Math 2263 Quiz 8 Solutions

1. (3 points) Evaluate $\int_C y \, ds$, where $C$ is the segment of the curve $x = 1 - t$, $y = 1 + 2t$ between the points $(1, 1)$ and $(-1, 5)$.

Solution: The point $(1, 1)$ on the curve $C$ corresponds to $t = 0$ and the point $(-1, 5)$ corresponds to $t = 2$. The bounds on $t$ are therefore from 0 to 2. Rewriting the integral $\int_C y \, ds$ in terms of $t$ gives

$$\int_C y \, ds = \int_0^2 y(t) \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt$$

$$= \int_0^2 (1 + 2t) \sqrt{(-1)^2 + 2^2} \, dt$$

$$= \left[ \sqrt{5}(t + t^2) \right]_0^2$$

$$= 6\sqrt{5}$$

Rubric:

(+1) for correct bounds on $t$
(+1) for correct integrand
(+1) for $2\sqrt{5}$

2. Consider the vector field $\mathbf{F}(x, y) = 3x^2 + 2xy, \ x^2 - 1$.

(a) (4 points) Is $\mathbf{F}$ conservative? If so, find a function $f(x, y)$ such that $\nabla f(x, y) = \mathbf{F}(x, y)$.

Solution: Because

$$\frac{\partial}{\partial y}(3x^2 + 2xy) = 2x = \frac{\partial}{\partial x}(x^2 - 1),$$

the vector field $\mathbf{F}$ is conservative.

If $f(x, y)$ is a function such that $\nabla f(x, y) = \mathbf{F}(x, y)$, then we need

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1$$

Integrating the first equation with respect to $x$ implies that $f(x, y) = x^3 + x^2y + g(y)$ for some function $g$. The derivative of this with respect to $y$ is $x^2 + g'(y)$, and comparing this to the second equation we see that we need $g'(y) = -1$. Therefore $g(y) = -y + C$ for some constant $C$. We can just take $C = 0$, so that $f(x, y) = x^3 + x^2y - y$.

Rubric:

(+1) for showing that $\mathbf{F}$ is conservative
(+1) for $\frac{\partial f}{\partial x} = 3x^2 + 2xy$ and $\frac{\partial f}{\partial y} = x^2 - 1$
(+1) for integrating with respect to $x$ to get $f(x, y) = x^3 + x^2y + g(y)$
(+1) for $f(x, y) = x^3 + x^2y - y$
(b) (3 points) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is the curve $\mathbf{r}(t) = \langle t - 1, 2t^3 \rangle$, $0 \leq t \leq 1$.

**Solution:** By part (a), we know that $\mathbf{F}(x, y) = \nabla f(x, y)$, where $f(x, y) = x^3 + x^2 y - y$. The Fundamental Theorem for Line Integrals then implies that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0))$$

$$= f(0, 2) - f(-1, 0)$$

$$= -2 - (-1)$$

$$= -1$$

**Rubric:**

(+2) for using the Fundamental Theorem for Line Integrals

(+1) for $-1$