Some Isoperimetric Inequalities for Membrane Frequencies and Torsional Rigidity

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1. Introduction

Let \( A \) denote the fundamental frequency of a two-dimensional membrane \( G \) fixed on its boundary. Let \( A \) be the area of \( G \), and \( L \) its perimeter. Makai [5, 6] has recently shown that if \( G \) is simply or doubly connected, the dimensionless quantity \( A^2 A^2 L^{-2} \) is at most 3. Pólya [7] has improved this result to

\[
A^2 \leq \left( \frac{2}{\pi} \right)^2 L^2 \cdot A^{-2}. \tag{1.1}
\]

The constant \((\frac{2}{\pi})^2\) is optimal, since equality is attained in the limiting case of an infinite rectangular strip. To obtain these results Makai and Pólya insert in the minimum principle for \( A^2 \) functions which depend only on the distance from the boundary.

In this paper we apply a similar method to a two-dimensional membrane \( G \) fixed on its exterior bounding curve \( C_0 \). The membrane is permitted to have interior bounding curves \( C_i \) (holes) along which it is free. We shall show that among all such membranes with given area \( A \) and given perimeter \( L \) of \( C_0 \) the highest fundamental frequency is attained when \( G \) is annular.

This fact gives the upper bound

\[
A \leq 2\pi L^{-1} \mu \tag{1.2}
\]

where \( \mu \) is the lowest root of the transcendental equation

\[
J_0(\mu)Y_1(\mu\psi) = Y_0(\mu)J_1(\mu\psi) \tag{1.3}
\]

with

\[
\psi^2 = 1 - 4\pi AL^{-1}. \tag{1.4}
\]

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The classical isoperimetric inequality [1, p. 831] shows that the expression on the right of (1.4) is always nonnegative, and vanishes if and only if \( G \) is a circle. The solution of (1.3) is graphed in Jahnke and Emde [3, pp. 207–208]. If \( G \) is simply-connected the inequality (1.2) is an improvement of (1.1).

The same method yields an isoperimetric inequality for membranes \( G \) which are elastically supported on \( C_0 \) and free along any inner boundaries \( C_i \). The annular membrane has the largest fundamental frequency among all such membranes of given area, perimeter of \( C_0 \), and elastic constant.

In a similar manner we find a lower bound for the torsional rigidity of a simply connected domain. Again we obtain an improvement of the inequalities of Makai [5,6] and Pólya [7].

The inequalities of Makai and Pólya for the fundamental frequency and torsional rigidity hold for doubly connected (ring-shaped) as well as simply connected domains \( G \).

Our bound (1.2) for the fundamental frequency applies when only the outer boundary \( C_0 \) of \( G \) is fixed. However, we may obtain a bound for a membrane \( G \) which is fixed along \( C_0 \) and along one or more inner boundaries \( C_i \). To do this, we replace \( G \) by a membrane \( G' \) which occupies the same domain and whose boundaries are fixed wherever those of \( G \) are fixed, as well as along straight-line paths connecting the fixed boundary components. Then the fundamental frequency \( \tilde{\lambda} \) of \( G' \) is greater than \( \lambda \). Moreover, \( G \) is fixed along a single curve \( C_0 \) consisting of the fixed boundary components of \( G \), together with the connecting paths, covered twice. The perimeter \( L \) of \( C_0 \) exceeds the total length \( L \) of the fixed boundary components of \( G \) by twice the total length of connecting lines. The area of \( G' \) is again \( A \).

Thus, we obtain the bound (1.2) with \( L \) replaced by \( L \) in (1.2) and (1.4). Whether or not this bound is better than (1.1) when \( G \) is ring-shaped depends upon the location of the hole.

Similar remarks apply to the torsional rigidity of multiply connected domains.

II. THE FUNDAMENTAL FREQUENCY

Let \( G \) be a plane domain lying inside a simple closed bounding curve \( C_0 \), and possibly having interior holes bounded by smooth curves \( C_i \).

Let \( \lambda^2 \) be the lowest eigenvalue of the membrane problem:

\[
\begin{align*}
\Delta u + \lambda^2 u &= 0 \quad \text{in } G, \\
u &= 0 \quad \text{on } C_0, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } C_i.
\end{align*}
\] (2.1)
It is well known [1, pp. 345–346; 9, p. 87] that

\[ A^2 \leq \frac{\iint |\nabla v|^2 \, dx \, dy}{\iint v^2 \, dx \, dy} \]  

(2.2)

where \( v \) is any piecewise continuously differentiable function vanishing on \( C_0 \).

We define \( C_\delta \) to be the curve consisting of points inside \( C_0 \) at distance \( \delta \) from \( C_0 \). It was shown by Sz.-Nagy [11] that the length \( l(\delta) \) of \( C_\delta \) is well defined for almost all values of \( \delta \), and that \( l(\delta) + 2\pi \delta \) is non-increasing in \( \delta \). Thus if \( l(\delta) \) is the length of the portion of \( C_\delta \) which lies in \( G \),

\[ l(\delta) \leq \bar{l}(\delta) \leq L - 2\pi \delta \]  

(2.3)

where \( L = \bar{l}(0) \) is the length of \( C_0 \).

Let \( a(\delta) \) be the area of the portion of \( G \) lying between \( C_0 \) and \( C_\delta \). Then

\[ a(\delta) = \int_0^\delta l(\delta) \, d\delta. \]  

(2.4)

Integrating (2.3) gives

\[ a(\delta) \leq L \delta - \pi \delta^2. \]  

(2.5)

Inserting (2.3) in this inequality yields

\[ \left( \frac{da}{d\delta} \right)^2 = l^2 \leq L^2 - 4\pi a(\delta). \]  

(2.6)

We define a function \( r(\delta) \) by

\[ 4\pi^2 r^2 = L^2 - 4\pi a(\delta). \]  

(2.7)

If we interpret this equation as a mapping of the portion of \( C_\delta \) in \( G \) onto the circle of radius \( r(\delta) \), we find that \( C_0 \) is mapped into a circle of equal perimeter and that the portion of \( G \) between \( C_0 \) and \( C_\delta \) goes into an annulus of equal area \( a(\delta) \). We differentiate (2.7) and use (2.6) and the fact that

\[ |\nabla \delta| = 1 \]  

(2.8)
almost everywhere to show that
\[ |\text{grad } r|^2 \leq 1 \]  
(2.9)

almost everywhere in \( G \).

We now let the function \( v \) in (2.2) depend only on \( r \). In view of (2.9),
\[ |\text{grad } r|^2 \leq \left( \frac{dv}{dr} \right)^2. \]  
(2.10)

Since the mapping (2.7) is area-preserving, (2.2) becomes
\[ A^2 \leq \frac{\int_{r_1}^r \left( \frac{dv}{dr} \right)^2 r \, dr}{\int_{r_1}^r v^2 r \, dr}, \]  
(2.11)

where
\[ r_1 = (L^2 - 4\pi A)^{1/2}/2\pi = L\mathcal{P}/2\pi, \]  
(2.12)

and \( v \) is any differentiable function of \( r \) satisfying
\[ v(r_2) = 0. \]  
(2.13)

The right-hand side of (2.11) is the Rayleigh quotient for the annular membrane \( \mathcal{G} \) whose area is \( A \) and whose outer boundary has perimeter \( L \). Its minimum under the condition (2.13) is the lowest eigenvalue for the membrane \( \mathcal{G} \) fixed on the outer boundary and free along the inner boundary. Thus we have established that \( \mathcal{G} \) has the highest fundamental frequency among all membranes \( G \) with given \( A \) and \( L \).

The minimum value of the expression on the right of (2.11) is attained for
\[ v = J_0(2\pi L^{-1} \mu r)Y_0(\mu) - Y_0(2\pi L^{-1} \mu r)J_0(\mu) \]  
(2.14)

where \( \mu \) is determined in such a way that \( v'(r_1) = 0 \). It is the lowest root of the Eq. (1.3) (cf. [3, pp. 207–208]), and therefore depends upon the dimensionless quantity \( \mathcal{P} \) defined by (1.4). Substituting (2.14) in (2.11) leads to the bound
\[ A \leq 2\pi L^{-1} \mu. \]  
(2.15)
If $G$ has no holes $C_0$, a lower bound for $A^2$ in terms of the area $A$ is given by the isoperimetric inequality of Faber [2] and Kranh [4].

$$A^2 \geq \pi \sqrt{2} A^{-1}.$$  \hspace{1cm} (2.16)

Here $j(\approx 2.4048)$ is the first zero of the Bessel function $I_0$. Equality in (2.16) is attained when $G$ is a circle.

If in (2.11) we choose

$$v = J_0(j[\pi A^{1/2}(r^2 - r_1^2)])$$

which satisfies (2.13), we obtain the upper bound

$$A^2 \leq \pi^2 / 2 A^{-1} [1 + (J_1^{-2}(j) - 1)\Psi^2 (1 - \Psi^2)^{-1}]$$

$$\leq \pi / 2 A^{-1} [1 + 2.712\Psi^2 (1 - \Psi^2)^{-1}]$$  \hspace{1cm} (2.18)

Here $\Psi$ is the dimensionless quantity defined by (1.4). Again equality is attained when $G$ is a circle.

The inequalities (2.16) and (2.18) show that if $G$ is simply connected and nearly circular in the sense that $\Psi$ is small, the fundamental frequency $\Lambda$ is near that of the circle of equal area.

Since the function (2.14) yields the best upper bound for $A^2$, the inequality (2.15) is in general sharper than (2.18).

III. The Elastically Supported Membrane

We consider the lowest eigenvalue $A^2(h)$ of the problem

$$\begin{align*}
Au + A^2 u &= 0 & \text{in } G, \\
\partial u / \partial n + ku &= 0 & \text{on } C_0, \\
\partial u / \partial n &= 0 & \text{on } C_1.
\end{align*}$$  \hspace{1cm} (3.1)

The elastic constant $h$ is positive. For any piecewise continuously differentiable function $v$ we have the inequality (cf. [1, pp. 345–346]).

$$A^2(h) \leq \frac{\iint_G |\text{grad } v|^2 \, dx \, dy + h \iint_C v^2 \, ds}{\iint_G v^2 \, dx \, dy}.$$  \hspace{1cm} (3.2)
We introduce the new variable \( r \) as in Section II and let \( v \) be a function of \( r \) only. This gives the upper bound

\[
A^2(k) \leq \int_{r_1}^{r_2} \left( \frac{dv}{dr} \right)^2 r \, dr + 2\pi hr_2 v^2(r_2)
\]

where \( r_1 \) and \( r_2 \) are given by (2.12). The right hand side of (3.3) is the Rayleigh quotient for the annular membrane \( G \) of area \( A \) elastically supported (with elastic constant \( k \)) on the outer boundary of perimeter \( L \), and free on the inner boundary. The minimum of the Rayleigh quotient is the lowest eigenvalue of this membrane. Thus we have shown that \( G \) gives the highest fundamental frequency among all membranes \( G \) of given \( A \), \( L \), and \( k \). This fact leads to the upper bound

\[
A(k) \leq 2\pi L^{-1} \mu
\]

where \( \mu \) is the lowest root of the equation

\[
Y_1(\mu Y)[kLJ_0(\mu) - 2\pi \mu J_1(\mu)] = J_1(\mu Y)[kLY_0(\mu) - 2\pi \mu Y_1(\mu)]
\]

If \( k \) in problem (3.1) is a nonnegative function of arc length rather than a constant, the inequality (3.4) still holds with \( kL \) in (3.5) replaced by \( \int_{C_0} k \, ds \).

IV. TORSIONAL RIGIDITY

Let \( G \) be a simply connected domain of area \( A \) bounded by the closed curve \( C_0 \) of perimeter \( L \). The torsional rigidity \( P \) of \( G \) is defined by [9, p. 87].

\[
P = \max \left[ \frac{2 \int_G v \, dx \, dy}{\int_G |\nabla v|^2 \, dx \, dy} \right]
\]

among sufficiently regular functions \( v \) which vanish on \( C_0 \).

We define the variable \( r \) as in section 2 and let

\[
v = \frac{1}{2} \left( r_2^2 - r^2 \right) + r_1^2 \log \frac{r}{r_2}
\]
Using the results of Section II leads immediately to the bound

$$P \geq \frac{A^2}{2\pi} \left[ 1 - 2\Psi^2(1 - \Psi^2)^{-1} - 4\Psi^4(1 - \Psi^2)^{-2}\log \Psi \right]$$

where \(\Psi\) is given by (1.4). An upper bound for \(P\) in terms of \(A\) is given by the isoperimetric inequality

$$P \leq A^2/2\pi$$

which was conjectured by St. Venant [10] and proved by Pólya [8]. Again we see that if \(G\) is nearly circular in the sense that \(\Psi\) is small, its torsional rigidity is close to that of the circle of equal area.

References