SIAM J. MATH. ANAL.
Vol. 13, No. 3, May 1982

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0036-1410/82/1303-0002 $01.00/0

LONG-TIME BEHAVIOR OF A CLASS OF BIOLOGICAL MODELS*

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Dedicated to my friend and collaborator Michael Golomb

Abstract. It is shown that many of the asymptotic properties of the Fisher model for population genetics and population ecology can also be derived for a class of models in which time is discrete and space may or may not be discrete. This allows one to discuss the behavior of models in which the data consist of occasional counts on survey tracts, as well as that of computer models.

1. Introduction. The purpose of this work is to present some techniques by which results on long-term behavior previously obtained for continuous models in population genetics and population ecology can be shown to apply to a larger class of models in which the time and space variables are allowed to be discrete. Such an extension is useful for several reasons. In the first place it usually is impractical to measure more than finitely many (and often relatively few) aggregate quantities. For instance, one can measure the populations of various species or genotypes in a few census tracts from time to time, but one cannot measure (or possibly even define) the population densities at all points at all times.

Secondly, any deterministic model must rely on the assumption that the inevitable random events found in nature are averaged out. This is much more likely to be true of aggregates over relatively large space-time domains than of limits such as population densities in a continuous space.

Lastly, computations for continuous models are usually done by approximating them with discrete ones. On the other hand, continuous models often appear as approximations to discrete ones. (See, e.g., [2], [38], [39].) It is useful to know that all these models have the same qualitative large-time behavior.

We shall deal only with models which are deterministic, and in which the independent variable is a scalar. Both of these restrictions are serious ones. We shall briefly discuss the possibility of applying our results to stochastic models at the end of § 4.

The fact that the independent variable must be scalar-valued prevents the consideration of any very sophisticated interactions with the environment, other species, or other age classes. In particular, we consider species with nonoverlapping generations, and this naturally introduces a discrete time interval, the generation time.

The most frequently used model for studying the spread of a mutant or a new population in a homogeneous environment is the Fisher equation

$$\frac{\partial u}{\partial t} = D\Delta u + f(u). \tag{1.1}$$

This equation in one dimension was introduced by R. A. Fisher [15] as a model for the spread of an advantageous form (allele) of a single gene in a population of diploid individuals. It is assumed that there are only two allelic forms A and a of the gene under study. The function $u(t, x)$ represents the gene fraction, that is, the ratio of the number of A alleles to the number of A and a alleles at the time t in the population near the point x.

Fisher found that when $f(u) = u(1-u)$ there is a constant c^* with the property that, when $|c| \geqq c^*$, (1.1) has a traveling wave solution $u(t, x) = W(x - ct)$ with speed c, while

* Received by the editors November 12, 1980, and in revised form February 24, 1981.

† School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455. This research was partly supported by the National Science Foundation under grant MCS-7812182. Part of the work was done while the author was a visitor at the University of Queensland.

for $|c| < c^*$ there is no such wave. He conjectured that the minimal wave speed c^* is also the speed at which a newly arrived advantageous allele spreads into an established population.

Kolmogoroff, Petrowsky, and Piscounoff [31] proved this conjecture in the one-dimensional case where $u(0, x)$ is 0 for $x < 0$ and 1 for $x > 0$ provided $f(0) = f(1) = 0$ and f is concave.

The Fisher equation (1.1) is also used as a model for the spread of a population when u is interpreted as a population density.

In this work we shall consider discrete time models of the form

$$u_{n+1} = Q[u_n], \tag{1.2}$$

where $u_n(x)$ represents the gene fraction or population density at time n at the point x of the habitat and Q is an operator on a certain set of functions on the habitat. The habitat may be one-, two-, or three-dimensional. It may be discrete, in which case x ranges over a discrete set of niches (census tracts).

The model (1.2) is deterministic in the sense that u_{n+1} is uniquely determined by u_n. We continue to assume that u_n is scalar-valued and that the habitat is homogeneous. We shall confine our attention to propagation phenomena and shall not be concerned with clines [8], [14], [16], [17], [21], [38], [46].

We shall show that a few biologically reasonable hypotheses about Q permit us to derive results of the Fisher and Kolmogoroff–Petrowsky–Piscounoff types for this class of models. More specifically, we shall show that for each unit vector ξ there is a wave speed $c^*(\xi)$ with the property that, in an asymptotic sense, a new mutant or population which is initially confined to a bounded set spreads like the solution of a wave equation whose plane wave solutions with normals ξ have the speeds $c^*(\xi)$. Under some additional conditions we also show that (1.2) has a nonincreasing traveling wave solution $u_n(x) = W(x \cdot \xi - nc)$ when $c \geqq c^*(\xi)$ and not when $c < c^*(\xi)$.

Equation (1.1) implies (1.2) when $Q[v]$ is defined to be the solution at some generation time τ of the initial value problem for (1.1) with initial values $v(x)$. Therefore our results generalize the one-dimensional results of Fisher [15], Kolmogoroff, Petrowsky, and Piscounoff [31], Kanel' [25], [26], Kametaka [24], and Aronson and Weinberger [4], as well as the multidimensional results of Aronson and Weinberger [5] for the rotationally symmetric model (1.1).

Translational invariance (homogeneity) does not imply rotational invariance (isotropy). Because prevailing winds and other phenomena can produce preferred directions, the analogue of (1.1) without rotational symmetry is also of interest. In this case the operator $D\Delta$ in (1.1) is replaced by an elliptic operator of the form

$$\sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i \frac{\partial}{\partial x_i}.$$

This problem can again be put into the form (1.2) by defining $Q[v]$ to be the solution at time τ of the initial value problem with initial values $v(x)$. We can, in fact allow for seasonal variation by permitting the coefficients and f to be periodic functions of t of period τ.

Since the migrational behavior can be affected by the genotype [41] or by the population density [2], [3], [19], it is also of interest to consider the problem in which the coefficients a_{ij} and b_i in the above operator depend upon u. This problem can again be converted to the form (1.2) and treated by our methods.

Our results also generalize work by Diekmann [10], [11], Diekmann and Kaper [12], Thieme [48], [49] and the author [53], [54] on one-dimensional and rotationally symmetric discrete time models of the form (1.2) where Q is a nonlinear integral operator.

We have not been able to generalize the more delicate results on convergence in shape obtained by Kolmogoroff, Petrowsky, and Piscounoff [31], Kanel' [25], [26], Fife and McLeod [13], Bramson [7], and Uchiyama [50], [51], which show that for the one-dimensional Fisher equation the width of the transition layer between the high and low parts of a spreading pulse remains bounded and which approximate the space-time path of this layer.

The results presented here can also be applied to certain models for the spread of epidemics [1], [10], [11], [28], [29], [37], [47], [56].

Some examples of biological models are given in § 2. A precise statement of the hypotheses on the recursion operator Q is in § 3. Section 4 introduces our mathematical tools. In § 5 we show that the rather small set (3.1) of hypotheses leads to a wave speed $c^*(\xi)$ for disturbances which depend only on the single variable $x \cdot \xi$, where ξ is any unit vector.

We state our results in § 6. Essentially these are that, in an asymptotic sense, an initially confined disturbance is propagated along the ray cone which corresponds to the wave speed $c^*(\xi)$, and that there are plane wave solutions of the form $W(x \cdot \xi - nc)$ exactly when $c \geqq c^*(\xi)$. The remainder of the paper is devoted to proofs of these theorems, together with some examples and counterexamples.

Our results show that a rather large class which contains both continuous and discrete models is robust with respect to qualitative asymptotic behavior. This robustness means, of course, that a very crude model can give correct predictions about the asymptotic behavior of an ecological system. The other side of the coin of robustness, of course, is that a model which is found to predict such behavior correctly may be far from being a good model for predicting other phenomena.

2. Some models in population genetics and population growth. To obtain an example of a recursion of the form (1.2), we consider the so-called stepping stone model [27], [30], [35], [40], [45] in population genetics. We classify the individuals of a certain diploid population according to their genotypes with respect to a single gene locus, which occurs in two allelic (that is, variant) forms, which we label A and a. There are then three genotypes: the homozygotes AA and aa, and the heterozygote Aa.

The habitat is either naturally or arbitrarily divided into discrete regions or niches. The population which lives in any one niche during the nonmigratory part of a life cycle is called a deme. We suppose that the generations are nonoverlapping, and that members of different demes do not mix or interact except during a brief period of migration at the end of the life cycle. Migration moves the individuals to new niches, so that it changes the memberships of the various demes. Just after migration the individuals form gametes and die. The individuals are assumed to be monoecious (that is, hermaphroditic). The gametes mate at random to form the new generation of diploids.

The ratio of the number of alleles of type A to the total number of alleles of types A and a in a certain deme is called its *gene fraction*. Let $u_n(i)$ denote the gene fraction in the newly born individuals of the nth generation in the ith deme.

The Hardy–Weinberg law states that the ratios of the numbers of newly born individuals of genotypes AA, aA, and aa, respectively, are $u_n(i)^2 : 2u_n(i) \times (1-u_n(i)) : (1-u_n(i))^2$. These individuals undergo various hazards during the growth

cycle, and we assume that their abilities to survive these hazards depend only on their genotypes with respect to the gene under consideration. If the fitnesses (that is, survival rates to the migration stage) of the three genotypes have the ratios $1+s_i:1:1+\sigma_i$, then the ratios of the numbers of survivors at the time of migration are $(1+\sigma_i)u_n(i)^2:2u_n(i)(1-u_n(i)):(1+s_i)(1-u_n(i))^2$. We suppose that the total number of individuals in the ith deme who survive to migrate is a fixed carrying capacity p_i, which does not depend upon the genotypic makeup of the population. We further suppose that a fraction l_{ji} of the population of each genotype in the ith deme migrates to become part of deme j. (Of course the fraction l_{ii} remains in the ith deme.) Under our assumptions the gene fraction $u_{n+1}(j)$ born in the jth deme in the next generation is equal to the gene fraction in the jth deme after migration. It is given by the formula

$$u_{n+1}(j)=\sum_i m_{ji}g_i(u_n(i)), \tag{2.1}$$

where

$$g_i(u)=\frac{2(1+s_i)u^2+2u(1-u)}{2[(1+s_i)u^2+2u(1-u)+(1+\sigma_i)(1-u)^2]} \tag{2.2}$$

is the gene fraction of the ith deme just before the migration and

$$m_{ji}=\frac{l_{ji}p_i}{\sum_k l_{jk}p_k} \tag{2.3}$$

is the fraction of those individuals who are in the jth deme after the migration who came from the ith niche.

Thus the function $\{u_n(i): i=1, 2, \cdots\}$ satisfies a recursion of the form (1.2) with

$$Q[u](j)=\sum_i m_{ji}g_i(u(i)). \tag{2.4}$$

In the present work we will only deal with a homogeneous habitat. By this we mean that all the niches are identical, that they are obtained by applying the elements of a group of translations to any one niche, and that the migration function l_{ij} depends only on the translation which takes the ith niche into the jth niche.

Suppose, for instance, that the organisms are living in a plane $\mathscr{R}^2$. This plane is divided into the squares $\{(x, y)|(k-\frac{1}{2})h \leqq x<(k+\frac{1}{2})h,\ (l-\frac{1}{2})h \leqq y<(l+\frac{1}{2})h;\ k, l=0, \pm1, \pm2, \cdots\}$ of side h, which are the niches. We label each niche by the coordinates of its center, which are integral multiples of h. Then we can think of the habitat as the set of points of $\mathscr{R}^2$ whose coordinates are integral multiples of h. We shall usually write these centers as vectors x. $\mathscr{H}$ can be obtained from any one niche by applying all translations by two-vectors whose components are multiples of h.

The fact that $\mathscr{H}$ is homogeneous means that the fitnesses s and σ and the adult population p are the same for all the niches, and the migration function l_{ij} depends only on the vector difference x_i-x_j between the centers of the niches. For the moment we also assume that s, σ, and p do not depend on u, so that they are constants. Since

$$\sum_k l_{jk}=\sum_k l(x_j-x_k)=\sum_i l(x_i)=\sum_i l_{io}=1$$

because all the individuals in niche x_o must go somewhere, (2.3) becomes

$$m_{ij}=l_{ij}\equiv m(x_i-x_j).$$

Thus the operator Q in the recursion (1.2) is

$$Q[u](x) = \sum_{y \in \mathcal{H}} m(x-y)g(u(y)), \tag{2.5}$$

where $\sum_{x \in \mathcal{H}} m(x) = 1$ and

$$g(u) = \frac{su^2 + u}{1 + su^2 + \sigma(1-u)^2}. \tag{2.6}$$

We see that

$$g(u) - u = \frac{u(1-u)[su - \sigma(1-u)]}{1 + su^2 + \sigma(1-u)^2}. \tag{2.7}$$

The definition of u shows that we must only consider functions $u(x)$ such that $0 \leqq u \leqq 1$. It is easily seen that g increases from 0 to 1 as u increases from 0 to 1. Therefore $Q[u]$ again has values on the interval [0, 1].

We see from (2.7) that there are three possible behaviors of the function g:

(i) If $s > 0 > \sigma$, the AA homozygotes are the most fit and the aa homozygotes are the least fit, and

$$g(u) > u \quad \text{for } 0 < u < 1. \tag{2.8}$$

This is called the *heterozygote intermediate* case. (Note that if $s < 0 < \sigma$, then $g(u) < u$. We can reduce this to the above case by replacing the variable u by $1 - u$, which amounts to interchanging the labels A and a.)

(ii) If s and σ are both negative, the heterozygote is more fit than either of the homozygotes, and we refer to this as the *heterozygote superior* case. (2.7) shows that this case is characterized by the fact that

$$g(u) \begin{cases} > u & \text{for } 0 < u < \pi_1, \\ < u & \text{for } \pi_1 < u < 1, \end{cases} \tag{2.9}$$

where

$$\pi_1 = \frac{\sigma}{s + \sigma}. \tag{2.10}$$

(iii) When s and σ are positive, we speak of the *heterozygote inferior* case. Here

$$g(u) \begin{cases} < u & \text{for } 0 < u < \pi_0, \\ > u & \text{for } \pi_0 < u < 1, \end{cases} \tag{2.11}$$

where π_0 is given by the same formula (2.10) as π_1.

There is, of course, no particular reason why the fitnesses $1 + s$ and $1 + \sigma$ should not depend upon u, that is, upon the composition of the competing population. As long as p is constant, we still obtain (2.5) and (2.6). The formula (2.7) shows that even when s and σ depend upon u, it is true that in the heterozygote intermediate case $s > 0 > \sigma$ the inequality (2.8) is satisfied.

However, the following example shows that the inequalities (2.8), (2.9), and (2.11) do not characterize the three cases when s and σ depend upon u.

Example 2.1. When $s = \varepsilon(1-u)(2 + \sin k\pi u)$, $\sigma = 2\varepsilon u$, where ε is any positive constant below $\frac{1}{3}$, the system is heterozygote inferior, but for $k = 1$ the corresponding g has the property (2.8), for $k = 2$ it has the property (2.9) with $\pi_1 = \frac{1}{2}$, and for larger k the

function $g-u$ has arbitrarily many sign changes. By multiplying $\sin k\pi u$ by $(-1)^k$ and making ε negative, one obtains a similar example for the heterozygote superior case.

The population size p before the migration may also depend on the genetic composition of the population and hence on u. Then (2.3) shows that in a homogeneous habitat m_{ji} is a function of $u(x_i)$ as well as of $x_j - x_i$. More generally, the migration pattern may depend upon genotype, as may the fertility.

If we assume no further selection after migration, one can hope to predict the numbers $l_A(x-y, u)$ of A gametes and $l_a(x-y, u)$ of a gametes produced in the niche at x by the deme born at y if its initial gene fraction is u. In this case we obtain the recursion (1.2) with the operator

$$Q[u](x) = \frac{\sum_{y\in\mathscr{K}} l_A(x-y, u(y))}{\sum_{y\in\mathscr{K}} [l_A(x-y, u(y)) + l_a(x-y, u(y))]}. \tag{2.12}$$

If there is nonrandom mating, the new gene fraction is a function of the right-hand side.

The above model can be modified in various other ways. The niches may consist of parallelograms, hexagons, or other regions instead of squares. We can consider the limiting case of very small regions, so that $u_n(x)$ becomes a function of the continuous variable x and the formula (2.5) for Q is replaced by

$$Q[u](x) = \int_{\mathscr{R}^2} m(x-y)g(u(y))\, dy. \tag{2.13}$$

If $m(x)$ depends only on the Euclidean distance $|x|$, the habitat is isotropic (rotationally symmetric). This case was treated in previous works [11], [12], [48], [49], [53], [54]. A discrete habitat cannot be rotationally symmetric[1], and therefore we need to treat nonisotropic operators Q here. Because of such factors as prevailing winds and the position of the sun, the migration may well be nonisotropic even in a continuous homogeneous habitat.

A continuous analogue of the more general operator (2.12) can also be obtained by replacing the sums by integrals.

Fisher's equation (1.1) can be obtained from the recursion (1.2) with Q defined by (2.5) by taking a suitable limit in which both the size of the niches and the length of the time step approach zero.

On the other hand, if we define $u_n(x) = u(n\tau, x)$, where $u(t, x)$ is a solution of Fisher's equation (1.1), then u_n satisfies the recursion (1.2) where $Q[v](x)$ is defined to be $u(\tau, x)$ with $u(t, x)$ the solution of the initial value problem

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \qquad u(0, x) = v(x). \tag{2.14}$$

This remark remains true when D and f are periodic functions of t of period τ, so that seasonal variations in the rates of migration and selection can be taken into account. Nonisotropy in the migration can be put into the model by replacing the Laplace operator by a more general second order elliptic operator, whose coefficients are independent of x but may depend periodically on t. In fact, these coefficients and, in particular, the diffusion constant D in (2.14), may also depend on u.

[1] Rotational symmetry in the natural habitat can be destroyed by aggregating over a discrete set of niches. For example, if the population density in $\mathscr{R}^2$ is $(x^2+y^2)^4$, which is radially symmetric, the population in a square of side h centered at (ih, jh) is a polynomial in i^2+j^2 plus a multiple of i^2j^2, so that it is not invariant under rotations about the origin.

All the above models involve various assumptions which may or may not be satisfied. One can think of the recursion (1.2) itself as the model. It says that the gene fraction of each deme of one generation is uniquely determined by the set of all gene fractions of the preceding generation. If the niches are not too small relative to the maximum distance of migration, $Q[u](x)$ only depends on the values of u at a few nearby niches, and the operator Q can be determined experimentally by varying these values and watching the outcome.

The major assumption inherent in the class of models (1.2) is that the number of individuals born in each deme is so large that the numbers of individuals of the three genotypes who survive to adulthood do not depend upon the sizes of the initial populations but only on their gene fraction. It is this assumption which allows us to work with a scalar-valued rather than a vector-valued dependent variable.

One must, of course, also assume that all the external influences on growth, selection, and migration remain the same for all generations.

Recursions of the form (1.2) can serve as models for many other biological processes. We may for example consider the population growth of a single migrating species with synchronized nonoverlapping generations. Let the habitat again consist of the plane $\mathcal{R}^2$ divided into squares centered at (ih, jh), and let $u_n(x)$ represent the size of the population which is in the square centered at x at a certain part of the life cycle of the nth generation. If one assumes that as far as interaction with the species under consideration is concerned, the rest of the world does not change in time, one obtains a model of the form (1.2) where Q may be measured experimentally.

If $u_n(x)$ is the number of individuals in the square centered at x just after migration, these individuals produce $g(u_n)$ new individuals and die, where $g(u)$ is a function whose graph is called the *reproduction curve* [44]. A fraction $m(y-x)$ of these new individuals migrates to y. Thus one obtains the model (2.5). One can again obtain space limits such as (2.13) by letting the size of the squares approach zero and partial differential equations like the Fisher equation (1.1) by letting both the square size and the length of the life cycle approach zero.

In all the above models we have made the assumption of determinism. That is, we assume that u_{n+1} is completely determined by the function u_n, the assignment being given by the operator Q. In nature things are, of course, not that simple, and the map from u_n to u_{n+1} is more likely to be a Markov process. That is, given the function u_n, one has a probability distribution for the function u_{n+1}.

It may, however, happen that, if u_n is known, one can say with probability one that $u_{n+1}(x)$ lies in a certain interval $[v^-(x), v^+(x)]$ for each value of x. The function $v^-(x)$ and $v^+(x)$ are determined by u_n and hence are given by applying two operators Q^- and Q^+ to u_n. Thus, instead of the recursion (1.2), we are led to the "interval statement"

$$Q^-[u_n] \leqq u_{n+1} \leqq Q^+[u_n]. \tag{2.15}$$

We shall show in § 4 that if the operators Q^- and Q^+ have some of the properties mentioned in this section, the theorems of § 6 will give information about the behavior of u_n for large n.

3. Formulation of the problem. In this section we shall formulate a mathematical model which contains many of the models discussed in the preceding section as special cases.

Since we wish to treat spaces of one, two, and three dimensions, we shall work in a general Euclidean space $\mathcal{R}^N$ of N dimensions. We shall often identify a point of $\mathcal{R}^N$

with the vector from the origin to this point, so that we can speak of the vector sum of two points and of the point αx where α is a scalar.

DEFINITION. A *habitat* $\mathcal{H}$ is defined to be a set of points in $\mathcal{R}^N$ with the following property: *If x and y are in $\mathcal{H}$, then $x+y$ and $x-y$ are also in $\mathcal{H}$.* This property implies that the origin 0 is in $\mathcal{H}$. (In fact, $\mathcal{H}$ is a group under addition.)

In order to avoid trivialities we shall assume that N is actually the dimension of $\mathcal{H}$. That is, we suppose that *there is no unit vector which is orthogonal to all of $\mathcal{H}$.* We also suppose that $N \geqq 1$ so that $\mathcal{H}$ contains a point other than the origin. Since it contains all multiples of this point, $\mathcal{H}$ is unbounded.

In all our applications the dependent variable u has only nonnegative values. If u is a gene fraction, its values lie on the interval $[0, 1]$. A population size will normally vary between 0 and some large upper bound π_+, but this upper bound can conceivably be $+\infty$.

DEFINITION. B is the set of continuous functions defined on $\mathcal{H}$ with values on the interval $[0, \pi_+]$ or, if $\pi_+ = \infty$, $[0, \infty)$.

For any y in $\mathcal{H}$ the translation operator

$$T_y u(x) \equiv u(x-y)$$

is defined.

Our basic evolution law is the recursion

$$u_{n+1} = Q[u_n],$$

where u_n and u_{n+1} are elements of B and Q is a given operator on B. The properties of the model are the properties of Q.

Since we always deal with nonnegative quantities, we assume that $Q[0] \geqq 0$.

The homogeneity of the habitat is equivalent to the assumption that Q commutes with any translation. That is, $Q[T_y[u]] = T_y[Q[u]]$ for all u in B.

A constant function α is clearly translation invariant. That is, $T_y\alpha = \alpha$ for all y. Consequently, $T_y Q[\alpha] = Q[\alpha]$ for all y, which means that $Q[\alpha]$ is again a constant. This simply states that in a homogeneous habitat the effects of migration cancel when u is constant. Thus, the properties of the model in the absence of migration can be found by looking at what Q does to constant functions.

Because $\sum_{x \in \mathcal{H}} m(x) = 1$ in the model (2.5), we see that $Q[\alpha] = g(\alpha)$ there. We shall therefore require the function $Q[\alpha]$ to have properties like those of the function g. In the genetics model with constant p, g is given by (2.6) and as long as the fitnesses are independent of u we have the three possibilities (2.8), (2.9) and (2.11). We can cover all three by saying that there are constants π_0 and π_1 with $0 \leqq \pi_0 < \pi_1 \leqq 1$ such that

$$Q[\alpha] \begin{cases} < \alpha & \text{for } \alpha \in (0, \pi_0), \\ > \alpha & \text{for } \alpha \in (\pi_0, \pi_1), \\ < \alpha & \text{for } \alpha \in (\pi_1, 1). \end{cases}$$

Then $\pi_0 = 0$, $\pi_1 = 1$ give the heterozygote intermediate case, $\pi_0 = 0$, $\pi_1 \in (0, 1)$ the heterozygote superior case, and $\pi_0 \in (0, 1)$, $\pi_1 = 1$ the heterozygote inferior case. In the absence of migration, π_1 is a stable equilibrium point and π_0 is an unstable one.

Example 2.1 shows that when the fitnesses vary with u these correspondences may no longer apply, and the function $g(u) - u$ may have more sign changes. We shall formulate our hypotheses so that we can also treat these cases by only requiring that $Q[\alpha] > \alpha$ in some interval (π_0, π_1) with $Q[\pi_0] = \pi_0$ and $Q[\pi_1] = \pi_1$. There may be more than one such interval.

The above properties of $Q[\alpha]$, with 1 replaced by some upper limit π_+, also include any reproduction curves $g(\alpha)$. In this case π_0 is a critical population size below which the population tends to decrease to 0 and π_1 is a carrying capacity, to which the population size tends in the absence of migration. For some populations $g(u)-u$ may have more than two sign changes so that there are two or more stable equilibria. See [33], [34] for an example of such a population.

We shall also assume that increasing u everywhere in one generation increases u everywhere in the next one. More precisely, if $u(x) \geqq v(x)$ for all x, then $Q[u](x) \geqq Q[v](x)$ for all x. In the case of the model (2.5), this means that the function $g(u)$ is nondecreasing. This is certainly true of the function (2.6) when s and σ do not depend upon u and $1+s$ and $1+\sigma$ are positive.

The operator (2.12) has the above property if the number $l_A(x-y, u)$ of A gametes produced at x by the deme at y is an increasing function of the gene fraction u at y while the number l_a of a gametes is decreasing in u. This is a plausible property, although one can certainly think of migratory behavior in which it is violated.

In the model (2.5) for population growth we shall assume that the reproduction curve is nonincreasing. This excludes the case of a humped reproduction curve, which even in the absence of migration leads to oscillations [42], [52] and sometimes to chaos [23], [32], [36]. A more complicated population growth model can be treated as long as an increase in $u_n(i)$ results in increased migration to every deme.

In addition to these assumptions we shall require $Q[u]$ to behave continuously with respect to changes in u. We summarize these basic hypotheses:

(i) $Q[u] \in B$ for all $u \in B$.

(ii) $Q[T_y[u]] = T_y[Q[u]]$ for all $u \in B$, $y \in \mathcal{H}$.

(iii) There are constants $0 \leqq \pi_0 < \pi_1 \leqq \pi_+$ such that

$$Q[\alpha] > \alpha \quad \text{for } \alpha \in (\pi_0, \pi_1), \quad Q[\pi_0] = \pi_0, \quad Q[\pi_1] = \pi_1 \quad \text{if } \pi_1 < \infty. \tag{3.1}$$

(iv) $u \leqq v$ implies that $Q[u] \leqq Q[v]$.

(v) $u_n \to u$ as $n \to \infty$ uniformly on each bounded subset of $\mathcal{H}$ implies that $Q[u_n](x) \to Q[u](x)$ for each $x \in \mathcal{H}$.

The operator Q will be understood to satisfy these conditions throughout the remainder of this work.

As we shall see, these hypotheses suffice to define wave speeds and to determine the large-time behavior of solutions u_n which for $n=0$ are less than π_1 and vanish outside a bounded set. If we wish to consider cases where u_0 is at least π_1 at some points, we need some further hypotheses about the action of Q on such functions. We first assume that

$$Q[\alpha] < \alpha \quad \text{for } \alpha \in (\pi_1, \pi_+) \quad \text{if } \pi_1 < \pi_+. \tag{3.2}$$

The following hypothesis is a generalized form of a strong maximum principle.

(3.3) There are two convex subsets K_1 and K_2 of $\mathcal{R}^N$ at a positive distance from each other and an ε_1 which is positive if $\pi_1 < \pi_+$ and 0 if $\pi_1 = \pi_+$ with the property that, if $u \leqq \pi_1 + \varepsilon_1$ on $\mathcal{H}$ and $u < \pi_1$ on either K_1 or K_2, then $Q[u](0) < \pi_1$.

This hypothesis excludes, of course, the trivial case of no migration at all.

Finally we may wish to restrict the range of migration of any individual. In any biologically realistic model this range is finite, which implies that there is a constant b

such that

$$u(x)=0 \quad \text{for } |x|\leqq b \Rightarrow Q[u](0)=0. \tag{3.4}$$

This condition implies that $Q[0]=0$, so that it excludes mutation into the allele A in the genetic model and spontaneous generation in the population model.

In order to allow Q to be the solution operator of (1.1), which permits a small leakage to infinity in a finite time, we can make the following weaker form of this hypothesis.

(3.5) There is a nonincreasing function $h(s)$ defined for $s\geqq 0$ such that (i) if, for some unit vector ξ and any sufficiently small positive β, $u(x)\geqq\beta$ for $x\cdot\xi\leqq 0$, then $Q[u]\geqq\beta h(x\cdot\xi)$ for all $x\in\mathcal{H}$ with $x\cdot\xi\geqq 0$; (ii) to each integer m and each $\alpha\in(0,\pi_1)$ there correspond a positive integer k_m and a constant β_m such that if $u_{n+1}=Q[u_n]$ and if, for some unit vector ξ, $u_0(x)=0$ for $x\cdot\xi\geqq 0$, then $u_m(x)\leqq\alpha[h(x\cdot\xi/k_m)]^{k_m}$ for $x\cdot\xi\geqq\beta_m$. Moreover, $h(\infty)=0$.

If (3.4) is valid, this hypothesis clearly holds with $h(s)\equiv 0$ and $\beta_m=mb$. When Q is the solution at the period time τ of Fisher's equation (1.1) with $D(t)\not\equiv 0$ and $f(v,t)$ bounded above and below by constant multiples of v, one can let h be a multiple of

$$\operatorname{erfc}\left[\frac{1}{2}s\left\{\int_0^\tau D(t)\,dt\right\}^{-1/2}\right] \quad \text{and} \quad k_m=2m.$$

If we allow mutation to the allele A or spontaneous generation so that $Q[0]>0$, we do not need any restriction on migration, provided we assume the technical condition that there is a positive constant $\bar{\gamma}$ such that

$$\gamma<Q[\gamma]<\bar{\gamma} \quad \text{for } \gamma\in[0,\bar{\gamma}), \qquad Q[\bar{\gamma}]=\bar{\gamma}. \tag{3.6}$$

In order to establish the existence of traveling waves (Theorem 6.6) we shall need the following compactness property:

(3.7) Every sequence v_n of functions in B with $v_n\leqq\pi_1$ has a subsequence v_{n_i} such that the sequence $Q[v_{n_i}]$ converges uniformly on every bounded subset of $\mathcal{H}$.

We are given an initial function u_0 in B and we shall be interested in predicting the behavior for large n of the sequence of functions $u_n(x)$ which is determined by the recursion

$$u_{n+1}=Q[u_n].$$

We shall also be interested in traveling wave solutions of this recursion. By this we mean solutions of the form

$$u_n(x)=W(x\cdot\xi-nc),$$

where W is a nonconstant function of one variable, ξ is a fixed unit vector (the normal to the waves), and c is a parameter called the wave speed. The function $W(s)$ needs only to be defined at those values of s which are of the form $x\cdot\xi-nc$ with x in $\mathcal{H}$ and $n=0,1,2,\cdots$.

We note that the operator Q is easily extended to an operator on the set $\tilde{B}$ of continuous functions on all of $\mathcal{R}^N$ with values in $[0,\pi_+]$ by means of the definition

$$\tilde{Q}[u](x_0)=Q[T_{-x_0}[u]](0) \quad \forall x_0\in\mathcal{R}^N. \tag{3.8}$$

The function $T_{-x_0}[u]$ on the right is to be interpreted as the restriction of the function $u(x-x_0)$ to $\mathcal{H}$. Because of the translation invariance (3.1.ii), the restriction of $\tilde{Q}[u]$ to $\mathcal{H}$ coincides with the function obtained by applying Q to the restriction of u to $\mathcal{H}$, so that $\tilde{Q}$ is indeed an extension of Q. It is easily verified that if Q has any of the properties (3.1), (3.2), (3.3), (3.4), (3.5), or (3.6), the same is true of the extension $\tilde{Q}$ when $\mathcal{H}$ is replaced by $\mathcal{R}^N$.

For this reason there is no harm in thinking of $\mathcal{H}$ as $\mathcal{R}^N$ except in the last two theorems, Theorems 6.5 and 6.6, in which further assumptions about Q are made which do not necessarily extend to $\tilde{Q}$. In particular, the property (3.7) need not be preserved by this extension.

4. Two basic propositions. In this section we shall state and prove some propositions which will be used in our proofs. The first of these is a comparison principle.

PROPOSITION 4.1. (comparison principle) *Let R be an operator from B to B which is order preserving in the sense that*

$$v \geqq w \Rightarrow R[v] \geqq R[w]. \tag{4.1}$$

If the sequence v_n satisfies the inequalities

$$v_{n+1} \geqq R[v_n] \tag{4.2}$$

while the sequence w_n satisfies

$$w_{n+1} \leqq R[w_n], \tag{4.3}$$

and if $v_0 \geqq w_0$, then $v_n \geqq w_n$ for all n.

Proof. Suppose that $v_n \geqq w_n$. Then by (4.2), (4.1) and (4.3)

$$v_{n+1} \geqq R[v_n] \geqq R[w_n] \geqq w_{n+1}.$$

Thus the proposition is proved by induction.

PROPOSITION 4.2. *Let R have the order-preserving property* (4.1), *and suppose that $R[w_0] \geqq w_0$. If the sequence w_n is defined by the recursion*

$$w_{n+1} = R[w_n],$$

then $w_{n+1} \geqq w_n$ for all n.

Proof. Let $v_n = w_{n+1}$. Then $v_0 \geqq w_0$, and the statement follows from Proposition 4.1.

Proposition 4.1 can often be used to obtain a robustness result for a stochastic model. Suppose that the evolution $u_n \to u_{n+1}$ is a Markov process which is bounded by two deterministic processes in the sense that the inequalities (2.15) are satisfied with probability one, and suppose that Q^+ and Q^- are order preserving operators.

We see from Proposition 4.1 that if the sequences u_n^- and u_n^+ are defined by

$$u_{n+1}^- = Q^-[u_n^-], \qquad u_{n+1}^+ = Q^+[u_n^+],$$

and if $u_0^- \leqq u_0 \leqq u_0^+$, then

$$u_n^- \leqq u_n \leqq u_n^+$$

for all n.

If the operators Q^- and Q^+ satisfy the hypotheses (3.1) and possibly some of the other hypotheses of § 3, then the theorems of § 6 can be combined with these inequalities to give information about the behavior of the random sequence u_n for large n. In particular, one can sometimes determine the qualitative behavior of u_n without

knowing much about the probability distribution of the Markov process. Such a result can be interpreted as a statement of robustness.

5. The wave speed. In this section we shall define a wave speed $c^*(\xi)$ corresponding to any operator Q which satisfies the hypotheses (3.1). The name wave speed will be justified by the theorems which are stated in the next section. For the moment, $c^*(\xi)$ will be defined as a scalar-valued function of unit vectors ξ. We can think of $c^*(\xi)$ as the speed of plane waves whose normal is in the direction ξ.

In order to define c^* we begin by choosing a function $\varphi(s)$ of one real variable with the properties

(5.1) (i) φ is continuous and nonincreasing,

(ii) $\varphi(-\infty)\in(\pi_0, \pi_1)$,

(iii) $\varphi(s)=0$ for $s\geqq 0$.

For any real number c and any unit vector ξ we define the operator

$$R_{c,\xi}[a](s)\equiv\max\{\varphi(s), Q[a(x\cdot\xi+s+c)](0)\} \tag{5.2}$$

on continuous functions of one real variable $a(s)$ with $0\leqq a\leqq\pi_1$. Here the maximum just means the larger of two numbers for each s. The function $a(x\cdot\xi+s+c)$ is to be regarded as a function of x in $\mathcal{H}$ with ξ, s, and c fixed.

We shall suppose that Q has the properties (3.1). Then $R_{c,\xi}$ still has the property

$$R_{c,\xi}[\alpha]>\alpha \quad \text{for } \alpha\in(\pi_0, \pi_1) \tag{5.3}$$

and the order preserving property (3.1.iv).

We now define a sequence $a_n(c, \xi; s)$ by the recursion

$$a_{n+1}=R_{c,\xi}[a_n], \qquad a_0=\varphi. \tag{5.4}$$

We begin by establishing some simple properties of this sequence.

LEMMA 5.1. *The sequence $a_n(c, \xi; s)$ is nondecreasing in n, nonincreasing in s and c, and continuous in c, ξ, and s.*

Proof. Because $a_0=\varphi$, it is clear that $R_{c,\xi}[a_0]\geqq a_0$. Therefore Proposition 4.2 implies that a_n is nondecreasing in n.

We shall prove the other properties by induction. Suppose that a_n is nonincreasing in s and c. Then if $c'\leqq c$ and $s'\leqq s$,

$$a_{n+1}(c', \xi; s')=\max\{\varphi(s'), Q[a_n(c', \xi; x\cdot\xi+s'+c')](0)\}.$$

Since $s'\leqq s$ and $c'\leqq c$, $\varphi(s')\geqq\varphi(s)$ and $a_n(c', \xi; x\cdot\xi+s'+c')\geqq a_n(c, \xi; x\cdot\xi+s+c)$. The order preserving property of Q shows that

$$a_{n+1}(c', \xi; s')\geqq a_{n+1}(c, \xi; s).$$

Since $a_0(c, \xi; s)=\varphi(s)$, which is nonincreasing in c and s, all the a_n are nonincreasing in c and s by induction.

Finally, let (c_ν, ξ_ν, s_ν) be any sequence which converges to (c, ξ, s) as $\nu\to\infty$. Suppose that a_n is continuous in all its variables. Then $a_n(x\cdot\xi_\nu+s_\nu+c_\nu)$ converges to $a_n(x\cdot\xi+s+c)$ uniformly for x in any bounded subset of $\mathcal{H}$, and $\varphi(s_\nu)$ converges to $\varphi(s)$. We see from the property (3.1v) that $a_{n+1}(c_\nu, \xi_\nu; s_\nu)$ converges to $a_{n+1}(c, \xi; s)$; that is, a_{n+1} is continuous.

Because $a_0=\varphi(s)$, which is continuous, the continuity of all the a_n follows by induction, and the lemma is proved.

We remark that since Q may only be defined on continuous functions, the continuity of the a_n as functions of s is needed in order that the recursion (5.4) really define the sequence a_n.

We now consider the limits of a_n at $s = \pm\infty$.

LEMMA 5.2. *Define the sequence of constants α_n and γ_n by the recursions*

$$\begin{aligned} \alpha_{n+1} &= Q[\alpha_n], \qquad \gamma_{n+1} = Q[\gamma_n], \\ \alpha_0 &= \varphi(-\infty), \qquad \gamma_0 = 0. \end{aligned} \tag{5.5}$$

Then α_n increases to π_1 as $n \to \infty$, γ_n increases to the smallest nonnegative solution $\bar{\gamma}$ of the equation $\bar{\gamma} = Q[\bar{\gamma}]$, and for all n, c, and ξ

$$a_n(c, \xi; -\infty) = \alpha_n, \qquad a_n(c, \xi; +\infty) = \gamma_n. \tag{5.6}$$

Proof. By the property (3.1iii) $Q[\alpha_0] > \alpha_0$, while by (3.1i) $Q[\gamma_0] \geqq \gamma_0$. Therefore by Proposition 4.2 the sequences α_n and γ_n are nondecreasing. By Proposition 4.1, $\gamma_n \leqq \pi_0 < \alpha_n \leqq \pi_1$. Therefore α_n has a limit $\bar{\alpha}$, which may be $+\infty$ if $\pi_1 = +\infty$ and γ_n has a limit $\bar{\gamma}$.

If $\bar{\alpha} < \pi_1$, then $Q[\bar{\alpha}] > \bar{\alpha}$ by (3.1iii) and by (3.1v) $\alpha_{n+1} = Q[\alpha_n] > \bar{\alpha}$ for sufficiently large n. This contradicts the fact that α_n is nondecreasing, and we conclude that $\bar{\alpha} = \pi_1$.

To prove the limit properties of a_n we again use induction. The statement (5.6) is certainly true for $n = 0$. Suppose that $a_n(c, \xi; -\infty) = \alpha_n$. Then the sequence

$$u_k(x) \equiv a_n(c, \xi; x \cdot \xi - k + c)$$

converges to α_n as $k \to \infty$, uniformly on bounded subsets of $\mathcal{H}$. Therefore by (3.1v)

$$\lim_{k \to \infty} Q[a_n(c, \xi: x \cdot \xi - k + c)] = Q[\alpha_n] = \alpha_{n+1}.$$

Because $\alpha_{n+1} \geqq \alpha_0 \geqq \varphi$, we find that

$$\lim_{k \to \infty} a_{n+1}(c, \xi; -k) = \alpha_{n+1}.$$

Since a_{n+1} is nonincreasing in s, we conclude that

$$a_{n+1}(c, \xi; -\infty) = \alpha_{n+1}.$$

The same argument with k replaced by $-k$ shows that $a_{n+1}(c, \xi; +\infty) = \gamma_n$. Thus, (5.6) is established by induction, and the lemma is proved.

It follows from Lemma 5.1 that the sequence a_n increases to a limit function $a(c, \xi; s)$ as n goes to infinity and that $a(c, \xi; s)$ is nonincreasing in s and in c. Because $a_n(c, \xi; s) \leqq a_n(c, \xi; -\infty) = \alpha_n \leqq \pi_1$, we find that $a \leqq \pi_1$. Lemma 5.2 shows that

$$a(c, \xi; -\infty) = \pi_1.$$

(If $\pi_1 = +\infty$, a may have the value $+\infty$ on an interval.)

The value $a(c, \xi; +\infty)$ may or may not be π_1. Since a is nonincreasing in s, $a(c, \xi; +\infty) = \pi_1$ if and only if $a \equiv \pi_1$. We give a criterion to determine whether or not this is the case.

LEMMA 5.3. *The value $a(c, \xi; +\infty) = \pi_1$ if and only if there is an n such that*

$$a_n(c, \xi; 0) > \varphi(-\infty). \tag{5.7}$$

Proof. If $a(c, \xi; +\infty) = \pi_1$, then $a(c, \xi; s) = \pi_1$ for all s, and in particular for $s = 0$.

Because $\pi_1 > \varphi(-\infty)$ and $a_n(c, \xi; 0)$ converges to $a(c, \xi; 0)$, there must be an n such that (5.7) is valid.

Suppose, on the other hand that (5.7) holds for $n = n_0$. Because a_n and φ are nonincreasing, $\varphi = 0$ for $s \geqq 0$, and $\varphi \leqq \varphi(-\infty)$, it follows from the continuity of a_n and φ that there is a positive constant δ such that

$$a_{n_0}(c, \xi; s+\delta) \geqq \varphi(s) = a_0(c, \xi; s)$$

for all s.

Suppose that for some $k > 0$

$$a_{n_0+k}(c, \xi; s+\delta) \geqq a_k(c, \xi; s). \tag{5.8}$$

Then

$$\begin{aligned} a_{n_0+k+1}(c, \xi; s+\delta) &= \max\{a_{n_0}(c, \xi; s+\delta), Q[a_{n_0+k}(c, \xi; x\cdot\xi + s + \delta + c)](0)\} \\ &\geqq \max\{\varphi(s), Q[a_k(c, \xi; x\cdot\xi + s + c)](0)\} \\ &= a_{k+1}(c, \xi; s), \end{aligned}$$

because $a_{n_0+k+1} \geqq a_{n_0}$. Thus we have established the inequality (5.8) by induction. We let $k \to \infty$ to see that

$$a(c, \xi; s+\delta) \geqq a(c, \xi; s)$$

for all s. Since $\delta > 0$ and a is nonincreasing in s, it follows that a is independent of s. Since $a(c, \xi; -\infty) = \pi_1$, we conclude that $a(c, \xi; s) = \pi_1$ for all s, and the lemma is proved.

We now observe that since $a_1(0, \xi; -\infty) = \alpha_1 > \varphi(-\infty)$, there must be a nonnegative constant $t = t(\xi)$ such that

$$a_1(0, \xi; -t) > \varphi(-\infty).$$

Then since $\varphi(-t) \leqq \varphi(-\infty)$, $a_1(0, \xi; -t) = Q[\varphi(x\cdot\xi - t)](0)$, and

$$a_1(-t, \xi; 0) = \max\{\varphi(0), Q[\varphi(x\cdot\xi - t)](0)\} = \max\{0, a_1(0, \xi; -t)\} > \varphi(-\infty).$$

Thus, by Lemma 5.3, $a(-t, \xi; +\infty) = \pi_1$. Hence $a(c, \xi; +\infty) = \pi_1$ for $c \leqq -t$.

We now define

$$c^*(\xi) \equiv \sup\{c \,|\, a(c, \xi; +\infty) = \pi_1\}. \tag{5.9}$$

If $a(c, \xi, +\infty) = \pi_1$ for all c, we set $c^*(\xi) = +\infty$.

Since $a(c, \xi; s)$ is the limit of a nondecreasing family of continuous functions, it is a lower semicontinuous function of c, ξ, and s. Lemma 5.3 shows that $a(c, \xi; +\infty) = \pi_1$ if and only if $a(c, \xi; 0) > \varphi(-\infty)$, and the lower semicontinuity of a shows that the set of (c, ξ) where this is valid is open. This fact has two consequences:

PROPOSITION 5.1. *$a(c, \xi; +\infty) = \pi_1$ if and only if $c < c^*(\xi)$ and $c^*(\xi)$ is a lower semicontinuous function of ξ.*

The sequence a_n depends upon the choice of the function $\varphi(s)$. Consequently $c^*(\xi)$ appears to depend upon this choice. The following lemma shows that this is not the case.

LEMMA 5.4. *Let $\hat{\varphi}(s)$ be any continuous nonincreasing function of s with the properties*

$$\hat{\varphi}(-\infty) \in (\pi_0, \pi_1), \qquad \hat{\varphi}(s) = 0 \quad \text{for } s \geqq 0. \tag{5.10}$$

Define the sequence $\hat{a}_n(c, \xi; s)$ by the recursion

$$\hat{a}_{n+1}(s) = \max\{\hat{\varphi}(s), Q[\hat{a}_n(x\cdot\xi + s + c)](0)\}, \qquad \hat{a}_0(s) = \hat{\varphi}(s), \tag{5.11}$$

so that $\hat{a}_n$ *increases to a nonincreasing limit function* $\hat{a}(c, \xi; s)$. *Then* $\hat{a}(c, \xi; +\infty) = a(c, \xi; +\infty)$.

Proof. Define the sequence of constants $\hat{\alpha}_n$ by the recursion

$$\hat{\alpha}_{n+1} = Q[\hat{\alpha}_n], \qquad \hat{\alpha}_0 = \hat{\varphi}(-\infty).$$

Then $\hat{\alpha}_n$ increases to π_1, and because $\varphi(-\infty) < \pi_1$ there is an integer n_0 such that $\hat{\alpha}_{n_0} > \varphi(-\infty)$. By Lemma 5.2, $\hat{a}_{n_0}(c, \xi; -\infty) = \hat{\alpha}_{n_0}$. Therefore if $n_0 \geqq 1$ there is a $t = t(c, \xi)$ such that

$$\hat{a}_{n_0}(c, \xi; -t) > \varphi(-\infty). \tag{5.12}$$

Because $\hat{a}_{n_0}$ and $\varphi = a_0$ are nonincreasing in s and $\varphi = 0$ for $s \geqq 0$, it follows that

$$\hat{a}_{n_0}(c, \xi; s - t) > \varphi(s). \tag{5.13}$$

Since $\hat{a}_n$ is nondecreasing in n, $\hat{a}_{n+1}(c, \xi; s - t) \geqq \hat{a}_{n_0}(c, \xi; s - t) \geqq \hat{\varphi}(s)$ for $n \geqq n_0$, and hence

$$\begin{aligned} \hat{a}_{n_0+k+1}(c, \xi; s - t) &= \max\{\hat{a}_{n_0}(c, \xi; s - t), Q[\hat{a}_{n_0+k}(c, \xi; x \cdot \xi + s - t + c)](0)\} \\ &\geqq \max\{\varphi(s), Q[\hat{a}_{n_0+k}(c, \xi; x \cdot \xi + s - t + c)](0)\} \\ &= R_{c,\xi}[\hat{a}_{n_0+k}(c, \xi; s - t)](s) \end{aligned}$$

for $k \geqq 0$. We now see from (5.13) and Proposition 4.1 that $\hat{a}_{n_0+k}(c, \xi; s - t) \geqq a_k(c, \xi; s)$ for $k \geqq 0$. By letting $k \to \infty$ and then $s \to \infty$ we obtain the inequality

$$\hat{a}(c, \xi; +\infty) \geqq a(c, \xi; +\infty).$$

We now use the same proof with the roles of φ and $\hat{\varphi}$ reversed to obtain the opposite inequality. Therefore $\hat{a}(c, \xi; +\infty) = a(c, \xi; +\infty)$, which proves the lemma.

Lemma 5.4 shows that the definition (5.9) gives the same value of $c^*(\xi)$ when a is replaced by $\hat{a}$, so that c^* does not depend on the choice of the function φ.

It is important to know the behavior of a as $s \to +\infty$ when $c \geqq c^*(\xi)$.

PROPOSITION 5.2.

$$\lim_{s \to +\infty} a(c, \xi; s) \leqq \pi_0$$

uniformly in c *and* ξ *on every set of the form* $\{(c, \xi) | c \geqq c^*(\xi), c^*(\xi) \leqq C, |\xi| = 1\}$ *where* C *is a constant.*

Proof. We have shown in the proof of Lemma 5.4 that for any (c, ξ) there exist n_0 and t such that the inequality (5.12) is valid, and that this inequality implies that

$$\hat{a}_{n_0+k}(c, \xi; s - t) \geqq a_k(c, \xi; s). \tag{5.14}$$

Because $\hat{a}_{n_0}$ is continuous in c and ξ, the inequality (5.12) is valid in a neighborhood of (c, ξ). The set $\{(c, \xi) | c = c^*(\xi) \leqq C, |\xi| = 1\}$ is bounded, and it is closed because $c^*(\xi)$ is lower semicontinuous. We cover it with finitely many of the neighborhoods and take the largest of the n_0 and the smallest of the t to obtain the inequality (5.12) and hence (5.14) for all (c, ξ) on this set. By Lemma 5.3 $\hat{a}_{n_0+k}(c^*(\xi), \xi; 0) \leqq \hat{\varphi}(-\infty)$. We set $s = t$ in (5.14) to see that if $c = c^*(\xi) \leqq C$

$$a_k(c, \xi; t) \leqq \hat{\varphi}(-\infty).$$

Since a_k is nonincreasing in c, this inequality is valid whenever $c \geqq c^*(\xi)$ and $c^*(\xi) \leqq C$.

We let $k \to \infty$ to see that there is a t such that

$$a(c, \xi; s) \leqq \hat{\varphi}(-\infty) \quad \text{for } s \geqq t$$

when $c \geqq c^*(\xi)$ and $c^*(\xi) \leqq C$.

We now recall that $\hat{\varphi}(-\infty)$ is any number in the interval (π_0, π_1). In particular it may be chosen arbitrarily close to π_0. Since a is nonnegative and nondecreasing, the statement of the proposition follows.

Proposition 5.2 implies that for $c \geqq c^*(\xi)$, $a(c, \xi; +\infty) \leqq \pi_0$. The following lemma gives more precise information.

LEMMA 5.5. *If π_1 is finite, then for any c and ξ the limit $a(c, \xi; +\infty)$ satisfies the equation*

$$a(c, \xi; +\infty) = Q[a(c, \xi; +\infty)].$$

Proof. Suppose that

$$Q[a(c, \xi; \infty)] > a(c, \xi; \infty).$$

By continuity there is a number β such that

$$\beta < a(c, \xi; \infty) < Q[\beta].$$

Choose a continuous nonincreasing function $\psi(s)$ such that

$$\psi(s) = \begin{cases} \beta & \text{for } s \leqq -1, \\ 0 & \text{for } s \geqq 0. \end{cases}$$

For each positive integer k one can find n_k so that

$$a_{n_k}(c, \xi; k) \geqq \beta.$$

Then

$$a_{n_k}(c, \xi; s) \geqq \psi(s - k).$$

Hence

$$a(c, \xi; s) \geqq a_{n_k+1}(c, \xi; s) \geqq Q[\psi(x \cdot \xi + s + c - k)](0).$$

We let $k \to \infty$ and use (3.1.v) to see that $a(c, \xi; s) \geqq Q[\beta]$ for all s. This contradicts the condition $a(c, \xi; \infty) < Q[\beta]$. We conclude that $Q[a(c, \xi; \infty)] \leqq a(c, \xi; \infty)$.

Suppose now that

$$Q[a(c, \xi; \infty)] < a(c, \xi; \infty).$$

We choose $\hat{\beta}$ so that

$$Q[\hat{\beta}] < a(c, \xi; \infty) < \hat{\beta} < \pi_1.$$

Since π_1 is finite, we can choose a continuous nonincreasing function $\hat{\psi}(s)$ such that

$$\hat{\psi}(s) = \begin{cases} \pi_1 & \text{for } s \leqq -1, \\ \hat{\beta} & \text{for } s \geqq 0. \end{cases}$$

We can now choose s_1 so that $a(c, \xi; s_1) \leqq \hat{\beta}$. Then

$$a_n(c, \xi; s) \leqq a(c, \xi; s) \leqq \hat{\psi}(s - 1 - s_1)$$

for all n. Hence

$$a_{n+1}(c, \xi; s) \leqq \max \{\varphi(s), Q[\hat{\psi}(x \cdot \xi + s + c - 1 - s_1](0)\}.$$

We let $n \to \infty$ and then $s \to \infty$ to see that $a(c, \xi; \infty) \leqq Q[\hat{\beta}]$, which contradicts the condition $Q[\hat{\beta}] < a(c, \varphi; \infty)$.

We have shown that $Q[a(c, \xi; \infty)]$ can be neither larger nor smaller than $a(c, \xi; \infty)$, which proves the lemma.

Remark. When π_1 is finite and $\mathcal{H}$ is discrete, then $a(c, \xi; x \cdot \xi + s + c)$ is automatically a continuous function of x in $\mathcal{H}$ and the sequence $a_n(c, \xi; x \cdot \xi + s + c)$ converges to it uniformly on bounded sets. In this case the lemma follows immediately if we let $n \to \infty$ and then $s \to \infty$ in (5.4).

While each $a_n(c, \xi; s)$ is continuous in c, ξ, and s, Lemma 5.3 shows that the limit function $a(c, \xi; s)$ is discontinuous in c at $c = c^*(\xi)$. Moreover, a may also be discontinuous in s. Consider, for example, the operator

$$Q[u](s) = \tfrac{1}{3}u(s+1)[2 - u(s+1)] + \tfrac{2}{3}u(s-1)[2 - u(s-1)]$$

on $\mathcal{H} = \mathcal{R}^1$.

It is obvious from the definition that

$$a(c, 1; s) = 0 \quad \text{for } c \geqq 1, \quad s \geqq 0.$$

Moreover, because

$$Q[u](s+1) \geqq \tfrac{2}{3}u(s)[2 - u(s)] > u(s) \quad \text{when } u(s) < \tfrac{1}{2},$$

we find that if $\varphi(s) > 0$ for $s < 0$,

$$a(1, 1; s) \geqq \tfrac{1}{2} \quad \text{for } s < 0.$$

Thus the function $a(1, 1; s)$ has a jump at $s = 0$.

It is easily seen that $a(c, 1; s) \equiv 1$ for $c < 1$, so that $c^*(1) = 1$ in this example.

Of course, since $a(c, \xi; s)$ is lower semicontinuous and nonincreasing in c and in s, it is continuous from the right in c and in s.

The following three properties of the wave speed give an intuitive feeling for this function. Their proofs are obvious.

PROPOSITION 5.3. *If $c^*(\xi)$ is the wave speed corresponding to the operator Q, then the wave speed corresponding to the operator $T_y Q$ is $c^*(\xi) + y \cdot \xi$.*

PROPOSITION 5.4. *If Q has the property that for some $\delta \in [0, \pi_0]$*

$$u(x) \leqq \delta \quad \textit{for } |x| < b \quad \textit{implies } Q[u](0) \leqq \delta,$$

then

$$c^*(\xi) \leqq b \quad \textit{for all } \xi.$$

PROPOSITION 5.5. *Let Q_1 and Q_2 be two operators with the properties* (3.1), *where π_0 and π_1 are replaced by $\pi_0^{(1)}$, $\pi_1^{(1)}$ for Q_1 and by $\pi_0^{(2)}$, $\pi_1^{(2)}$ for Q_2. Let $c_i^*(\xi)$ be the wave speed corresponding to Q_i. If $\pi_0^{(2)} \leqq \pi_0^{(1)} < \pi_1^{(1)} \leqq \pi_1^{(2)}$ and*

$$Q_1[u] \leqq Q_2[u]$$

for all continuous functions u on $\mathcal{H}$ with values on $[0, \pi_1^{(1)}]$ then

$$c_1^*(\xi) \leqq c_2^*(\xi)$$

for all ξ.

6. Statement of the theorems. For the convenience of the reader we shall state our results in this section. The proofs will be presented in the subsequent sections.

For any set V of vectors in $\mathscr{R}^N$ we define

$$nV = \{v_1 + v_2 + \cdots + v_n | v_j \in V \text{ for } j = 1, \cdots, n\}. \tag{6.1}$$

It is easily seen that if V is convex, then

$$nV = \{nv | v \in V\}.$$

We denote the unit sphere $\{\xi \in \mathscr{R}^N \| \xi| = 1\}$ by S^{N-1}.

We shall show that, in an asymptotic sense, $c^*(\xi)$ is a propagation speed for arbitrary initial distrubances.

We define the convex set

$$\mathscr{S} = \{x \in \mathscr{R}^N | x \cdot \xi \leqq c^*(\xi) \forall \xi \in S^{N-1}\}. \tag{6.2}$$

If $c^*(\xi)$ were a true propagation speed, so that $u = 0$ for $x \cdot \xi \leqq 0$ implied that $Q[u] = 0$ for $x \cdot \xi \leqq c^*(\xi)$, but not that $Q[u] = 0$ for $x \cdot \xi \leqq c$ when $c < c^*(\xi)$, then it would follow that, if the initial disturbance u_0 is concentrated at the origin, the support of $u_1 = Q[u_0]$ is $\mathscr{S}$. More generally it would follow that if u_n is defined by the recursion $u_{n+1} = Q[u_n]$, the support of u_n is $n\mathscr{S} + \text{supp}\,(u_0)$. (Note that $n\mathscr{S} = \{x | x \cdot \xi \leqq nc^*(\xi) \forall \xi \in S^{N-1}\}$.)

Our first two theorems say that this property is approximately true when n is large. The first theorem says that if the support of u_0 is bounded, then at time n very little disturbance lies far beyond $n\mathscr{S}$, while the second theorem states that the disturbance at time n fills most of $n\mathscr{S}$.

In all the theorems the sequence u_n of functions in B is a solution of the recursion $u_{n+1} = Q[u_n]$, and Q satisfies the hypotheses (3.1).

THEOREM 6.1. *Suppose that the set $\mathscr{S}$ defined by* (6.2) *is bounded and not empty*[2]. *Let $\mathscr{S}'$ be any open set which contains $\mathscr{S}$. Suppose that $u_0 = 0$ outside a bounded set. If $u_0 < \pi_1$ or if $u_0 < \pi_+$ and Q satisfies the additional hypotheses* (3.2), (3.3) *and one of the hypotheses* (3.4), (3.5), *or* (3.6), *then*

$$\limsup_{n \to \infty} \max_{x \notin n\mathscr{S}'} u_n(x) \leqq \pi_0. \tag{6.3}$$

If $\mathscr{S}$ is empty and c^* is bounded, (6.3) *holds when the maximum is over the whole space $\mathscr{H}$.*

Note that if $\pi_0 = 0$, the statements say that the limit is 0.

THEOREM 6.2. *Suppose that the interior of $\mathscr{S}$ is not empty and let $\mathscr{S}''$ be any closed bounded subset of the interior of $\mathscr{S}$. For any $\sigma > \pi_0$ there exist a radius r_σ with the property that if $u_0(x) \geqq \sigma$ on a ball of radius r_σ and if $u_{n+1} = Q[u_n]$, then*

$$\liminf_{n \to \infty} \min_{x \in n\mathscr{S}''} u_n(x) \geqq \pi_1. \tag{6.4}$$

If $u_0 \leqq \alpha < \pi_+$ and if Q satisfies the additional hypothesis (3.2), *then*

$$\lim_{n \to \infty} \min_{x \in n\mathscr{S}''} u_n(x) = \lim_{n \to \infty} \max_{x \in \mathscr{H}} u_n(x) = \pi_1. \tag{6.5}$$

The next two theorems show how to estimate the wave speed in some cases.

[2] For the analogous results when $\mathscr{S}$ is unbounded or when $\mathscr{S}$ is empty but c^* is unbounded see [55].

THEOREM 6.3. *If $m(x, dx)$ is a bounded nonnegative measure on $\mathcal{H}$ with the property that for all continuous u with $0 \leqq u \leqq \pi_1$*

$$Q[u](x) \leqq \int u(x-y)m(y, dy), \tag{6.6}$$

then

$$c^*(\xi) \leqq \inf_{\mu>0} \frac{1}{\mu} \log \int e^{\mu x \cdot \xi} m(x, dx). \tag{6.7}$$

THEOREM 6.4. *Suppose that $l(x, dx)$ is a bounded nonnegative measure on $\mathcal{H}$ with the properties that*

$$\int l(x, dx) > 1$$

and that there is a positive ε such that for all continuous u with $0 \leqq u \leqq \varepsilon$

$$Q[u](x) \geqq \int u(x-y)l(y, dy).$$

Then $\mathcal{S}$ is not empty and

$$c^*(\xi) \geqq \inf_{\mu>0} \frac{1}{\mu} \log \int e^{\mu x \cdot \xi} l(x, dx), \tag{6.8}$$

where the right-hand side is $+\infty$ if the integral on the right diverges for all positive μ.

If the measure l is not concentrated on any hyperplane $x \cdot \xi =$ constant, then $\mathcal{S}$ has interior points, and the radius r_σ in Theorem 6.2 can be chosen so that it does not depend upon σ.

We note the following immediate corollary of Theorems 6.3 and 6.4.

COROLLARY. *If the nonnegative bounded measure $m(x, dx)$ satisfies (6.6) when $0 \leqq u \leqq \pi_1$ and if for every positive δ there is a positive ε such that*

$$Q[u](x) \geqq (1-\delta) \int u(x-y)m(y, dy) \quad \text{when } 0 \leqq u \leqq \varepsilon, \tag{6.9}$$

then

$$c^*(\xi) = \inf_{\mu>0} \frac{1}{\mu} \int e^{\mu x \cdot \xi} m(x, dx) \tag{6.10}$$

for all unit vectors ξ.

The hypotheses of Theorem 6.4 and the corollary imply that $\pi_0 = 0$. This condition is also need for the last two theorems.

THEOREM 6.5. (hairtrigger effect) *Let Q and l satisfy the hypotheses of Theorem 6.4 and suppose that the support of l contains a set $\mathcal{W}$ of vectors in $\mathcal{H}$ with the property that any bounded subset of $\mathcal{H}$ is contained in a translate of the set $n\mathcal{W}$ for some integer n. Then if u_0 is not identically zero on $\mathcal{H}$ and $\mathcal{S}''$ is as in Theorem 2,*

$$\liminf_{n\to\infty} \min_{x \in n\mathcal{S}''} u_n(x) \geqq \pi_1. \tag{6.11}$$

This theorem states that r_σ in Theorem 6.2 may be chosen to be arbitrarily small, regardless of the value of σ.

The last theorem asserts the existence of a traveling wave of speed c with normal ξ whenever $c \geqq c^*(\xi)$. We again require π_0 to be 0 and we need the property (3.7).

THEOREM 6.6. *Suppose that* $\pi_0 = 0$ *and that, in addition to* (3.1), Q *has the compactness property* (3.7). *Then if* $c \geqq c^*(\xi)$, *there is a nonincreasing function* $W(s)$ *which is defined for all* s *of the form* $x \cdot \xi - nc$ *with* x *in* $\mathcal{H}$ *and* n *an integer such that the sequence* $u_n(x) = W(x \cdot \xi - nc)$ *satisfies the recursion* (1.2), $W(-\infty) = \pi_1$, *and* $W(+\infty) = 0$.

7. The rate of propagation. (Proof of Theorem 6.1) We shall use the construction of § 5 in order to prove Theorem 6.1. We begin with an easy lemma.

LEMMA 7.1. *Suppose* $0 \leqq u_0(x) \leqq \alpha < \pi_1$, *and that for some finite collection of unit vectors* $\xi_1, \cdots, \xi_K$ *there is a* ρ *such that for each* i

$$u_0(x) = 0 \quad \text{for } x \cdot \xi_i \geqq \rho - 1.$$

For a positive ε *define the set*

$$\mathcal{M} = \{x | x \cdot \xi_i \leqq c^*(\xi_i) + \varepsilon,\ i = 1, \cdots, K\} \tag{7.1}$$

Then, if u_n *is the solution of the recursion* $u_{n+1} = Q[u_n]$,

$$\limsup_{n \to \infty} \max_{x \notin n\mathcal{M}} u_n(x) \leqq \pi_0. \tag{7.2}$$

Proof. We choose a function $\varphi(s)$ with the properties (5.1) and the additional property

$$\varphi(s) = \alpha \quad \text{for } s \leqq -1.$$

Then

$$u_0(x) \leqq \varphi(x \cdot \xi_i - \rho) \quad \text{for } i = 1, \cdots, K. \tag{7.3}$$

We define the sequence $a_n(c^*(\xi_i), \xi_i; s)$ by the recursion (5.4). Then the sequence

$$v_n(x) = a_n(c^*(\xi_i), \xi_i; x \cdot \xi_i - nc^*(\xi_i) - \rho)$$

satisfies the recursion

$$v_{n+1}(x) = \max\{\varphi(x \cdot \xi_i - (n+1)c^*(\xi_i) - \rho), Q[v_n](x)\} \geqq Q[v_n](x),$$

while (7.3) shows that $v_0(x) \geqq u_0(x)$.

Proposition 4.1 implies that

$$\begin{aligned} u_n(x) \leqq v_n(x) &= a_n(c^*(\xi_i), \xi_i; x \cdot \xi_i - nc^*(\xi_i) - \rho) \\ &\leqq a(c^*(\xi_i), \xi_i; x \cdot \xi_i - nc^*(\xi_i) - \rho) \end{aligned} \tag{7.4}$$

for all n and for $i = 1, \cdots, K$.

If $x \notin n\mathcal{M}$, then $x \cdot \xi_i > n(c^*(\xi_i) + \varepsilon)$ for some i. Because a is nonincreasing in s, we find that

$$\max_{x \notin n\mathcal{M}} u_n \leqq \max_{1 \leqq i \leqq K} a(c^*(\xi_i), \xi_i; n\varepsilon - \rho). \tag{7.5}$$

The statement of the lemma now follows from Proposition 5.2.

In order to use Lemma 5.1 we need to show that there is a set $\mathcal{M}$ of the form (7.1) inside $\mathcal{S}'$.

LEMMA 7.2. *If $\mathcal{S}$ is bounded and nonempty, and if the open set $\mathcal{S}'$ contains $\mathcal{S}$, then there is a set $\mathcal{M}$ of the form* (7.1) *such that*

$$\mathcal{S} \subset \mathcal{M} \subset \mathcal{S}'.$$

Proof. We begin with the well-known fact [6, § 27] that a bounded convex set $\mathcal{S}$ can be approximated by an open convex polyhedron

$$P = \{x | x \cdot \eta_\alpha < b_\alpha, \alpha = 1, \cdots, L\}$$

in the sense that if $\mathcal{S}$ lies in the interior of $\mathcal{S}'$, there is a P such that $\mathcal{S} \subset P \subset \mathcal{S}'$. Because $\mathcal{S}$ is bounded, it lies in the interior of a closed ball B_R, and we may choose P so that its closure lies in the interior of B_R.

Because $\mathcal{S}$ lies in P, its intersection with each half-space $\{x | x \cdot \eta_\alpha \geqq b_\alpha\}$ is empty. Because of the definition (6.2) of $\mathcal{S}$ and because $\mathcal{S} \subset B_R$, we can write this fact in the form

$$\bigcap_{\substack{|\xi|=1 \\ \delta>0}} [B_R \cap \{x | x \cdot \eta_\alpha \geqq b_\alpha, x \cdot \xi \leqq c^*(\xi) + \delta\} = \varnothing].$$

Because of the presence of the set B_R, each of the sets in brackets is closed and bounded. Helly's theorem (the finite intersection property) (see, e.g., [6, p. 3]) states that there is a finite collection of the sets in brackets whose intersection is empty. That is, there are unit vectors $\xi_{1\alpha}, \cdots, \xi_{K_\alpha \alpha}$ and positive numbers $\delta_{1\alpha}, \cdots, \delta_{K_\alpha \alpha}$ such that

$$B_R \cap \{x | x \cdot \xi_{j\alpha} \leqq c^*(\xi_{j\alpha}) + \delta_{j\alpha}, j = 1, \cdots, K_\alpha\} \subset \{x | x \cdot \eta_\alpha < b_\alpha\}.$$

It follows that

$$B_R \cap \{x | x \cdot \xi_{j\alpha} \leqq c^*(\xi_{j\alpha}) + \delta_{j\alpha}, j = 1, \cdots, K_\alpha, \alpha = 1, , \cdots, L\} \subset P.$$

Because the closure of P lies in the interior of B_R and all the sets involved are convex, it follows that the set in braces is a subset of P. If we let $\xi_1, \cdots, \xi_K$ be an ordering of the vectors $\xi_{j\alpha}$ and $\varepsilon = \min \delta_{j\alpha}$ we find that the set $\mathcal{M}$ defined by (7.1) lies in P.

Because $P \subset \mathcal{S}'$, we have $\mathcal{M} \subset \mathcal{S}'$. It is obvious that $\mathcal{M} \supset \mathcal{S}$, so the lemma is proved.

If $u_0 < \pi_1$ and $u_0 = 0$ outside a bounded set, then because u_0 is continuous there is an $\alpha \in (\pi_0, \pi_1)$ such that $u_0 \leqq \alpha$. Thus the statement (6.3) of Theorem 6.1 is an immediate consequence of Lemmas 7.1 and 7.2 in this case.

Suppose now only that u_0 is less than π_+ and vanishes outside a bounded set, and that the hypotheses (3.2) and (3.3) are valid. Because u_0 is continuous there is a $\nu \in [\pi_1, \pi_+)$ such that $u_0 \leqq \nu$.

If we define the sequence of constants ν_n by the recursion $\nu_{n+1} = Q[\nu_n]$, $\nu_0 = \nu$, this sequence decreases to π_1 and there is an integer m_0 which depends only on ν such that $\nu_{m_0} \leqq \pi_1 + \varepsilon_1$, where ε_1 is the constant in (3.3). By Proposition 4.1,

$$u_n \leqq \pi_1 + \varepsilon_1 \quad \text{for } n \geqq m_0.$$

We recall the nondecreasing sequence γ_n defined in (5.5). We wish to show that $u_n \geqq \gamma_n$ and

$$\lim_{|x| \to \infty} u_n(x) = \gamma_n$$

for all n. Suppose this is true for some value of n. Then $u_{n+1} \geqq \gamma_{n+1}$.

Assume that there is a positive δ and a sequence x_k with $|x_k| \to \infty$ such that

$$u_{n+1}(x_k) \geqq \gamma_{n+1} + \delta.$$

By taking a subsequence if necessary, we may assume that $x_k/|x_k|$ converges to a unit vector η. Choose a nonincreasing continuous function $\psi(s)$ such that $\psi(\infty) = \gamma_n$ and

$$u_n(x) \leqq \psi(x \cdot \eta).$$

Then

$$u_{n+1}(y) \leqq Q[\psi(x \cdot \eta)](y) = Q[\psi(x \cdot \eta + y \cdot \eta)](0).$$

In particular,

$$u_{n+1}(x_k) \leqq Q[\psi(x \cdot \eta + x_k \cdot \eta)](0).$$

Since $x_k \cdot \eta \to \infty$, the sequence of functions $\psi(x \cdot \eta + x_k \cdot \eta)$ converges to γ_n uniformly on bounded sets. Thus by (3.1.v),

$$\limsup_{k \to \infty} u_{n+1}(x_k) \leqq \gamma_{n+1}$$

contrary to the assumption that $u_{n+1}(x_k) \geqq \gamma_n + \delta$. We conclude that $u_{n+1}(x)$ converges to γ_{n+1} as $|x| \to \infty$. Since $u_0 = \gamma_0 = 0$ outside a bounded set, we have shown by induction that

$$\lim_{|x| \to \infty} u_n(x) = \gamma_n \quad \text{for } n \geqq 0.$$

Since $\gamma_n \leqq \pi_0 < \pi_1$, we see that $u_{m_0}(x) < \pi_1$ outside a bounded set, whose diameter we call D.

Suppose that for some $m_1 > m_0$ there is a point $\bar{x}$ such that $u_{m_1}(\bar{x}) \geqq \pi_1$. By (3.3) and the translation invariance of Q, there must be vectors x_1 in K_1 and y_1 in K_2 such that $u_{m_1-1}(\bar{x} + x_1) \geqq \pi_1$ and $u_{m_1-1}(\bar{x} + y_1) \geqq \pi_1$. By the same reasoning, there must be x_2 in K_1 and y_2 in K_2 such that $u_{m_1-2}(\bar{x} + x_1 + x_2) \geqq \pi_1$ and $u_{m_1-2}(\bar{x} + y_1 + y_2) \geqq \pi_1$. Proceeding in this way, we find that $u_{m_0}(\bar{x} + x_1 + x_2 + \cdots + x_{m_1-m_0}) \geqq \pi_1$ and $u_{m_0}(\bar{x} + y_1 + \cdots + y_{m_1-m_0}) \geqq \pi_1$ where $x_i \in K_1$ and $y_i \in K_2$.

Because K_1 and K_2 are convex, $(m_1 - m_0)^{-1}(x_1 + \cdots + x_{m_1-m_0}) \in K_1$ and $(m_1 - m_0)^{-1}(y_1 + \cdots + y_{m_1-m_0}) \in K_2$. If d is the positive distance from K_1 to K_2, then the distance from $\bar{x} + x_1 + \cdots + x_{m_1-m_0}$ to $\bar{x} + y_1 + \cdots + y_{m_1-m_0}$ is a least $(m_1 - m_0)d$. Consequently, we must have $(m_1 - m_0)d \leqq D$. Thus we find that if we choose

$$m_1 = m_0 + \frac{D}{d} + 1,$$

then

$$u_{m_1} < \pi_1. \tag{7.6}$$

Because u_{m_1} is continuous and its limit superior at infinity is at most π_0, there is an $\alpha \in (\pi_0, \pi_1)$ such that $u_{m_1} \leqq \alpha$.

If the condition (3.4) is valid, (6.3) now follows from applying Lemmas 7.1 and 7.2 to the sequence u_{m_1+n}.

If (3.6) holds, the sequence γ_n is strictly increasing. We recall that $u_{m_1} \to \gamma_{m_1}$ as $|x| \to \infty$, that $a_{m_1+1}(c^*(\xi), \xi; +\infty) = \gamma_{m_1+1} > \gamma_{m_1}$, and that $u_{m_1}(x) \leqq \alpha < Q[\alpha] \leqq a(c^*(\xi), \xi; -\infty)$. It follows that there are k and ρ such that

$$u_{m_1}(x) \leqq a_{m_1+k}(c^*(\xi), \xi; x \cdot \xi - m_1 c^*(\xi) - \rho)$$

and hence by Proposition 4.2 that

$$u_n(x) \leqq a_{n+k}(c^*(\xi), \xi; x\cdot\xi - nc^*(\xi) - \rho)$$
$$\leqq a(c^*(\xi), \xi; x\cdot\xi - nc^*(\xi) - \rho)$$

for $n \geqq m_1$. Thus we obtain (7.4) and the rest of the proof proceeds as before.

Finally suppose that (3.5) is valid. We see from (3.5i) that if $\varphi(s) = \alpha$ for $s \leqq -1$ then for a sufficiently small β

$$a_1(c^*(\xi), \xi; s) \geqq \beta h(s + c^*(\xi) + 1) \quad \text{for } s \geqq -c^*(\xi) - 1.$$

Since a_1 is nonincreasing, we have for any $s_0 > \max\{0, -c^*(\xi)\}$

$$a_1(c^*(\xi), \xi; s) \geqq \beta h(s_0 + c^*(\xi)) \quad \text{for } s \leqq s_0 - 1.$$

If $h(s_0 + c^*(\xi)) \leqq 1$, then by (3.5.i)

$$a_2(c^*(\xi), \xi; s) \geqq \beta h(s_0 + c^*(\xi)) h(s + c^*(\xi) - s_0 + 1)$$
$$\geqq \beta h(s_0 + c^*(\xi))^2 \quad \text{for } s \leqq 2s_0 - 1.$$

By continuing in this manner and setting $s_0 = (s+1)/k$, we find that

$$a_k(c^*(\xi), \xi; s) \geqq \beta\left[h\left(\frac{s + 1 + kc^*(\xi)}{k}\right)\right]^k \tag{7.7}$$

for all unit vectors ξ as long as $h((s + 1 + kc^*(\xi))/k) \leqq 1$, and $s > \max\{0, -kc^*(\xi)\}$.

From (3.5.ii) and the fact that $u_0 = 0$ for $|x| \geqq r$ we see that

$$u_{m_1}(x + r\xi) \leqq \beta h\left(\frac{x\cdot\xi}{k_{m_1}}\right)^{k_{m_1}} \quad \text{for } x\cdot\xi \geqq \beta_{m_1}.$$

Thus (7.7) shows that there is a constant E such that for every ξ

$$u_{m_1}(x) \leqq a_{k_{m_1}}(c^*(\xi), \xi; x\cdot\xi - k_{m_1}c^* - r - 1) \quad \text{for } x\cdot\xi \geqq E. \tag{7.8}$$

Since $k_{m_1} \geqq 1$,

$$a_{k_{m_1}}(c^*(\xi), \xi; -\infty) \geqq Q[\alpha] > \alpha \geqq u_{m_1}.$$

Since a_k is nonincreasing in s, and continuous in c and ξ, we can find a ρ so that for any given set $\xi_1, \cdots, \xi_K$ of unit vectors

$$u_{m_1}(x) \leqq a_{k_{m_1}}(c^*(\xi_i), \xi_i; x\cdot\xi_i - \rho)$$

for $x\cdot\xi_i \leqq E$ and $i = 1, \cdots, K$. By (7.8) this inequality then holds for all x.

Proposition 4.1 now shows that

$$u_n(x) \leqq a_{k_{m_1}+n-m_1}(c^*(\xi_i), \xi_i; x\cdot\xi_i - (n - m_1)c^*(\xi_i) - \rho)$$
$$\leqq a(c^*(\xi_i), \xi_i; x\cdot\xi_i - nc^*(\xi_i) + m_1c^*(\xi_i) - \rho),$$

as in (7.4), and (6.3) follows from the remainder of the proof of Lemma 7.1 and from Lemma 7.2.

If c^* is bounded and $\mathcal{S}$ is empty, let $\xi_1, \cdots, \xi_{2N}$ be the unit vectors in the coordinate directions and their negatives. We have the representation

$$\mathcal{S} = \bigcap_{\substack{\xi\in S^{N-1} \\ \varepsilon>0}} \{x | x\cdot\xi_i \leqq c^*(\xi_i) + \varepsilon, i = 1, \cdots, 2N, x\cdot\xi \leqq c^*(\xi) + \varepsilon\}$$

of $\mathscr{S}$ as an intersection of closed bounded sets. Since $\mathscr{S}$ is empty, Helly's theorem (the finite intersection property) shows that there are finitely many unit vectors $\xi_{2N+1}, \cdots, \xi_K$ and a positive ε such that the intersection of the halfspaces $\{x | x \cdot \xi_i \leqq c^*(\xi_i)+\varepsilon\}$, $i=1, \cdots, K$ is empty. Then for each y in R^N there is an i such that $y \cdot \xi_i > c^*(\xi_i)+\varepsilon$. We take any x, set $y = n^{-1}x$ and insert the resulting inequality into (7.4). Since a is nonincreasing in s, we find that for every x in $\mathscr{H}$

$$u_n(x) \leqq \max_{1 \leqq i \leqq K} a(c^*(\xi_i), \xi_i; \varepsilon n - \rho),$$

and the last statement of Theorem 6.1 follows from Proposition 5.2.

Thus Theorem 6.1 is proved.

Remark. When the dimension N is one, the argument of Lemma 7.1 gives the slightly stronger version of (6.3)

$$\limsup_{\sigma \to \infty} \max_{d(x, n\mathscr{S}) \geqq \sigma} u_n(x) \leqq \pi_0.$$

The statement (6.3) of Theorem 6.1 is still valid when $\mathscr{S}$ is unbounded, provided $\mathscr{S}'$ satisfies slightly stronger conditions. The proof can be found elsewhere [55].

The following example shows that the set $\mathscr{S}$ may, indeed, be empty.

Example 7.1. Let $Q[v]$ be $u(1, x)$, where u is the solution of the initial value problem (2.14) and

$$f(u) = u(1-u)(u-\tfrac{3}{4}).$$

Phase plane analysis (see, e.g., [5]) shows that $c^*(\xi) < 0$ for all ξ, so that $\mathscr{S}$ is empty. The results of [5] also show that if $0 \leqq u \leqq 1$ and $u \not\equiv 1$, then v approaches zero as $t \to \infty$. In particular, $u_n(x) = v(n, x)$ approaches zero as $n \to \infty$.

We also present an example to show that extra conditions such as (3.3) and (3.4), (3.5) or (3.6) are needed to keep initial data which lie above π_1 from propagating at a speed higher than $c^*(\xi)$.

Example 7.2. Let $N = 1$ and let $\mathscr{H}$ be the set of integers. Define

$$Q[u](x) = q(u(x)) + [4q(\tfrac{1}{4}) - 1] \sum_{k=1}^{\infty} (4q(\tfrac{1}{4}))^{-k} p(u(x-k)),$$

where

$$q(u) = \begin{cases} 2u - 3u^2 + 2u^3, & 0 \leqq u \leqq \tfrac{1}{2}, \\ \tfrac{1}{2}(u + \tfrac{1}{2}), & \tfrac{1}{2} \leqq u \leqq 1 \end{cases}$$

and

$$p(u) = \begin{cases} 0, & 0 \leqq u \leqq \tfrac{1}{2}, \\ (u - \tfrac{1}{2})^2, & \tfrac{1}{2} \leqq u \leqq 1. \end{cases}$$

Here $\pi_0 = 0$, $\pi_1 = \tfrac{1}{2}$, $\pi_+ = 1$.

Since $p = 0$ for $u < \tfrac{1}{2} = \pi_1$, we see that $c^*(\pm 1) = 0$. However, if

$$u_0(x) = \begin{cases} \tfrac{3}{4}, & \text{for } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$u_1(k) = [4q(\tfrac{1}{4}) - 1] p(\tfrac{3}{4}) [4q(\tfrac{1}{4})]^{-k} \quad \text{for } k > 0. \tag{7.9}$$

Because $q'' < 0$ for $u < \frac{1}{2}$, we see that $q(u) \geqq 4q(\frac{1}{4})u$ for $u \leqq \frac{1}{4}$. Consequently, if $u_n(k) \leqq \frac{1}{4}$ for $n < k$, then $u_{n+1}(k) \geqq 4q(\frac{1}{4})u_n(k)$. Starting with the values (7.9) at $n = 1$, we find that

$$u_k(k) \geqq [4q(\tfrac{1}{4}) - 1]p(\tfrac{3}{4})[4q(\tfrac{1}{4})]^{-1}$$

for all positive k. Thus an impulse of magnitude at least $\frac{3}{4}$ concentrated at 0 is propagated with a speed at least equal to 1, so that Theorem 6.1 is not valid.

It is not known whether this phenomenon can also occur when $u_0 \leqq \pi_1$.

8. Convergence to the equilibrium value. (Proof of Theorem 6.2). According to the hypotheses of Theorem 6.2 $\mathcal{S}$ has at least one interior point y_0. Then

$$y_0 \cdot \xi < c^*(\xi)$$

for all unit vectors ξ.

If $u_n(x)$ satisfies the recursion $u_{n+1} = Q[u_n]$ and if we define the sequence

$$v_n(x) = u_n(x + ny_0),$$

then v_n satisfies the recursion

$$v_{n+1} = T_{-y_0}Q[v_n].$$

By Proposition 5.3 $T_{-y_0}Q$ has the wave speed $c^*(\xi) - y_0 \cdot \xi$ in the direction ξ. Thus, the introduction of moving coordinates replaces Q by $T_{-y_0}Q$, $c^*(\xi)$ by $c^*(\xi) - y_0 \cdot \xi$, and the set $\mathcal{S}$ by its translate by $-y_0$. We then also translate $\mathcal{S}''$ by $-y_0$. It is easily seen that Theorem 6.2 for v_n in terms of $T_{-y_0}Q$ implies Theorem 6.2 for u_n in terms of Q.

Consequently we shall assume without loss of generality that $x = 0$ is an interior point of $\mathcal{S}$, so that

$$c^*(\xi) > 0$$

for all unit vectors ξ.

We now note that Theorem 6.2 is strengthened when $\mathcal{S}''$ is enlarged. We add to $\mathcal{S}''$ a closed neighborhood of the origin which also lies in the interior of $\mathcal{S}$. Because $\mathcal{S}$ is convex, we may replace this union by its convex hull and still obtain a closed bounded set in the interior of $\mathcal{S}$. It is well known that a closed bounded convex set may be approximated by a larger closed convex set whose boundary is analytic, has positive Gaussian curvature, and lies within an arbitrarily small distance of the original set (see [6, § 27]). Consequently we shall assume without loss of generality that *$\mathcal{S}''$ is a closed bounded subset of the interior of $\mathcal{S}$ whose boundary is analytic and has positive Gaussian curvature, and $x = 0$ is an interior point of $\mathcal{S}''$.*

We define the distance function

$$D(x) = \inf\left\{d > 0 \,\middle|\, \frac{1}{d}x \in \mathcal{S}''\right\},$$

$D(0) = 0$, so that $\mathcal{S}'' = \{x \mid D(x) \leqq 1\}$, and the boundary of $\mathcal{S}''$ has the polar coordinate representation $|x| = 1/D(x/|x|)$. We also define the vector field

$$\tau(x) = \frac{1}{|\operatorname{grad} D(x)|} \operatorname{grad} D(x).$$

Because the boundary of $\mathcal{S}''$ is analytic and has positive Gaussian curvature, $D(x)$ and $\tau(x)$ are analytic for $x \neq 0$. Moreover, $D(x)$ is positive homogeneous of degree one and $\tau(x)$ is positive homogeneous of degree zero. In fact, $\tau(x)$ is the unit outward

normal vector at that boundary point of $\mathscr{S}''$ which lies on the ray from the origin through x.

Because the boundary of $\mathscr{S}''$ has positive curvature, the implicit function theorem shows that the equation $D(x)\tau(x)=\xi$ can be inverted, and that the inverse function $x=\sigma(\xi)$ is analytic for $\xi\neq 0$. Since $|\tau(x)|=1$, $D(x)=|\xi|$. Hence, if $|\xi|=1$, $\sigma(\xi)$ lies on the boundary of $\mathscr{S}''$. It is the unique boundary point at which the unit outward normal is ξ.

The support function of $\mathscr{S}''$ is defined for $|\xi|=1$ by

$$S(\xi)=\max_{x\in\mathscr{S}''} x\cdot\xi.$$

It is easily seen that this maximum is attained at $x=\sigma(\xi)$. Thus, $S(\xi)=\xi\cdot\sigma(\xi)$, so that $S(\xi)$ is analytic.

Because of the definition of $S(\xi)$ and the homogeneity of $D(x)$,

$$\frac{x\cdot\xi}{D(x)S(\xi)}\leqq 1$$

for $x\neq 0$. Hence

$$D(x)=\max_{|\xi|=1}\frac{x\cdot\xi}{S(\xi)},$$

with equality when $\xi=\tau(x)$.

We suppose that $\mathscr{S}''$ contains a ball of radius ρ and lies in a ball of radius R, both centered at the origin. Then

$$\rho\leqq S(\xi)\leqq R \quad \text{for } |\xi|=1,$$

$$\frac{|x|}{R}\leqq D(x)\leqq\frac{|x|}{\rho} \quad \text{for all } x.$$

Because $\mathscr{S}''$ lies in the interior of $\mathscr{S}$, there is a positive constant ε such that the dilatation $(1+\varepsilon)\mathscr{S}''$ still lies in the interior of $\mathscr{S}$. It follows that

$$(1+\varepsilon)S(\xi)<c^*(\xi) \quad \text{for } |\xi|=1.$$

We shall begin the proof of Theorem 6.2 by constructing a comparison function which will force u_n to be uniformly positive on the set $n(1+\varepsilon)\mathscr{S}''$. This comparison function will be constructed out of the functions $a_n(c,\xi;s)$ which were defined in § 5.

We first choose some $\alpha\in(\pi_0,\pi_1)$ and a smooth nonincreasing function $\varphi(s)$ with the properties (5.1) and the additional property

$$\varphi(s)=\alpha \quad \text{for } s\leqq -1.$$

The family of nonincreasing functions $a_n((1+\varepsilon)S(\xi),\xi;s)$ is defined by the recursion (5.4) with $c=(1+\varepsilon)S(\xi)$. Since this c is less than $c^*(\xi)$, $a_n((1+\varepsilon)S(\xi),\xi;s)$ increases to the constant π_1 as $n\to\infty$. Because $a_n((1+\varepsilon)S(\xi),\xi;0)$ is continuous in ξ, Dini's theorem [9, p. 106] shows that the convergence is uniform on the bounded set $|\xi|=1$. Therefore there is an integer n_0 such that

$$a_n((1+\varepsilon)S(\xi),\xi;0)>\alpha \quad \text{for } n\geqq n_0, \quad |\xi|=1. \tag{8.1}$$

We wish to construct an N-dimensional comparison function of bounded support by patching together functions of one variable. For this purpose we shall first approximate $a_n((1+\varepsilon)S(\xi),\xi;s)$ by a function which is constant for small s and zero for large s. We begin by constructing a family of operators Q_k on B in the following manner.

Let $\zeta(s)$ be a smooth nonincreasing function of one variable with the properties

$$\zeta(s) = \begin{cases} 1 & \text{for } s \leqq \frac{1}{2}, \\ 0 & \text{for } s \geqq 1. \end{cases} \tag{8.2}$$

For any u in B define

$$Q_k[u](y) = Q\left[u(x+y)\zeta\left(\frac{|x|}{k}\right)\right](0), \tag{8.3}$$

where y is fixed and the argument of Q is considered as a function of x. We state the properties of this family of operators as a lemma.

LEMMA 8.1. *The family Q_k takes B into itself and has the following properties.*

(i) *Q_k has the properties* (3.1.i), (3.1.ii), (3.1.iv) *and* (3.1.v).

(ii) *For each u in B, the sequence $Q_k[u]$ is nondecreasing in k and converges to $Q[u]$ as $k \to \infty$.*

(iii) *$Q_k[u](x_0)$ depends only on the values of u in the ball of radius k centered at x_0. That is, if $u(x) = v(x)$ for $|x - x_0| \leqq k$, then*

$$Q_k[u](x_0) = Q_k[v](x_0).$$

Proof. Property (iii) is obvious from the definition. Property (i) follows immediately from the definition and the fact that Q satisfies (3.1). Property (ii) follows from the fact that for each fixed y the sequence $u(x+y)\zeta(|x|/k)$ increases to $u(x+y)$ uniformly on each bounded set as $k \to \infty$ together with (3.1.iv) and (3.1.v).

We now define the sequence $a_n^{(k)}(c, \xi; s)$ by means of the recursion

$$\begin{aligned} a_{n+1}^{(k)}(c, \xi; s) &= \max\{\varphi(s), Q_k[a_n^{(k)}(x \cdot \xi + s + c)](0)\}, \\ a_0^k(c, \xi; s) &= \varphi(s). \end{aligned} \tag{8.4}$$

Proposition 4.2 and the proof of Lemma 5.1 show that $a_n^{(k)}((1+\varepsilon)S(\xi), \xi; s)$ is continuous in ξ and s, nonincreasing in s, and nondecreasing in n. By Lemma 8.1, $a_n^{(k)}((1+\varepsilon)S(\xi), \xi; s)$ increases to $a_n((1+\varepsilon)S(\xi), \xi; s)$ as $k \to \infty$. Since all these functions are continuous, Dini's theorem states that the convergence is uniform on any bounded set of (ξ, s). In particular, we see from (8.1) that there is an integer k_0 such that

$$a_{n_0}^{(k_0)}((1+\varepsilon)S(\xi), \xi; 0) > \alpha \quad \text{for } |\xi| = 1. \tag{8.5}$$

We define the sequence $\alpha_n^{(k)}$ of constants by the recursion

$$\begin{aligned} \alpha_{n+1}^{(k)} &= Q_k[\alpha_n^{(k)}], \\ \alpha_0^{(k)} &= \alpha = \varphi(s) \quad \text{for } s \leqq -1. \end{aligned} \tag{8.6}$$

Then $\alpha_n^{(k)}$ is nondecreasing in k and converges to α_n, which is defined by the recursion $\alpha_{n+1} = Q[\alpha_n]$, $\alpha_0 = \alpha$, as $k \to \infty$.

By replacing the operator Q by $\min\{Q[u], \pi_1 - \tau(\pi_1 - u)\}$ with τ positive and small if necessary, we assume without loss of generality that $Q[\beta] < \pi_1$ for $\beta < \pi_1$. Then α_n is strictly increasing. In particular, $\alpha_{n_0+1} > \alpha_{n_0} \geqq \alpha_{n_0}^{(k)}$ for all k. We choose k_0 so large that in addition to (8.5) the inequality

$$\alpha_{n_0+1}^{(k_0)} > \alpha_{n_0} \geqq \alpha_{n_0}^{(k_0)} \tag{8.7}$$

is satisfied.

The advantage of $a_n^{(k_0)}$ over a_n is that it is constant for large s and for small s.

LEMMA 8.2. *$\alpha_n^{(k_0)}$ is increasing in n and for each n*

$$a_n^{(k_0)}(c, \xi; s) = \begin{cases} \alpha_n^{(k_0)} & \text{for } s \leqq -1 - n[k_0 + c], \\ 0 & \text{for } s \geqq n[k_0 - c]. \end{cases} \tag{8.8}$$

Proof. We note that, if $Q_k[\alpha] \leqq \alpha$, the proof of Proposition 4.2 shows that $\alpha_n^{(k)} \leqq \alpha$ for all n. On the other hand, by (3.1iii) $Q[\alpha] > \alpha$, so that $\alpha_n > \alpha$ for $n \geqq 1$. In particular $\alpha_{n_0} > \alpha$. Therefore we see from (8.7) that $Q_{k_0}[\alpha] > \alpha$ and then from Proposition 4.2 that $\alpha_n^{(k_0)}$ is increasing in n.

It is easily seen from the property (iii) of Lemma 8.1 and the fact that $\alpha_n^{(k_0)} \geqq \alpha \geqq \varphi$ that if (8.8) holds for n, it also holds for $n+1$. Since $a_0^{(k_0)}(c, \xi; s) = \varphi(s)$, about which we have assumed that (8.8) with $n = 0$ is valid, the lemma is proved.

Since $\alpha_n^{(k_0)} \leqq \alpha_n \leqq \pi_1$, and $\alpha_n^{(k_0)}$ is nondecreasing in n, the limit

$$\alpha^{(k_0)} = \lim_{n \to \infty} \alpha_n^{(k_0)} \tag{8.10}$$

exists. The proof of Lemma 5.3 shows that because of (8.5)

$$\lim_{n \to \infty} a_n^{(k_0)}((1+\varepsilon)S(\xi), \xi; s) \equiv \alpha^{(k_0)}. \tag{8.11}$$

The inequality (8.5) also implies that

$$a_n^{(k_0)}((1+\varepsilon)S(\xi), \xi; s) \geqq \varphi(s) \quad \text{for } n \geqq n_0.$$

Therefore the recursion (8.4) becomes

$$\begin{aligned} &a_{n+1}^{(k_0)}((1+\varepsilon)S(\xi), \xi; s) \\ &\quad = Q_{k_0}[a_n^{(k_0)}(x \cdot \xi + s + (1+\varepsilon)S(\xi))](0) \quad \text{for } n \geqq n_0, |\xi| = 1. \end{aligned} \tag{8.12}$$

We shall need the following lemma, in which we replace the variable s by $t = s/S(\xi)$.

LEMMA 8.3. *There is an integer $n_1 > n_0$ such that*

$$a_{n_1}^{(k_0)}((1+\varepsilon)S(\xi), \xi; S(\xi)t) \geqq a_{n_0}^{(k_0)}((1+\varepsilon)S(\xi'), \xi'; S(\xi')t) \quad \text{for } t \in \mathcal{R}, |\xi| = |\xi'| = 1. \tag{8.13}$$

Proof. We see from (8.7), (8.11), and Dini's theorem that there is an integer $n_1 > n_0$ such that

$$a_{n_1}^{(k_0)}\left((1+\varepsilon)S(\xi), \xi; n_0 k_0 \frac{R}{\rho}\right) > \alpha_{n_0}^{(k_0)} \quad \text{for } |\xi| = 1.$$

Because $a_{n_1}^{(k_0)}$ is nonincreasing in s, it follows that the same inequality holds for $s \leqq n_0 k_0 R/\rho$. Since $\rho \leqq S(\xi) \leqq R$ and $a_{n_0}^{(k_0)} \leqq \alpha_{n_0}^{(k_0)}$, the inequality (8.13) is then valid when $t \leqq n_0 k_0/\rho$. When $t \geqq n_0 k_0/\rho$, then $S(\xi')t \geqq n_0 k_0$ and because $c = (1+\varepsilon)S(\xi') > 0$ we see from (8.8) that the right-hand side of (8.13) is zero. Thus (8.13) is true for all t.

We recall that the ratio $x \cdot \xi/S(\xi)$ is homogeneous of degree zero in ξ and takes on its maximum value $D(x)$ when ξ is equal to the unit vector $\tau(x)$. Moreover, $S(\xi)$ is positive and analytic for $\xi \neq 0$. Then Taylor's theorem shows that $x \cdot \xi/S(\xi) = D(x) + O(|\xi - \tau(x)|^2)$ for ξ near $\tau(x)$. By looking at the bounded set $|\xi| = 1$, $|x| \leqq 1$ and using the homogeneity in x, we find that there is a constant m such that

$$\frac{x \cdot \xi}{S(\xi)} \geqq D(x) - m|x|\,|\xi - \tau(x)|^2 \quad \text{for } |\xi| = 1. \tag{8.14}$$

The function $\tau(x)$ is smooth for $x \neq 0$ and homogeneous of degree zero. Therefore if x and y are any two nonzero points in $\mathscr{R}^N$

$$|\tau(x)-\tau(y)| = \left|\tau\left(\frac{x}{|x|}\right)-\tau\left(\frac{y}{|y|}\right)\right| \leqq M\left|\frac{x}{|x|}-\frac{y}{|y|}\right|.$$

for some constant M. But

$$\left|\frac{x}{|x|}-\frac{y}{|y|}\right|^2 = 2\left(1-\frac{x\cdot y}{|x|\,|y|}\right) = \frac{1}{|x|\,|y|}[|x-y|^2-(|x|-|y|)^2] \leqq \frac{|x-y|^2}{|x|\,|y|},$$

so that

$$|\tau(x)-\tau(y)| \leqq M|x|^{-1/2}|y|^{-1/2}|x-y|. \tag{8.15}$$

We now choose a constant A which satisfies

$$A \geqq \rho^{-1}+n_1[k_0\rho^{-1}+1+\varepsilon]+(n_1-n_0)k_0\rho^{-1}+2mM^2\rho^{-1}\varepsilon^{-1}(n_1-n_0)^2k_0^2, \tag{8.16}$$

and define the comparison sequence

$$e_n(x) = a_{n_1}^{(k_0)}((1+\varepsilon)S(\tau(x)),\ \tau(x);\ S(\tau(x))[D(x)-A-(1+\tfrac{1}{2}\varepsilon)n]), \qquad n = 0, 1, 2, \cdots. \tag{8.17}$$

We recall that $\tau(x)$ is the unit outward normal to $\mathscr{S}''$ at the boundary point on the ray from the origin through x. Because $\rho \leqq S \leqq R$ and $c = (1+\varepsilon)S$, (8.8) shows that

$$e_n(x) = \begin{cases} \alpha_{n_1}^{(k_0)} & \text{for } D(x) \leqq A-\rho^{-1}-n_1[k_0\rho^{-1}+1+\varepsilon]+n(1+\tfrac{1}{2}\varepsilon), \\ 0 & \text{for } D(x) \geqq A+n_1[k_0\rho^{-1}-1-\varepsilon]+n(1+\tfrac{1}{2}\varepsilon). \end{cases} \tag{8.18}$$

Since $R^{-1}|x| \leqq D(x) \leqq \rho^{-1}|x|$, we see from (8.16) that $e_n = \alpha_{n_1}^{(k_0)}$ near $x = 0$, so that e_n is continuous, and e_n vanishes outside a bounded set. The most important property of e_n is given by the following lemma. As usual, we define $Q_{k_0}^r$ to be the rth iterate of the operator Q_{k_0}.

LEMMA 8.4. *If A satisfies the inequality* (8.16), *then the sequence $e_n(x)$ satisfies the inequality*

$$e_{n+n_1-n_0} \leqq Q_{k_0}^{n_1-n_0}[e_n] \quad \textit{for } n = 0, 1, 2, \cdots. \tag{8.19}$$

Proof. Since e_n is obtained from e_0 by replacing A by $A+(1+\frac{1}{2}\varepsilon)n$, which is at least as large as A, it is sufficient to prove (8.19) for $n = 0$.

Because $D(x) \leqq |x|/\rho$, we see from (8.18) that if

$$|x_0| \leqq \rho\{A-\rho^{-1}-n_1[k_0\rho^{-1}+1+\varepsilon]\}-(n_1-n_0)k_0,$$

then $e_0(x) = \alpha_{n_1}^{(k_0)}$ for $|x-x_0| \leqq (n_1-n_0)k_0$. Because of the definition (8.3) of Q_k we find that for such x_0

$$Q_{k_0}^{n_1-n_0}[e_0](x_0) = \alpha_{2n_1-n_0}^{(k_0)} \geqq \alpha_{n_1}^{(k_0)} \geqq e_{n_1-n_0}(x_0),$$

so that (8.19) with $n = 0$ is true.

We now consider a point x_0 where

$$|x_0| > \rho\{A-\rho^{-1}-n_1[k_0\rho^{-1}+1+\varepsilon]\}-(n_1-n_0)k_0, \tag{8.20}$$

and let x be any point such that

$$|x-x_0| \leqq (n_1-n_0)k_0.$$

We then see from (8.15) and (8.16) that

$$m|x|\,|\tau(x)-\tau(x_0)|^2 \leqq \tfrac{1}{2}\varepsilon.$$

The latter, together with (8.14), shows that

$$D(x) \leqq \frac{x\cdot\tau(x_0)}{S(\tau(x_0))}+\frac{1}{2}\varepsilon \quad \text{for } |x-x_0| \leqq (n_1-n_0)k_0.$$

Since $a_{n_1}^{(k_0)}$ is nonincreasing in s, it follows that when x_0 satisfies (8.20),

$$e_0(x) \geqq a_{n_1}^{(k_0)}\left((1+\varepsilon)S(\tau(x),\tau(x);S(\tau(x)))\left[\frac{x\cdot\tau(x_0)}{S(\tau(x_0)}-A+\frac{1}{2}\varepsilon\right]\right)$$
$$\text{for } |x-x_0| \leqq (n_1-n_0)k_0.$$

We now use the inequality (8.13) to see that

$$e_0(x) \geqq a_{n_0}^{(k_0)}((1+\varepsilon)S(\tau(x_0)),\tau(x_0);x\cdot\tau(x_0)-(A-\tfrac{1}{2}\varepsilon)S(\tau(x_0))$$
$$\text{for } |x-x_0| \leqq (n_1-n_0)k_0.$$

Since $x_0\cdot\tau(x_0)=S(\tau(x_0))D(x_0)$, it follows from the order-preserving property of Q_{k_0} and the recursion (8.12) that

$$\begin{aligned} Q_{k_0}^{n_1-n_0}[e_0](x_0) &\geqq a_{n_1}^{(k_0)}((1+\varepsilon)S(\tau(x_0)),\tau(x_0);x_0\cdot\tau(x_0) \\ &\quad -(A-\tfrac{1}{2}\varepsilon)S(\tau(x_0))-(n_1-n_0)(1+\varepsilon)S(\tau(x_0)) \\ &\geqq a_{n_1}^{(k_0)}((1+\varepsilon)S(\tau(x_0)),\tau(x_0);S(\tau(x_0)[D(x_0)-A-(n_1-n_0)(1+\tfrac{1}{2}\varepsilon)] \\ &= e_{n_1-n_0}(x_0). \end{aligned}$$

We have established the inequality (8.19) for all points x_0, and, if we replace A by $A+(1+\frac{1}{2}\varepsilon)n$, for all n, so that the lemma is proved.

Next we show that if u_0 is uniformly positive on a sufficiently large ball, then the sequence e_n can be used as a comparison sequence.

LEMMA 8.5. *For any $\sigma\in(\pi_0,\pi_1)$ there are a radius r_σ and an integer l_σ such that, if $u_0(x)\geqq\sigma$ on the ball $\{x\,|\,|x|\leqq r_\sigma\}$ and if $u_{n+1}=Q[u_n]$, then*

$$u_l(x) \geqq e_0(x) \quad \text{for } l_\sigma \leqq l < l_\sigma + n_1 - n_0.$$

Proof. Define the sequence of constants μ_n by the recursion

$$\mu_{n+1}=Q[\mu_n], \qquad \mu_0=\sigma.$$

Then μ_n increases to π_1. Choose an integer l_σ such that

$$\mu_{l_\sigma} > \alpha_{n_1}^{(k_0)} \geqq e_0(x). \tag{8.21}$$

We use a nonnegative smooth function $\zeta(s)$ with the properties (8.2) to define the one parameter family of functions $v_n^{(r)}$ by the recursion

$$v_{n+1}^{(r)}=Q[v_n^{(r)}], \qquad v_0^{(r)}(x)=\sigma\zeta\left(\frac{|x|}{r}\right).$$

By (3.1.v), $v_l^{(r)}(x)$ increases to μ_l as $r\to\infty$, and by Dini's theorem the convergence is uniform on bounded sets. Because of the inequality (8.21) there is a value r_σ of r such that

$$v_l^{(r_\sigma)} \geqq e_0 \quad \text{for } l_\sigma \leqq l < l_\sigma + n_1 - n_0$$

on the bounded set where $e_0 > 0$. Since $v_l^{(r_\sigma)}$ is nonnegative, these inequalities then hold for all x.

Since $v_0^{(r_\sigma)} = 0$ for $|x| \geqq r_\sigma$ and $v_0^{(r_\sigma)} \leqq \sigma$, Proposition 4.1 shows that if $u_0 \geqq \sigma$ for $|x| \leqq r_\sigma$, then $u_l \geqq v_l^{(r_\sigma)} \geqq e_0$ for $l_\sigma \leqq l \leqq l_\sigma + n_1 - n_0$, which proves the lemma.

We now show that if $u_0 \geqq \sigma$ on a ball of radius r_σ, then for all sufficiently large n the function u_n is uniformly positive on a large set.

LEMMA 8.6. *Suppose that, for some $\sigma \in (0, \pi_0)$ and some $\bar{x}$ in $\mathcal{R}^N$, $u_o(x) \geqq \sigma$ for $|x - \bar{x}| \leqq r_\sigma$. Then for all sufficiently large n*

$$u_n(x) \geqq \alpha_{n_1}^{(k_0)} \quad \text{when } D(x) \leqq (1 + \tfrac{1}{4}\varepsilon)n. \tag{8.22}$$

Proof. Since $Q[u] \geqq Q_{k_0}[u]$ for all u, we see from Lemma 8.5, (8.19), and Proposition 4.1 applied to the operator $Q_{k_0}^{n_1 - n_0}$ that for $0 \leqq q < n_1 - n_0$ and $j \geqq 0$

$$u_{l_\sigma + q + j(n_1 - n_0)}(x) \geqq e_{j(n_1 - n_0)}(x - \bar{x}).$$

By (8.18)

$$u_n(x) \geqq \alpha_{n_1}^{(k_0)} \tag{8.23}$$

$$\text{for } D(x - \bar{x}) \leqq A - \rho^{-1} - n_1[k_0\rho^{-1} + 1 + \varepsilon] + (n - l_\sigma - n_1)(1 + \tfrac{1}{2}\varepsilon).$$

We note that

$$D(x - \bar{x}) = \max_{|\xi|=1} \frac{(x - \bar{x}) \cdot \xi}{S(\xi)} \leqq \max_{|\xi|=1} \frac{x \cdot \xi}{S(\xi)} + \max \frac{-\bar{x} \cdot \xi}{S(\xi)} = D(x) + D(-\bar{x}). \tag{8.24}$$

That is, the function D is subadditive. Hence if $D(x) \leqq (1 + \frac{1}{4}\varepsilon)n$ and n is sufficiently large, the condition on $D(x - \bar{x})$ in (8.23) is satisfied, which proves the lemma.

We have shown that, if $u_0 > \sigma$ on a ball of radius r_σ, then when n is large u_n is uniformly greater than π_0 on the set $(1 + \frac{1}{4}\varepsilon)n\mathcal{S}''$. We shall now show that this fact implies that u_n is arbitrarily near π_1 on the set $n\mathcal{S}''$.

LEMMA 8.7. *Suppose that $u_0(x) \geqq \sigma \in (\pi_0, \pi_1)$ on a ball $|x - \bar{x}| \in r_\sigma$. Then for any positive δ there is an n_δ such that if $m \geqq n_\delta$ and $D(x) \leqq m$, then $u_m(x) \geqq \pi_1 - \delta$.*

Proof. Because the sequence α_n defined by the recursion $\alpha_{n+1} = Q[\alpha_n]$, $\alpha_0 = \alpha$ converges to π_1, there is an integer n_2 such that $\alpha_{n_2} > \pi_1 - \delta$.

Let the sequence $w_n^{(r)}$ be defined by the recursion

$$w_{n+1}^{(r)} = Q[w_n^{(r)}], \qquad w_0^{(r)}(x) = \alpha\zeta(|x|/r)$$

where ζ is a smooth nonnegative function with the properties (8.2). Then $w_{n_2}^{(r)}(0)$ converges to α_{n_2} as $r \to \infty$. Hence there is an $\bar{r}$ such that

$$w_{n_2}^{(\bar{r})}(0) > \pi_1 - \delta.$$

Proposition 4.1 and the translation invariance of Q now show that if $u_n(x) \geqq \alpha$ on the ball $|x - x_1| \leqq \bar{r}$, then

$$u_{n+n_2}(x_1) > \pi_1 - \delta. \tag{8.25}$$

Because of the subadditivity (8.24), and the bound $D(x) \leqq |x|/\rho$, the inequalities $D(x_1) \leqq n + n_2$ and $|x - x_1| \leqq \bar{r}$ imply that

$$\begin{aligned} D(x - \bar{x}) &\leqq D(x_1) + D(x - x_1) + D(-\bar{x}) \\ &\leqq n + n_2 + \rho^{-1}(\bar{r} + |\bar{x}|). \end{aligned}$$

When n is sufficiently large, say $n \geqq n_\delta - n_2$, this inequality implies the condition in (8.23). Therefore $u_n(x) \geqq \alpha_{n_1}^{(k_0)} \geqq \alpha$ for $|x - x_1| \leqq \bar{r}$, and the inequality (8.25) follows.

Thus when $n+n_2 \geqq n_\delta$ and $D(x_1) \leqq n+n_2$, we have $u_{n+n_2}(x_1) \geqq \pi_1 - \delta$. This is the statement of the lemma with $m = n+n_2$.

Equation (6.4) is an immediate consequence of this lemma. Equation (6.5) is an immediate consequence of (6.4), (3.2), and Proposition 4.1. Thus, Theorem 6.2 is proved.

The following example of Fujita [18] shows that the radius r_σ in Theorem 6.2 must, in general, depend upon σ.

Example 8.1. Let $Q[u](x) = v(x, 1)$, where $v(x, t)$ is the solution of the N-dimensional Fisher equation

$$\frac{\partial v}{\partial t} = \Delta v + f(v)$$

with the initial values $v(x, 0) = u(x)$. If $f(0) = f(1) = 0$ and $0 < f(v) \leqq v^4$, the hypotheses (3.1), (3.3) and (3.5) are valid with $\pi_0 = 0$, $\pi_1 = \pi_+ = 1$. The function

$$w(x, t) = [6(t+1)]^{-1/3}\, e^{-|x|^2/4(t+1)}$$

satisfies the differential inequality $w_t \geqq \Delta w + f(w)$. It follows from the maximum principle for parabolic equations [43, Thm. 3.10] that if $u(x_0) = w(x, 0)$, then $u_n(x) \leqq w(x, n)$. Thus u_n does not approach π_1 and, in fact, goes to zero. The explanation is that for each σ the set where $u_0 \geqq \sigma$ does not contain a sufficiently large ball. We conclude that $r_\sigma > 2|\log 6^{1/3}\sigma|^{1/2}$, which approaches infinity as σ goes to 0.

The following variant of Theorem 6.2 gives a condition under which r_σ does not need to vary with σ. It will be used in the proof of Theorem 6.4.

LEMMA 8.8. *Suppose that Q satisfies the conditions of Theorem* 6.2 *and that it has the additional property that*

$$Q[\rho u] \geqq \rho Q[u] \tag{8.26}$$

for every constant ρ with $0 \leqq \rho \leqq 1$. Then there is a fixed radius r such that Theorem 6.2 *is valid when r_σ is replaced by r for any positive σ.*

Proof. Theorem 6.2 was proved by showing that when $u_0 \geqq \sigma$ on a ball of radius σ, then a translate of some u_0 lies above the first member e_0 of a sequence with the properties (8.18) and (8.19).

The additional property (8.26) shows that once one has constructed such a sequence, the sequence ρe_n with $\rho \in (0, 1)$ again satisfies (8.19) and (8.18) with the positive constant $\alpha_{n_1}^{(k_0)}$ replaced by the positive constant $\rho\alpha_{n_1}^{(k_0)}$.

Let $e_0 = 0$ outside the ball $|x| \leqq r$. Then if u_0 is uniformly positive in a ball $|x - \bar{x}| \leqq r$ of radius r, there is a positive ρ such that

$$u_0(x) \geqq \rho e_0(x - \bar{x}).$$

Proposition 5.1 then shows that

$$u_{m(n_1-n_0)})(x) \geqq \rho e_{m(n_1-n_2)})(x - \bar{x}) \quad \text{for } m \geqq 0.$$

Because of (8.19) it follows that there is an m_0 such that for $u_{m_0}(n_1 - n_0) \geqq \sigma$ on a ball of radius r_σ, where $\sigma = \rho\alpha_{n_1}^{(k_0)}$.

The proof of Lemma 8.6 still gives (8.22) for sufficiently large n, and the proof of Theorem 6.2 is finished as above.

9. Bounds for the wave speed. (Proof of Theorems 6.3 and 6.4). Theorems 6.3 and 6.4 will follow from Proposition 5.5 when we determine the wave speed of the

comparison operator

$$M[u](x) \equiv \min\left\{\beta, \int_{\mathscr{H}} u(x-y)m(y, dy)\right\}. \tag{9.1}$$

Here m is a nonnegative measure on $\mathscr{H}$ with the properties $\int_{\mathscr{H}} m(x, dx) < \infty$ and

$$\int_{\mathscr{H}} m(x, dx) > 1, \tag{9.2}$$

and β is any positive constant or $+\infty$. (In the latter case, M is linear.) M is defined on the set of continuous nonnegative functions. It has the properties (3.1) with $\pi_0 = 0$ and $\pi_1 = \beta$, so that its wave speed $\bar{c}^*(\xi)$ is defined. We shall obtain a formula for this wave speed.

LEMMA 9.1. *For each unit vector ξ the wave speed $\bar{c}^*(\xi)$ of M is given by the formula*

$$\bar{c}^*(\xi) = \inf_{\mu>0}\left\{\frac{1}{\mu}\log\int e^{\mu x\cdot\xi} m(x, dx)\right\}, \tag{9.3}$$

which is independent of β.

Moreover the set

$$\bar{\mathscr{S}} = \{x \mid x\cdot\xi \leqq \bar{c}^*(\xi) \forall \xi \in S^{N-1}\}$$

is not empty, and its interior is empty if and only if the measure m is concentrated on a hyperplane $x\cdot\xi =$ constant.

Proof. Consider a fixed unit vector ξ. For every $\mu > 0$. define

$$\Phi(\mu) = \frac{1}{\mu}\log\int e^{\mu x\cdot\xi} m(x, dx) \tag{9.4}$$

with $\Phi(\mu) = +\infty$ if the integral on the right diverges.

For any μ such that $\Phi(\mu)$ is finite, consider the function of one variable

$$w(s) = \min\{\beta, e^{-\mu s}\}.$$

Then $w(s) \leqq e^{-\mu s}$, and hence

$$M[w(x\cdot\xi + s + \Phi(\mu))](0) \leqq \int e^{-\mu(-y\cdot\xi + s + \Phi(\mu))} m(y, dy) = e^{-\mu s}$$

because of the definition (9.4) of $\Phi(\mu)$. The definition (9.1) of M also shows that $M[w(x\cdot\xi + s + \Phi(\mu))] \leqq \beta$. Consequently

$$M[w(x\cdot\xi + s + \Phi(\mu))](0) \leqq w(s).$$

If $\varphi(s)$ is any function with the properties (5.1) such that $\varphi(s) \leqq w(s)$, then also

$$\max\{\varphi(s), M[w(x\cdot\xi + s + \Phi(\mu))](0)\} \leqq w(s).$$

Therefore if the sequence $a_n(\Phi(\mu), \xi; s)$ is defined by the recursion (5.4), Proposition 4.1 shows that $a_n \leqq w$ for all n. Hence $a(\Phi(\mu), \xi; s) \leqq w(s)$. It follows that $a(\Phi(\mu), \xi; +\infty) = 0$. We conclude that

$$\bar{c}^*(\xi) \leqq \Phi(\mu) \tag{9.5}$$

for all positive μ, so that inf $\Phi(\mu)$ provides an upper bound for the wave speed $\bar{c}^*$.

We now wish to show that the infimum of $\Phi(\mu)$ is equal to $\bar{c}^*$. We shall assume for the moment that the measure m vanishes outside a ball $|x| \leqq b$. Then $\Phi(\mu)$ is

analytic in μ. We define the function

$$\psi(\mu) = (\mu\Phi)' = \frac{\int x \cdot \xi\, e^{\mu x \cdot \xi} m(x, dx)}{\int e^{\mu x \cdot \xi} m(x, dx)}. \tag{9.6}$$

Differentiation shows that

$$\psi'(\mu) = e^{-\mu\Phi(\mu)} \int [x \cdot \xi - \psi(\mu)]^2 e^{\mu x \cdot \xi} m(x, dx) \geqq 0, \tag{9.7}$$

so that ψ is nondecreasing. Moreover by (9.6)

$$\Phi'(\mu) = \frac{1}{\mu}[\psi - \Phi], \tag{9.8}$$

so that

$$(\mu^2\Phi')' = \mu\psi' \geqq 0. \tag{9.9}$$

We see from the definition (9.4) and the condition (9.2) that $\Phi(\mu) \to +\infty$ as μ decreases to zero, while $\psi(\mu)$ remains bounded as $\mu \to 0$ because the support of m is bounded. Then (9.8) shows that Φ is decreasing near $\mu = 0$, and (9.9) shows that Φ' changes its sign at most once.

It is easily seen that if

$$K = \sup \{x \cdot \xi | x \in \text{supp } m\},$$

where supp m is the support of m, then

$$\lim_{\mu \to +\infty} \Phi(\mu) = \lim_{\mu \to +\infty} \psi(\mu) = K. \tag{9.10}$$

Moreover,

$$\Phi(\mu) = K + \frac{1}{\mu} \log \left\{ \int_{x \cdot \xi < K} e^{\mu(x \cdot \xi - K)} m(x, dx) + \int_{x \cdot \xi = K} m(x, dx) \right\},$$

and the first integral approaches zero as $\mu \to \infty$. Consequently, there are two possibilities:

(a) If

$$\int_{x \cdot \xi = K} m(x, dx) \geqq 1,$$

then $\Phi(\mu) \geqq K \geqq \psi(\mu)$ for all μ. Φ is nonincreasing by (9.8), and

$$\inf_{\mu > 0} \Phi(\mu) = \Phi(\infty) = K = \psi(\infty).$$

(b) If

$$\int_{x \cdot \xi = K} m(x, dx) < 1,$$

then $\Phi(\mu) < K$ for all sufficiently large μ. Hence there is a $\mu^* > 0$ such that $\Phi' < 0$ for $\mu < \mu^*$ and $\Phi' \geqq 0$ for $\mu > \mu^*$. Then

$$\inf_{\mu > 0} \Phi(\mu) = \Phi(\mu^*) = \psi(\mu^*), \tag{9.11}$$

and by (9.8)

$$\Phi(\mu) > \psi(\mu) \quad \text{for } \mu < \mu^*. \tag{9.12}$$

For the case (a) we define $\mu^* = +\infty$, so that (9.11) and (9.12) are also valid for case (a).

We now choose any $\mu \in (0, \mu^*)$, and introduce a positive parameter γ which is so small that the integral

$$\int e^{\mu y \cdot \xi} \cos \gamma y \cdot \xi m(y, dy)$$

is positive. We define the constant

$$z = \frac{1}{\gamma} \tan^{-1} \frac{\int e^{\mu y \cdot \xi} \sin \gamma y \cdot \xi m(y, dy)}{\int e^{\mu y \cdot \xi} \cos \gamma y \cdot \xi m(y, dy)}. \tag{9.13}$$

Clearly,

$$\lim_{\gamma \to 0} z = \psi(\mu).$$

We suppose that γ is so small that

$$\gamma(b + |z|) < \pi, \tag{9.14}$$

where $m(x, dx) = 0$ for $|x| \geqq b$.

For any ε such that $0 < \varepsilon \leqq \beta$ we define the function

$$v(s) = \begin{cases} \varepsilon e^{-\mu s} \sin \gamma s & \text{for } 0 \leqq s \leqq \dfrac{\pi}{\gamma}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$M[v(x \cdot \xi + s + z)](0) = \min \left\{ \beta, \int v(-y \cdot \xi + s + z)\, m(y, dy) \right\}.$$

When $0 \leqq s \leqq \pi/\gamma$ and $|y| \leqq b$, (9.14) implies that $-\pi/\gamma \leqq -y \cdot \xi + s + z \leqq 2\pi/\gamma$ so that

$$v(-y \cdot \xi + s + z) \geqq \varepsilon e^{-\mu(-y \cdot \xi + s + z)} \sin \gamma(-y \cdot \xi + s + z).$$

We then see from the definition (9.13) of z that for $0 < s < \pi/\gamma$

$$M[v(x \cdot \xi + s + z)] \geqq \min \left\{ \beta, \varepsilon \int e^{-\mu(-y \cdot \xi + s + z)} \sin \gamma(-y \cdot \xi + s + z)\, m(y, dy) \right\}$$

$$= \min \left\{ \beta, e^{-\mu z} \sec \gamma z \int e^{\mu y \cdot \xi} \cos \gamma y \cdot \xi m(y, dy) v(s) \right\}.$$

As γ goes to zero, the coefficient of $v(s)$ approaches $e^{\mu[\Phi(\mu) - \psi(\mu)]}$, which is greater than 1 because $\mu < \mu^*$. Since $v \leqq \varepsilon \leqq \beta$, we find that when γ is sufficiently small,

$$M[v(x \cdot \xi + s + z)] \geqq v(s) \tag{9.15}$$

for $0 \leqq s \leqq \pi/\gamma$. Since v vanishes outside this interval, this inequality is valid for all s.

The function v attains its maximum at a unique point $\bar{s} = \gamma^{-1} \tan^{-1}(\gamma/\mu)$. We define the function

$$\varphi(s) = \begin{cases} v(\bar{s}) & \text{for } s \leqq -\dfrac{\pi}{\gamma} + \bar{s}, \\ v\left(s + \dfrac{\pi}{\gamma}\right) & \text{for } s \geqq -\dfrac{\pi}{\gamma} + \bar{s}, \end{cases}$$

which clearly has the properties (5.1).
Moreover,

$$\varphi(s) = \max\left\{ v(s-t) \middle| t \leqq -\frac{\pi}{\gamma} \right\}.$$

Therefore (9.15) shows that

$$M[\varphi(x \cdot \xi + s + z)](0) \geqq \max\left\{ v(s-t) \middle| t \leqq -\frac{\pi}{\gamma} \right\} = \varphi(s).$$

For any $c < z$ we define the sequence $a_n(c, \xi; s)$ by the recursion (5.4) with Q replaced by M. Then

$$\begin{aligned} a_1(c, \xi; s) &= \max\{\varphi(s), M[\varphi(x \cdot \xi + s + c)]\} \geqq \max\{\varphi(s), \varphi(s + c - z)\} \\ &= \varphi(s + c - z) = a_0(c, \xi; s + c - z). \end{aligned}$$

Proposition 4.1 now shows that $a_{n+1}(c, \xi; s) \geqq a_n(c, \xi; s + c - z)$, so that $a(c, \xi; s) \geqq a(c, \xi; s + c - z)$. Since a is nonincreasing and $c - z < 0$, this implies that a is a constant. Therefore the inequality $c < z$ implies that $c < \bar{c}^*(\xi)$.

We conclude that when γ is sufficiently small, $z \leqq \bar{c}^*(\xi)$. Taking the limit of z as γ decreases to 0, we find that when $\mu < \mu^*$

$$\bar{c}^*(\xi) \geqq \psi(\mu).$$

We recall that as μ increases to μ^*, $\psi(\mu)$ increases to the infimum of $\Phi(\mu)$.

Thus we have shown that when the measure m has bounded support,

$$\bar{c}^*(\xi) \geqq \inf_{\mu > 0} \Phi(\mu). \tag{9.16}$$

If m does not have bounded support we choose a nonincreasing function ζ with the properties (8.2) and approximate m by the nonnegative measure

$$m_k(x, dx) = \zeta\left(\frac{|x|}{k}\right) m(x, dx),$$

which has bounded support. Clearly the sequence of measures m_k increases to m and for all sufficiently large k the measure m_k has the property (9.2). We define the operator M_k and the function $\Phi_k(\mu)$ by replacing m by m_k in the definitions (9.1) and (9.4) and let $\bar{c}_k^*(\xi)$ be the wave speed of M_k. Obviously, $M \geqq M_k$. Therefore by Proposition 5.5 $\bar{c}^*(\xi) \geqq \bar{c}_k^*(\xi)$, so that by (9.16)

$$\bar{c}^*(\xi) \geqq \inf_{\mu > 0} \Phi_k(\mu) \tag{9.17}$$

for all k. The right-hand side is nondecreasing in k, and its limit again provides a lower bound for $\bar{c}^*(\xi)$.

Suppose that ρ is any number which is above this limit. That is, suppose that

$$\rho > \inf_{\mu>0} \Phi_k(\mu)$$

for all k. Then the sets

$$S_k = \{\mu > 0 | \Phi_k(\mu) \leqq \rho\}$$

are nonempty, and S_k contains S_{k+1}. When k is so large that m_k satisfies (9.2), $\Phi_k(0) = +\infty$, so that S_k is closed.

If $x \cdot \xi$ is not bounded above on the support of m, we see from (9.10) that for sufficiently large k the set S_k is bounded above. We then see from Helly's theorem (the finite intersection property) that the intersection S_∞ of all the S_k is not empty. Since Φ_k converges to Φ, we see that $\Phi(\mu) \leqq \rho$ for $\mu \in S_\infty$; so that $\inf \Phi \leqq \rho$.

We conclude that if $x \cdot \xi$ is not bounded above on the support of m,

$$\inf_{\mu>0} \Phi(\mu) = \lim_{k\to\infty} \inf_{\mu>0} \Phi_k(\mu). \tag{9.18}$$

If, on the other hand, $x \cdot \xi$ is bounded above on the support of m, then for all sufficiently large k, $\sup\{x \cdot \xi | x \in \operatorname{supp} m_k\} = K$ which is independent of k. There are then two possibilities:

If

$$\int_{x\cdot\xi=K} m(x, dx) < 1,$$

then by the argument we have used earlier $\inf \Phi_k \leqq \inf \Phi < \Phi(\infty) = K = \Phi_k(\infty)$ for large k. Therefore we may carry out the above argument when

$$\inf \Phi_k < \rho < K \quad \text{for all } k \tag{9.19}$$

to reach the conclusion (9.18).

When

$$\int_{x\cdot\xi=K} m(x, dx) \geqq 1,$$

we easily see that $\Phi(\mu) \geqq K$ and $\Phi(\infty) = K$. If there were a ρ which satisfies (9.19), the same argument would lead to the conclusion that $\inf \Phi(\mu) \leqq \rho$, which is incompatible with (9.19) since $\inf \Phi = K$. Therefore (9.19) leads to a contradiction, and we conclude that

$$\lim_{k\to\infty} \inf_{\mu>0} \Phi_k(\mu) = K = \inf_{\mu>0} \Phi.$$

We have thus proved (9.18) in all cases.

The formula (9.3) now follows from (9.5), (9.17) and (9.18). Note that our proof is valid even when $\bar{c}^*(\xi) = +\infty$.

To prove that $\bar{\mathcal{F}}$ is not empty we again begin by considering the case where m has bounded support. We define the "drift velocity"

$$V = \frac{\int x m(x, dx)}{\int m(x, dx)}. \tag{9.20}$$

We note that for any unit vector ξ, $\psi(0) = V \cdot \xi$.

We recall that there is a $\mu^* \in (0, \infty]$ such that $\bar{c}^*(\xi) = \psi(\mu^*)$. Since (9.7) shows that ψ is nondecreasing, we conclude that

$$V \cdot \xi \leqq \bar{c}^*(\xi) \tag{9.21}$$

for all ξ. This shows that V lies in $\bar{\mathcal{S}}$ so that $\bar{\mathcal{S}}$ is not empty.

If the support of m is not concentrated on a plane normal to ξ, then (9.7) shows that ψ is strictly increasing so that

$$V \cdot \xi < c^*(\xi). \tag{9.22}$$

If the support of m is not contained in any plane, this inequality is valid for all ξ so that V is an interior point of $\bar{\mathcal{S}}$.

If the support of m is unbounded, we again define the measure $m_k = \zeta(|x|/k)m$ of bounded support and choose k so large that $\int m_k > 1$. If the support of m does not lie in any plane, we choose k so large that the same is true of m_k. We let

$$V_k = \frac{\int x m_k(x, dx)}{\int m_k(x, dx)}.$$

Then by the above arguments

$$V_k \cdot \xi \leqq \bar{c}_k^*(\xi) \leqq \bar{c}^*(\xi)$$

for all ξ, so that $\bar{\mathcal{S}}$ is not empty. If the support of m_k is not concentrated on any plane, then

$$V_k \cdot \xi < \bar{c}_k^*(\xi) \leqq \bar{c}^*(\xi)$$

for all ξ, so that V_k is an interior point of $\bar{\mathcal{S}}$.

The formula (9.3) shows that if the support of m lies in the plane $x \cdot \xi_0 = A$, then $\bar{c}^*(\xi_0) = A$ and $\bar{c}^*(-\xi_0) = -A$. Therefore $\bar{\mathcal{S}}$ again lies in the plane $x \cdot \xi_0 = A$, so that $\bar{\mathcal{S}}$ cannot have interior points.

We have thus proved all the statements of the lemma.

Theorem 6.3 is now an immediate consequence of Lemma 9.1 and Proposition 5.5 with $Q_1 = Q$ and $Q_2 = M$ where $\beta = \pi_1$.

The first two parts of Theorem 6.4 follow from Lemma 9.1 and Proposition 5.5 with $Q_2 = Q$ and $Q_1 = M$ where m is replaced by l and β by ε in the definition (9.1) of M. We must, of course, use the observation that increasing the function c^* enlarges the set $\mathcal{S}$.

To prove the last statement of Theorem 6.4 we first note that the definition (9.1) implies that $M[\rho u] \geqq \rho M[u]$ when ρ is any constant such that $0 \leqq \rho \leqq 1$. Thus M satisfies the conditions of Lemma 8.8. Because $Q \geqq M$, Lemma 8.8 implies that if u_0 is uniformly positive on a ball of radius r, then $u_n \geqq \frac{1}{2}\varepsilon$ on a set $n\bar{\mathcal{S}}''$ when n is sufficiently large. In particular, there is an m_0 such that $u_{m_0} \geqq \frac{1}{2}\varepsilon$ on a ball of radius $r_{\varepsilon/2}$, so that the proof of Lemma 8.6 still implies (8.22). The remainder of the proof of Theorem 6.2 then yields the last part of Theorem 6.4.

If, as in the corollary, the same measure m gives both the upper bound (6.6) and the lower bound (6.9) for arbitrarily small δ, then the linear operator M given by (9.1) with $\beta = \infty$ is the Fréchet derivative of Q at $u = 0$ in the space $C[\mathcal{H}]$ of continuous functions with the maximum norm.

For example if $Q[v]$ is again the value $u(\tau, x)$ of the solution u of (2.14) with $u(0, x) = v(x)$, its Fréchet derivative $M[v]$ is the solution at $t = \tau$ of the corresponding problem in which $f(u)$ is replaced by $f'(0)u$. The maximum principle for parabolic

equations [43, Thm. 3.6.10] shows that if $f(u) \leqq f'(0)u$ for $u \geqq 0$, then (6.7) and (6.9) are valid with $m(x, dx) = C_N \tau^{-N/2} \exp[f'(0)\tau - |x|^2/4\tau]dx$ where C_N is a known constant. The formula (6.10) gives $c^* = 2\sqrt{f'(0)\tau}$, which becomes the usual wave speed for the Fisher equation [15], [31] when τ is set equal to 1.

In the case of various integral operators the formula (6.10) has been found by Aronson [1], Diekmann [10], [11], Diekmann and Kaper [12], Thieme [48], [49] and the author [53], [54]. In a different context it was discovered by Hammersley [22].

In fact, as far as I know, the only explicit formula for an asymptotic speed which does not come from the corollary is one given by Hadeler and Rothe [20] for the Fisher equation in which $f(u)$ is a cubic polynomial.

The following example shows that the Fréchet derivative of Q at $u = 0$ need not satisfy the inequality (6.9) for arbitrarily small δ, even if it satisfies (6.6).

Example 9.1. Let $N = 1$, $\mathscr{H} = \{0, \pm 1, \cdots\}$. Define

$$Q[u](k) = \sinh 1 \sum_{j=-\infty}^{\infty} e^{-|j|}(1 - e^{-|j|u(k-j)}),$$

which satisfies the hypotheses (3.1) with $\pi_0 = 0$, $\pi_1 = \pi_+ = 1$. The Fréchet derivative at $u = 0$ is easily found to be

$$M[u](k) = \sinh 1 \sum_{j=-\infty}^{\infty} |j| e^{-|j|} u(k-j)$$

and the bound (6.6) is a consequence of the fact that the second derivative of the function $1 - e^{-t}$ is negative.

Let $\varepsilon > 0$, $\delta \in (0, 1)$, and $u(j) = \varepsilon \delta_{j0}$. Then

$$Q[u](k) - (1-\delta)M[u](k) = \sinh 1 \, e^{-|k|}(1 - (1-\delta)\varepsilon|k| - e^{-\varepsilon|k|}),$$

which is negative for all sufficiently large values of $|k|$. Thus the inequality (6.9) is not true for any positive value of ε.

10. The hairtrigger effect. (Proof of Theorem 6.5). The proof of Theorem 6.5 is based on the simple observation that when u and l are nonnegative

$$\operatorname{supp}\left\{\int u(x-y)l(y, dy)\right\} = \operatorname{supp}(u) + \operatorname{supp}(l). \tag{10.1}$$

(We define the sum $V + W$ of two sets of vectors by

$$V + W = \{v + w | v \in V, w \in W\}.)$$

We see from (10.1) and Proposition 4.1 that, if $u_{n+1} = Q[u_n]$ and Q satisfies the inequality $Q[u] \geqq \int u(x-y)l(y, dy)$ for $0 \leqq u \leqq \varepsilon$, then

$$\operatorname{supp}(u_n) \supset \operatorname{supp}(u_0) + n \operatorname{supp}(l).$$

Suppose that m_0 is so large that the support of e_0, the function defined by (8.17), lies in the interior of a translate of $m_0 \operatorname{supp}(l)$. Then if $u_0 \not\equiv 0$ there are a $\rho \in (0, 1]$ and an $\bar{x}$ in R^N such that $u_{m_0}(x - \bar{x}) \geqq \rho e_0(x)$. Since $M[\rho u] \geqq \rho M[u]$ for $\rho \in [0, 1]$, the proof of Lemma 8.8 gives Theorem 6.5.

The following example shows that the extra condition in Theorem 6.5 is needed to produce the hairtrigger.

Example 10.1. If $N = 1$, $\mathscr{H} = \{0, \pm 1, \pm 2, \cdots\}$, and

$$Q[u](x) = \tfrac{1}{2}u(x-1)[2 - u(x-1)] + \tfrac{1}{2}u(x+1)[2 - u(x+1)],$$

Q satisfies the hypotheses of Theorem 6.4 with $\pi_0 = 0$, $\pi_1 = 1$, $\varepsilon = \frac{1}{2}$ and

$$l(x, dx) = \tfrac{3}{4}[\delta(x-1) + \delta(x+1)]$$

where δ is the usual Dirac measure. The set n supp (l) consists of those points whose coordinates are integers of the same parity as n. Hence the hypotheses of Theorem 6.5 are not satisfied. In fact, if $u_0(x) = 1$ for $x = 0$ and 0 elsewhere, $u_n(x) = 0$ when $n + x$ is odd. However, if u_0 is positive at two points of opposite parities, and in particular, at two adjacent points, the above argument leads to the conclusion of Theorem 6.4.

Example 10.2. If $\mathscr{H} = \{0, \pm 1, \pm 2, \cdots\}$,

$$Q[u] = \tfrac{1}{3}u(x-1) + \tfrac{1}{3}u(x)[2 - u(x)] + \tfrac{1}{3}u(x+1)$$

and

$$l = \tfrac{1}{3}[\delta(x-1) + \tfrac{3}{2}\delta(x) + \delta(x+1)],$$

the hypotheses of Theorem 6.5 are satisfied with $\varepsilon = \frac{1}{2}$, $\pi_0 = 0$ and $\pi_1 = 1$. However if $\mathscr{H}$ is the whole real line the hypotheses of Theorem 6.4 are satisfied but those of Theorem 6.5 are not. In fact, if $u_0(x)$ is positive for $-\frac{1}{4} < x < \frac{1}{4}$ and zero elsewhere, $u_n(x)$ is zero when x lies at distance at least $\frac{1}{4}$ from any integer, so that the statement of Theorem 6.5 is not valid for this Q.

Thus while the hypotheses of Theorems 6.1 to 6.4 are preserved when Q is extended from $\mathscr{H}$ to $\mathscr{R}$, the same need not be true of the extra hypothesis in Theorem 6.5.

11. The existence of traveling waves. (Proof of Theorem 6.6). We choose a function φ with the properties 5.1. For each positive integer k we define the sequence $a_n(c, \xi, k; s)$ by the recursion

$$\begin{aligned} &a_{n+1}(c, \xi, k; s) = \max\{k^{-1}\varphi(s), Q[a_n(c, \xi, k; x \cdot \xi + s + c](0)\}, \\ &a_0(c, \xi, k; s) = k^{-1}\varphi(s). \end{aligned} \tag{11.1}$$

As in § 5, we find that $a_n(c, \xi, k; s)$ is nonincreasing in c, k, and s and nondecreasing in n. As $n \to \infty$ it converges to a limit function $a(c, \xi, k; s)$, which is nonincreasing in c, k, and s. By Proposition 5.2

$$\begin{aligned} &\lim_{s \to -\infty} a(c, \xi, k; s) = \pi_1, \\ &\lim_{s \to +\infty} a(c, \xi, k; s) = 0 \quad \text{for } c \geqq c^*(\xi). \end{aligned} \tag{11.2}$$

We now use the additional property (3.7): Every sequence $Q[v_n]$ with $v_n \leqq \pi_1$ has a subsequence $Q[v_{n_i}]$ which converges uniformly on each bounded subset of $\mathscr{H}$.

Then for any given real number t there is a sequence n_i such that the sequence $Q[a_{n_i}(c, \xi, k; x \cdot \xi + t + c)](y)$ converges uniformly for y on bounded subsets of $\mathscr{H}$. Since the sequence a_n is nonincreasing in n and Q is order preserving, it follows that the whole sequence $Q[a_n(c, \xi, k; x \cdot \xi + t + c)](y)$ converges uniformly on bounded sets. Because of the translation invariance of Q, (11.1) then shows that for each t the sequence $a_n(c, \xi, k; y \cdot \xi + t)$ converges to $a(c, \xi, k; y \cdot \xi + t)$, uniformly for y in any bounded subset of $\mathscr{H}$. In particular, $a(c, \xi, k; y \cdot \xi + t)$ is a continuous function of y on $\mathscr{H}$.

Furthermore, because of the continuity property (3.1v) of Q we may now take limits in (11.1) to see that

$$a(c, \xi, k; s) = \max\{k^{-1}\varphi(s), Q[a(c, \xi, k; x \cdot \xi + s + c)](0)\}. \tag{11.3}$$

We choose $y_0 \in \mathcal{H}$ such that $y_0 \cdot \xi > 0$. For every integer l and any $c \geqq c^*(\xi)$ we define the sequence

$$K_k(l) = \tfrac{1}{2}[a(c, \xi, k; ly_0 \cdot \xi) + a(c, \xi, k; (l+1)y_0 \cdot \xi)]. \tag{11.4}$$

Then $K_k(l)$ is nonincreasing in l, $K_k(-\infty) = \pi_1$, and $K_k(+\infty) = 0$. Since a decreases from π_1 to 0 as s goes from $-\infty$ to $+\infty$,

$$K_k(l) - K_k(l-1) = \tfrac{1}{2}[a(c, \xi, k; (l+1)y_0 \cdot \xi) - a(c, \xi, k; (l-1)y_0 \cdot \xi)] \leqq \tfrac{1}{2}\pi_1.$$

Consequently there is an integer, which we call l_k, such that

$$\tfrac{1}{4}\pi_1 \leqq K_k(l_k) \leqq \tfrac{3}{4}\pi_1. \tag{11.5}$$

We now consider the sequence $a(c, \xi, k; x \cdot \xi + l_k y_0 \cdot \xi)$, $k = 1, 2, \cdots$.

Because of the equation (11.3) and the hypothesis (3.7) there is a subsequence k_i of the integers such that $a(c, \xi, k_i; x \cdot \xi + l_{k_i} y_0 \cdot \xi)$ converges uniformly for x on bounded subsets of $\mathcal{H}$ to a function $W(x \cdot \xi)$ defined on $\mathcal{H}$. This subsequence has a subsequence k_i' such that $a(c, \xi, k_i'; x \cdot \xi + l_{k_i'} y_0 \cdot \xi + c)$ also converges uniformly on bounded subsets of $\mathcal{H}$ to a function $W(x \cdot \xi + c)$. (Of course, if c is of the form $z \cdot \xi$ for some z in $\mathcal{H}$, $W(x \cdot \xi + c)$ is already defined and the convergence is implied by the convergence of the previous sequence, but c need not be of this form.) We take further subsequences to get uniform convergence on bounded subsets of $\mathcal{H}$ of $a(c, \xi, k_i^{(m)}; x \cdot \xi + l_{k_i^{(m)}} y_0 \cdot \xi + mc)$ to a function $W(x \cdot \xi + mc)$ for any (positive or negative) integer m. By taking a diagonal sequence $\bar{k}_i$, we obtain simultaneous convergence for all m. Because the convergence is uniform on bounded subsets of $\mathcal{H}$, we may take limits in the equation (11.3) with $k = \bar{k}_i$ and $s = y \cdot \xi + l_{\bar{k}_i} y_0 \cdot \xi - (n+1)c$ to find that

$$W(y \cdot \xi - (n+1)c) = Q[W(x \cdot \xi - nc)](y). \tag{11.6}$$

Thus $u_n(x) = W(x \cdot \xi - nc)$ is a traveling wave solution of the recursion $u_{n+1} = Q[u_n]$.

The definition (11.4) shows that the sequence $K_{\bar{k}_i}(l_{\bar{k}_i})$ converges to $\frac{1}{2}[W(0) + W(y_0 \cdot \xi)] \geqq W(y_0 \cdot \xi)$. By (11.5) this quantity must be at most $\frac{3}{4}\pi_1$. Theorem 6.2 proves that $W(-\infty) = \pi_1$, and therefore W is not constant. Because the functions a are nonincreasing in s, the same is true of $W(s)$. Therefore $W(s)$ has a limit as $s \to \infty$. By letting $n \to -\infty$ in (11.6) we see that $W(\infty) = Q[W(\infty)]$. Therefore $W(\infty)$ is a fixed point of Q whose value is below π_1. Hence it must be 0. This proves Theorem 6.6.

Remarks. 1. If $\mathcal{H}$ is a discrete set and π_1 is finite, the extra condition (3.7) is automatically satisfied. If $\mathcal{H}$ has limit points and π_1 is finite, the condition is normally verified by showing that the set of functions $\{Q[u] | u \in B\}$ is equicontinuous and applying Ascoli's theorem.

2. The set of points of the form $x \cdot \xi + mc$ where $W(s)$ is defined may depend upon ξ and c. As an example, let $\mathcal{H}$ be the set of points with integral coordinates in the plane $\mathcal{R}^2$. If ξ_1, ξ_2, and c are all integral multiples of a fixed real number ρ, then the set of points of the form $x \cdot \xi + mc$ lies in the discrete set of integral multiples of ρ, while if the ratio of two of these three numbers is irrational, the set of points of the form $x \cdot \xi + mc$ is dense on the real line.

If there is a y in $\mathcal{H}$ such that $y \cdot \xi = rc$ for some integer r, then increasing n by r in the function $W(x \cdot \xi - nc)$ translates the function by y. If there is no y in $\mathcal{H}$ such that $y \cdot \xi$ is an integral multiple of c, then the arguments of $W(x \cdot \xi - nc)$ for two different values of n are all different. However, the evolution of the nonincreasing function $W(x \cdot \xi - nc)$ with increasing n will still have the general appearance of a traveling wave.

3. The uniform convergence on bounded subsets of $\mathscr{H}$ does not imply that the limit functions a and W are continuous functions of s.

For example, let $\mathscr{H}$ be the set of points with integral coordinates in $\mathscr{R}^2$ and

$$Q[u] = u(x_1+1, x_2-2)[2-u(x_1+1, x_2-2)].$$

It is easy to see that $c^*(\xi) = -\xi_1 + 2\xi_2$, that if $\varphi(s) > 0$ for $s < 0$

$$a(c^*(\xi), k; s) = \begin{cases} 1 & \text{for } s < 0, \\ 0 & \text{for } s \geqq 0, \end{cases}$$

and that for $c = c^*(\xi)$

$$W(s) = \begin{cases} 1 & \text{for } s < 0, \\ 0 & \text{for } s \geqq 0. \end{cases}$$

Note that if ξ_2/ξ_1 is irrational, there is a sequence x_ν such that $x_\nu \cdot \xi$ decreases to zero. However, $|x_\nu|$ is unbounded so that the sequence x_ν does not lie in any bounded set and does not have a convergent subsequence.

4. It is known [13], [25], [26] that in the heterozygote inferior case of Fisher's equation (1.1) there is a monotone traveling wave when $c = c^*$, but not, in general, for $c > c^*$. Thus the statement of Theorem 6.6 can be false if the condition $\pi_0 = 0$ is dropped. We do not know whether a monotone traveling wave always exists for $c = c^*(\xi)$ when this condition is dropped.

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