Integration in Multivariable Calculus

1. Integrals over 1-dimensional objects.
   - "Calculus I" Integrals: These are integrals of a function \( f(x) \) over a 1-dimensional interval \([a, b]\) in \( \mathbb{R}^1 \). Geometrically, \( \int_a^b f(x) \, dx \) computes the (signed) area between the graph of the function \( f(x) \) and the interval \([a, b]\).
   - Line Integrals: These are integrals over a curve \( C \). If \( C \) is a curve in \( \mathbb{R}^2 \), we will suppose it has a parametrization of the form \( r(t) = \langle x(t), y(t) \rangle \), where \( a \leq t \leq b \); if \( C \) is a curve in \( \mathbb{R}^3 \), we will suppose that \( r(t) = \langle x(t), y(t), z(t) \rangle \), where \( a \leq t \leq b \).
     - For a function:
       - With respect to arc length: \( \int_C f \, ds = \int_a^b f(r(t)) \left| r'(t) \right| \, dt \)
       - With respect to \( x \) (or \( y \) or \( z \)): \( \int_C f \, dx = \int_a^b f(r(t)) \frac{dx}{dt} \, dt \) or \( \int_a^b f(r(t)) \frac{dy}{dt} \, dt \) or \( \int_a^b f(r(t)) \frac{dz}{dt} \, dt \)
     - For a vector field: \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(r(t)) \cdot r'(t) \, dt \)
   * Special cases:
     - \( \int_a^b 1 \, dx \) equals the length of \([a, b]\)
     - \( \int_C 1 \, ds \) equals the length of \( C \)

2. Integrals over 2-dimensional objects.
   - Double Integrals: These are integrals of a function \( f(x, y) \) over a 2-dimensional region \( D \) in \( \mathbb{R}^2 \). Geometrically, \( \int_D \int f(x, y) \, dA \) computes the (signed) volume between the graph of the function \( f(x, y) \) and the region \( D \).
   - Surface Integrals: These are integrals over a surface \( S \) in \( \mathbb{R}^3 \). We will suppose it has a parametrization of the form \( r(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \) where \( u \) and \( v \) lie in some 2-dimensional parameter domain \( D \).
     - For a function: \( \iint_S f \, dS = \iint_D f(r(u, v)) \left| r_u \times r_v \right| \, dA \)
     - For a vector field: \( \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(r(u, v)) \cdot (r_u \times r_v) \, dA \)
   * Special cases:
     - \( \iint_D 1 \, dA \) equals the area of \( D \)
     - \( \iint_S 1 \, dS \) equals the (surface) area of \( S \)

3. Integrals over 3-dimensional objects.
   - Triple Integrals: These are integrals of a function \( f(x, y, z) \) over a 3-dimensional solid region \( E \) in \( \mathbb{R}^3 \). Geometrically, \( \iiint_E f(x, y, z) \, dV \) computes the (signed) 4-dimensional volume between the graph of the function \( f(x, y, z) \) and the region \( E \).
   * Special case: \( \iiint_E 1 \, dV \) equals the volume of \( E \).
* Relevant Theorems:

- **The Fundamental Theorem of Line Integrals:** If \( \mathbf{F} \) is a conservative vector field (so that \( \mathbf{F} = \nabla f \) for some function \( f \)) and if \( C \) is a smooth curve with parametrization \( \mathbf{r}(t) \) \((a \leq t \leq b)\), then
  \[
  \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))
  \]
  This specializes to **The Fundamental Theorem of Calculus:** If \( g \) is an “antidifferentiable” function (so that \( g = f' \) for some function \( f \)) and if \([a, b]\) is some interval, then
  \[
  \int_a^b g(x) \, dx = \int_a^b \frac{d}{dx} f(x) \, dx = f(b) - f(a)
  \]

- **Stokes’ Theorem:** If \( \mathbf{F}(x, y, z) \) is a vector field (with all three component functions differentiable) and if \( C \) is a simple, piecewise-smooth, positively-oriented, and closed curve in \( \mathbb{R}^3 \) that forms the boundary of an oriented smooth surface \( S \), then
  \[
  \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}
  \]
  This specializes to **Green’s Theorem:** If \( \mathbf{F}(x, y) \) is a vector field (with both component functions \( P(x, y) \) and \( Q(x, y) \) differentiable) and if \( C \) is a simple, piecewise-smooth, positively-oriented, and closed curve in \( \mathbb{R}^2 \) that forms the boundary of a region \( D \), then
  \[
  \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_C \mathbf{F} \cdot d\mathbf{r}
  \]

- **The Divergence Theorem:** If \( \mathbf{F}(x, y, z) \) is a vector field (with all three component functions differentiable) and if \( S \) is a smooth, positively-oriented, closed surface in \( \mathbb{R}^3 \) that forms the boundary of a solid region \( E \), then
  \[
  \iiint_E \text{div}(\mathbf{F}) \, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}
  \]

Note: In each case, these say that an integral of some type of “derivative” over a region equals an integral over the boundary of that region:

- Fundamental Theorem of Line Integrals
  “Derivative”: Gradient
  Region: Curve (with boundary a pair of points: the start point and end point of the curve)

- Stokes’ Theorem
  “Derivative”: Curl
  Region: Surface (with boundary a closed curve)

- Divergence Theorem
  “Derivative”: Divergence
  Region: Solid 3-dimensional region (with boundary a closed surface)