:

$$\limsup_{T \to \infty} |g_T(0)|$$
$$= \limsup_{T \to \infty} |J_1 + J_2 + J_3|$$
$$\leq \frac{B}{R} + 0 + \frac{B}{R} = \frac{2B}{R}$$

akin to Lec 28 p. 19 (line −3). Since R is arbitrary, $\lim_{T \to \infty} g_T(0) = 0 = g(0)$ and we are done. ▥

After completing this (last) proof, it pays to step back and note how the correctness of B4 (box) for a large class of functions f, together with B5 (top) + B8 (lines 4 ~ 8), clearly engender a kind of "moral encouragement" that a limit theorem like Newman's General Thm [or B14] might well prove feasible on a relatively simple technical level.

The miraculous cancellations that appeared "along the way" are perhaps best viewed in this light.