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An extension of Fichera's eigenvalue bounds to the Neumann problem.

Hans F. Weinberger
School of Mathematics
University of Minnesota

1. Introduction. The Rayleigh-Ritz method provides an easily computed means of obtaining arbitrarily close upper bounds for the eigenvalues of a self-adjoint ordinary or partial differential operator. In contrast, it is usually difficult to compute good lower bounds for these eigenvalues. Most known methods for finding complementary bounds, including the intermediate problem methods of Weinstein and Aronszajn (see [12]) and its simplifications (see, e.g, [9]) require knowledge of the eigenvalues of a base operator and the repeated calculation of either the determinants or the eigenvalues of large symmetric matrices.

In 1965 Gaetano Fichera [3] introduced a method of finding lower bounds which requires less a priori knowledge. This method has been extended in a number of directions. (See [1,10], and the bibliographies in these papers.) It works very well when the boundary conditions are of Dirichlet type, but great difficulties arise when some or all of the boundary conditions are "natural".

Fichera's method is based on his observation that an inequality of Trefftz can be used to reduce the problem of finding an arbitrarily close lower bound for each eigenvalue to that of finding an arbitrarily close upper bound for the trace of the square of the Green's (inverse) operator of the differential operator. Fichera then created a procedure for finding such a bound. As was pointed out in [10], the procedure is closely related to the hypercircle method of Prager and Synge [6,7] and the equivalent Schwarz's inequality method of Diaz and Weinstein [2]. This relation permits one to generalize Fichera's ideas. Such a generalization allows the treatment of problems with natural boundary conditions such as Neumann conditions. In order to keep this paper as simple as possible, we shall only treat the special case of the Neumann problem for the Laplace operator. We shall only consider the trace of the square of the Green's operator, although the same procedure works for any of the monotone orthogonal invariants studied by Fichera (see [1]). A more general form of our results will be published elsewhere [11].

The Rayleigh-Ritz method and the Trefftz inequality will be explained in Section 2. Section 3 will present the extension of Fichera's method to the Neumann problem.

2. The Rayleigh-Ritz method and the Trefftz inequality. We shall consider the Neumann eigenvalue problem

$$\begin{aligned} -\Delta u &= \mu u \qquad \text{in } \Omega \\ \partial u/\partial n &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

where Ω is a bounded plane domain, and $\partial u/\partial n$ is the outward normal derivative on the boundary $\partial\Omega$.

It is useful to choose a positive constant κ, and add the term κu to both sides of the differential equation. This results in the equivalent problem

$$\begin{aligned} L[u] &:= -\Delta u + \kappa u = \lambda u \text{ in } \Omega \\ \partial u/\partial n &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where the eigenvalue parameter μ is replaced by the parameter $\lambda = \mu + \kappa$. The advantage is that with respect to Neumann boundary conditions, the operator $-\Delta + \kappa$ has an inverse, while $-\Delta$ does not.

We define the usual L_2 scalar product

$$(v, w) := \int_\Omega vw d\mathbf{x}.$$

Integration by parts permits one to write

$$(v, L[w]) = \mathcal{A}(v, w), \tag{2.2}$$

where

$$\mathcal{A}(v, w) = \int_\Omega \{\nabla v \cdot \nabla w + \kappa vw\} \, d\mathbf{x}, \tag{2.3}$$

provided $\partial w/\partial n = 0$ on the boundary. Clearly

$$\mathcal{A}(v, v) \geq \kappa(v, v) \tag{2.4}$$

for all v, whether or not they satisfy the boundary conditions.

Let the eigenvalues of the problem (2.1) be written in nondecreasing order $\lambda_1 \leq \lambda_2 \leq \cdots$. H. Poincaré [5] proved the Poincaré principle, which characterizes λ_n as the minimum with respect to choices of n-spaces of the maximum of the Rayleigh quotient $\mathcal{A}(v, v)/(v, v)$ with v ranging over a chosen n-space[1]. It is easily seen from the Poincaré principle that if $v_1, v_2, \cdots, v_n$ are any n linearly independent functions which satisfy the boundary conditions, and if the eigenvalues of the matrix problem

$$\sum_{\nu-1}^{n} \mathcal{A}(v_\mu, v_\nu) b_\nu = \lambda \sum_{\nu-1}^{n} (v_\mu, v_\nu) b_\nu \tag{2.5}$$

are written as $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_n^{(n)}$, then

$$\lambda_\alpha \leq \lambda_\alpha^{(n)} \qquad \text{for } \alpha = 1, \cdots, n. \tag{2.6}$$

[1] Poincaré claimed to show in [5] that each eigenvalue of the matrix problem (2.5) is an upper bound for the corresponding eigenvalue of A. While Poincaré's proof is only correct for the largest eigenvalue, this is sufficient to establish the Poincaré principle, which, in turn, implies the more general statement.

These upper bounds were used for computational purposes by Lord Rayleigh and by W. Ritz and are usually called the Rayleigh-Ritz upper bounds. They are quite satisfactory from a computational point of view. The hard part is obtain good lower bounds for the eigenvalues, or, equivalently, upper bounds for the errors $\lambda_\alpha^{(n)} - \lambda_\alpha$.

The scalar product $\mathcal{A}(v, w)$ defines a Hilbert space $H_{\mathcal{A}}$ whose members are the square integrable functions with square integrable gradients. The inequality (2.4) shows that for any fixed $v \in L_2(\Omega)$, the bilinear form (v, w) is a bounded linear functional of $w \in H_{\mathcal{A}}$. The Riesz representation theorem then implies that there is a bounded linear transformation G from L_2 to $H_{\mathcal{A}}$ such that

$$(f, v) = \mathcal{A}(G[f], v) \tag{2.9}$$

for all v in $V_{\mathcal{A}}$. We see from the property (2.2) of $\mathcal{A}$ that $w = G[f]$ is the weak solution of the equation $L[w] = f$ with the Neumann boundary conditions. We call G the **Green's operator** of L on with the Neumann boundary condition. Since L is uniformly elliptic, G is an integral operator

$$G[f](\mathbf{x}) := \int_\Omega g(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}.$$

Its kernel g is called the **Green's function** of $L = -\Delta + \kappa$ with the Neumann boundary conditions (or, more commonly, the Neumann function of L). Because $(G[f], h) = \mathcal{A}(G[f], G[h])$ by (2.9), the kernel $g(\mathbf{x}, \mathbf{y})$ is symmetric in $\mathbf{x}$ and $\mathbf{y}$.

One can write this eigenvalue problem (2.1) in the form

$$G[u] = \frac{1}{\lambda} u. \tag{2.10}$$

That is, the eigenvalues of the integral operator G with the kernel $g(\mathbf{x}, \xi)$ are the reciprocals of those of L.

The trace of G^2 is the sum of its eigenvalues λ_α^{-2}, and it can be computed as the integral of the square of the kernel g with respect to $\mathbf{x}$ and $\mathbf{y}$. Therefore,

$$\int_\Omega \int_\Omega g(\mathbf{x}, \mathbf{y})^2 d\mathbf{x} d\mathbf{y} - \sum_{\alpha=1}^{n} [\lambda_\alpha^{(n)}]^{-2} \geq \sum_{\alpha=1}^{n} \{\lambda_\alpha^{-2} - [\lambda_\alpha^{(n)}]^{-2}\}.$$

Since (2.6) shows that each term on the right is nonnegative, we can bound any one of them by the left-hand side. This leads to the Trefftz inequality [8]

$$\lambda_\beta \geq \left\{ [\lambda_\beta^{(n)}]^{-2} + \int_\Omega \int_\Omega g(\mathbf{x}, \mathbf{y})^2 d\mathbf{x} d\mathbf{y} - \sum_{\alpha=1}^{n} [\lambda_\alpha^{(n)}]^{-2} \right\}^{-1/2}.$$

It is not difficult to see that if the v_i in the Rayleigh-Ritz method are complete in $H_{\mathcal{A}}$, then the right-hand side of this inequality is very close to the upper bound $\lambda_\beta^{(n)}$ when n is sufficiently large.

When Trefftz presented this inequality, he immediately observed that it is not practical because g is unknown. In contrast, Fichera [F] observed that the method becomes practical if one can find an arbitrarily close upper bound for the trace of G^2, and showed how to find such an upper bound.

3. An extension of Fichera's method. In order to obtain an upper bound for the trace of G^2, we observe that if v is any solution of the differential equation $-\Delta v + \kappa v = f$ which also satisfies the boundary condition $\partial v/\partial n = 0$, then integration by parts shows that

$$\mathcal{A}(v, G[f]) = (f, G[f]).$$

In particular, since $G[f]$ satisfies the above conditions, we have

$$\mathcal{A}(G[f], G[f]) = (f, G[f]) = \mathcal{A}(v, G[f]). \tag{3.1}$$

An application of Schwarz's inequality now yields the inequality[2]

$$(f, G[f]) = \mathcal{A}(G[f], G[f]) \leq \mathcal{A}(v, v). \tag{3.2}$$

Thus if the linear integral transformation $R[f]$ has the property that $v = R[f]$ satisfies the conditions on v, then the quadratic form $\mathcal{A}(R[f], R[f])$ is an upper bound for $(f, G[f])$ for all f, so that the trace of the square of the corresponding operator K gives an upper bound for the trace of G^2.

This is precisely what Fichera did. In the case of Dirichlet boundary conditions, the function $v = R[f]$ has to satisfy the differential equation but no boundary conditions, so that R can be taken to be an integral operator whose kernel is any fundamental solution of the equation. In the case of Neumann conditions, the function $v = R[f]$ is required to satisfy both the differential equation $-\Delta u + \kappa v = f$ and the boundary condition $\partial v/\partial n = 0$. It is easy to see that these define a unique function, so that the only admissible operator R is the operator whose kernel is the Neumann function itself. Since this function is usually not known, the method cannot be used for this case[3].

To circumvent this problem, we generalize the minimum principle in a way which is well known from the hypercircle method. Recall that Thomson's principle is formulated in terms of a vector-valued function which approximates the electric field rather than in terms of the gradient of an approximation of the electrostatic potential. In this spirit, we replace the function v and its gradient by a 3-vector valued function $\mathbf{p} \sim \{\tilde{p}^0, \tilde{p}^1, \tilde{p}^2\}$. We extend the form $\mathcal{A}$ to such vectors in an obvious manner. Namely, we set

$$\mathcal{B}(\mathbf{p}, \mathbf{q}) := \int_\Omega \left[\sum_{i=1}^{2} \tilde{p}^i \tilde{q}^i + \kappa \tilde{p}^0 \tilde{q}^0 \right] d\mathbf{x}, \tag{3.3}$$

so that $\mathcal{B}(\{v, \nabla v\}, \{w, \nabla w\}) = \mathcal{A}(v, w)$.

[2] This method of finding an upper bound for $\mathcal{A}(G[f], G[f])$ was found by Diaz and Weinstein [2], and an equivalent derivation was given by Prager and Synge [6]. The latter introduced the term hypercircle method [7] for this idea.

[3] For a simply connected domain, Fichera [4] suggested overcoming this difficulty by introducing a smooth mapping with smooth inverse of Ω to a disc, whose Neumann function is known.

Suppose now that

$$\begin{aligned} \tilde p^0 - \tilde p^1_{,x_1} - \tilde p^2_{x_2} &= f \text{ in } \Omega \\ \tilde p^1 n_1 + \tilde p^2 n_2 &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.4}$$

Then integration by parts shows that

$$\mathcal{B}(\mathbf{p}, \{G[f], \nabla G[f]\}) = (f, G[f]) = \mathcal{B}(\{G[f], \nabla G[f]\}, \{G[f], \nabla G[f]\}).$$

By Schwarz's inequality

$$(f, G[f]) = \mathcal{B}(\{G[f], \nabla G[f]\}, \{G[f], \nabla G[f]\}) \leq \mathcal{C}(\mathbf{p}, \mathbf{p})). \tag{3.5}$$

Thus if we can find a linear integral transformation $\mathbf{T}$ from scalar-valued functions to 3-vectors with the property that $\mathbf{p} = \mathbf{T}[f]$ satisfies the conditions (3.4), then

$$(f, G[f]) \leq \mathcal{B}(\mathbf{T}[f], \mathbf{T}[f]). \tag{3.6}$$

The integral operator whose quadratic form is the right-hand side gives an upper bound for the Green's operator G, and hence the trace of its square is an upper bound for the trace of G^2. If we write

$$\mathbf{T}[f] = \left\{ \int_\Omega t^\alpha(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y} \right\}$$

with $\alpha = 0, 1, 2$, then the fact that $\mathbf{T}[f]$ satisfies (3.4) for all f means that for each $\mathbf{y} \in \mathbf{\Omega}$ the functions t^α must satisfy the conditions

$$\begin{aligned} &t^0(\mathbf{x}, \mathbf{y}) - t^1_{,x_1}(\mathbf{x}, \mathbf{y}) - t^2_{,x_2}(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \text{ for } \mathbf{x} \in \Omega \\ &t^1(\mathbf{x}, \mathbf{y}) n_1(\mathbf{x}) + t^2(\mathbf{x}, \mathbf{y}) n_2(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \partial\Omega, \end{aligned}$$

where δ is the Dirac distribution.

It is easy to find a particular solution of the first equation by letting $\phi(\mathbf{x})$ be a fundamental solution of $-\Delta$ such as $(1/2\pi)\ln(1/|\mathbf{x}|)$, and setting $t_i = \phi_{,x_i}(|\mathbf{x} - \mathbf{y}|)$ for $i = 1, 2$ and $t_0 = 0$. However, this function does not satisfy the boundary conditions. If $\mathbf{y}$ is away from the boundary, we can choose a cutoff function $\psi(\mathbf{x}; \mathbf{y})$ which is 1 and has the gradient zero at $\mathbf{y}$, and which vanishes on the boundary. We then set $t_i = \psi\phi_{,x_1}$ for $i = 1, 2$ and $t_0 = \nabla_\mathbf{x}\psi \cdot \nabla_\mathbf{x}\phi$. This gives a solution which satisfies the boundary conditions. However, as $\mathbf{y}$ approaches the boundary, the function ψ must go from 1 to 0 in a short distance, and it follows that t_0 approaches infinity. For a very small class of domains it is possible to avoid this problem by using reflection. For most domains we need a different extension of $\mathcal{A}$.

In order to obtain this extension, we first choose a smooth vector field $\mathbf{r}$ with the property that $\mathbf{r} \cdot \mathbf{n}$ is uniformly positive on the boundary. By the divergence theorem we may write the functional $\mathcal{A}$ defined in (2.3) in the form

$$\mathcal{A}(v, w) = \int_\Omega [\nabla v \cdot \nabla w + \kappa v w - \nabla \cdot (v w \mathbf{r})] d\mathbf{x} + \int_{\partial\Omega} \mathbf{r} \cdot \mathbf{n} v w ds. \tag{3.7}$$

By collecting the coefficients of v and its derivatives, we can write this in the form

$$\mathcal{A}(v,w) = \int_\Omega \{v\tilde{S}^0[w] + v_{,x_i}\tilde{S}^i[w]\}d\mathbf{x} + \int_{\partial\Omega} v\hat{S}[w]ds, \tag{3.8}$$

where the vector-valued transformation $\mathbf{S}[w]$ is defined by

$$\begin{aligned} \tilde{S}^0[w] &= -\mathbf{r}\cdot\nabla w + (\kappa - \nabla\cdot\mathbf{r})w \text{ in } \Omega \\ \tilde{S}^i[w] &= w_{,x_i} - wr_i \text{ in } \Omega \\ \hat{S}[w] &= \mathbf{r}\cdot\mathbf{n}\ w \text{ on } \partial\Omega. \end{aligned} \tag{3.9}$$

In (3.8) and (3.9) we have used the summation convention with i and j running from 1 to 2, and we shall continue to do so in the rest of this paper.

We assume that the positive constant κ is chosen so large that the function

$$C(\mathbf{x}) := \kappa - \nabla\cdot\mathbf{r} - |\mathbf{r}|^2 \tag{3.10}$$

is uniformly positive. Then we can solve the definitions (3.9) for w and its gradient and boundary values:

$$\begin{aligned} w &= (1/C)(\tilde{S}^0[w] + r_j\tilde{S}^j[w]) \\ w_{,x_i} &= \tilde{S}^i[w] + (1/C)(\tilde{S}^0[w] + r_j\tilde{S}^j[w])r_i \\ w\big|_{\partial\Omega} &= (1/\mathbf{r}\cdot\mathbf{n})\hat{S}[w]. \end{aligned}$$

By expressing v and its gradient and boundary value in this way, we may write

$$\mathcal{A}(v,w) = \mathcal{C}(\mathbf{S}[v],\mathbf{S}[w]), \tag{3.11}$$

where we have defined the bilinear form

$$\mathcal{C}(\mathbf{p},\mathbf{q}) := \int_\Omega [((1/C)(\tilde{p}^0 + r_i\tilde{p}^i)(\tilde{q}^0 + r_j\tilde{q}^j) + \tilde{p}^i\tilde{q}^i]d\mathbf{x} + \int_{\partial\Omega}(1/\mathbf{r}\cdot\mathbf{n})\hat{p}\hat{q}ds \tag{3.12}$$

for all 4-vector fields $\mathbf{p} \sim (\tilde{p}^0,\tilde{p}^1,\tilde{p}^2,\hat{p})$ and $\mathbf{q} \sim (\tilde{q}^0,\tilde{q}^1,\tilde{q}^2,\hat{q})$. The components $\hat{p}$ and $\hat{q}$ are, of course, only defined on the boundary.

$\mathcal{C}$ is another extension of $\mathcal{A}$ to vector fields. By construction,

$$\mathcal{C}(\mathbf{p},\mathbf{S}[w]) = \int_\Omega [\tilde{p}^0 w + \tilde{p}^i w_{,x_i}]d\mathbf{x} + \int_{\partial\Omega}\hat{p}w ds.$$

Hence integration by parts shows that if

$$\begin{aligned} \tilde{p}^0 - \tilde{p}^i_{,x_i} &= f \text{ in } \Omega \\ \tilde{p}^i n_i + \hat{p} &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.13}$$

then

$$\mathcal{C}(\mathbf{p},\mathbf{S}[G[f]]) = (f,G[f]) = \mathcal{C}(\mathbf{S}[G[f]],\mathbf{S}[G[f]]).$$

As before, Schwarz's inequality shows that

$$(f, G[f]) \leq \mathcal{C}(\mathbf{p}, \mathbf{p}). \tag{3.14}$$

If $\mathbf{T}[f]$ is any linear vector-valued integral transformation which takes any f into a 4-vector field which satisfies the conditions (3.13), then the quadratic form $\mathcal{C}(\mathbf{T}[f], \mathbf{T}[f])$ is an upper bound for the quadratic form of G. Hence the trace of the square of the integral operator whose quadratic form is $\mathcal{C}(\mathbf{T}[f], \mathbf{T}[f])$ is an upper bound for the trace of G^2.

We note that for each $\mathbf{y}$ in Ω the components of the kernel of the operator $\mathbf{T}$ must only satisfy the conditions

$$\begin{aligned} &\tilde{t}^0(\mathbf{x}, \mathbf{y}) - \tilde{t}^i_{,x_i}(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \text{ for } \mathbf{x} \in \Omega \\ &\tilde{t}^i(\mathbf{x}, \mathbf{y}) n_i(\mathbf{x}) + \hat{t}(\mathbf{x}, \mathbf{y}) = 0 \text{ for } \mathbf{x} \in \partial\Omega. \end{aligned} \tag{3.15}$$

If $\tilde{t}^i$ is the gradient of a parametrix $\phi(\mathbf{x}, \mathbf{y})$ of Laplace's equation, we can satisfy the first condition by solving it for $\tilde{t}^0$ and the second condition by solving it for $\hat{t}$. Thus it is easy to find such a kernel. There is still the question of what needs to be done with the kernel $\mathbf{t}$.

We wish to find the kernel $k(\mathbf{x}, \mathbf{y})$ whose quadratic form is $\mathbf{C}(\mathbf{T}[f], \mathbf{T}[f])$. A formal interchange of integration, which can be justified, shows that

$$k(\mathbf{x}, \mathbf{y}) = \mathcal{C}(\mathbf{t}(\cdot, \mathbf{x}), \mathbf{t}(\cdot, \mathbf{y})).$$

Thus we see from (3.12) that

$$\begin{aligned} &k(\mathbf{x}, \mathbf{y}) \\ &\quad = \int_\Omega \{(1/C(\mathbf{z})[\tilde{t}^0(\mathbf{z}, \mathbf{x}) + r_i(\mathbf{z})\tilde{t}^i(\mathbf{z}, \mathbf{x})][\tilde{t}^0(\mathbf{z}, \mathbf{y}) + r_i(\mathbf{z})\tilde{t}^i(\mathbf{z}, \mathbf{y})] + \tilde{t}^i(\mathbf{z}, \mathbf{x})\tilde{t}^i(\mathbf{z}, \mathbf{y})\} d\mathbf{z} \\ &\quad + \int_{\partial\Omega} [1/\mathbf{r}(\mathbf{z}) \cdot \mathbf{n}(\mathbf{z})] \hat{t}(\mathbf{z}, \mathbf{x}), \hat{t}(\mathbf{z}, \mathbf{y}) ds_{\mathbf{z}}. \end{aligned} \tag{3.16}$$

Once this quantity is found for each $\mathbf{x}$ and $\mathbf{y}$, one still needs to calculate the trace

$$\int_\Omega \int_\Omega k(\mathbf{x}, \mathbf{y})^2 d\mathbf{x} d\mathbf{y}. \tag{3.17}$$

which is an upper bound for the trace of G^2 in the Trefftz inequality (2.10).

In order to improve the upper bound for the trace of G^2 obtained in this way, we recall that the kernel $\mathbf{t}$ only needs to satisfy the conditions (3.15). If $\mathbf{t}$ satisfies these conditions and if the vector field $\mathbf{q}(\mathbf{x}, \mathbf{y})$, as a function of $\mathbf{x}$, satisfies the corresponding homogeneous conditions

$$\begin{aligned} &\tilde{q}^0(\mathbf{x}, \mathbf{y}) - \tilde{q}^i_{,x_i}(\mathbf{x}, \mathbf{y}) = 0 \text{ for } \mathbf{x} \in \Omega \\ &\tilde{q}^i(\mathbf{x}, \mathbf{y}) n_i(\mathbf{x}) + \hat{q}(\mathbf{x}, \mathbf{y}) = 0 \text{ for } \mathbf{x} \in \partial\Omega \end{aligned} \tag{3.18}$$

for each $\mathbf{y}$, then $\mathbf{t}-\mathbf{q}$ again satisfies these conditions. In particular, if $\mathbf{q}=\mathbf{t}-\mathbf{S}[g]$, then $k=g$, so that one obtains the exact value of the trace of G^2. We cannot, of course, find this $\mathbf{q}$. Instead, Fichera suggested applying a Galerkin method.

Let $\mathbf{q}_1(\mathbf{x}), \mathbf{q}_2(\mathbf{x}), \mathbf{q}_L(\mathbf{x})$ be L vector fields which satisfy the conditions (3.18). We may then replace $\mathbf{t}$ by $\mathbf{t}-\sum_{\mu=1}^{L} c_\mu(\mathbf{y})\mathbf{q}_\mu(\mathbf{x})$ and determine the coefficients c_μ to make the corresponding quadratic form $\mathcal{C}(T[f]-\sum_{\mu=1}^{L}(c_\mu,f)\mathbf{q}_\mu, T[f]-\sum_{\mu=1}^{L}(c_\mu,f)\mathbf{q}_\mu)$ as small as possible. This form is a quadratic function of the numbers (c_ν, f), and we only need to set its gradient with respect to these numbers equal to zero to find the condition that $\mathcal{C}(T[f]-\sum_{\mu=1}^{L}(c_\mu,f)\mathbf{q}_\mu, q_\nu)=0$ for $\nu=1,\cdots,L$. If $\Phi^{\mu\nu}$ denotes the inverse of the $L\times L$ matrix $\mathcal{C}(\mathbf{p}_\mu,\mathbf{p}_\nu)$, we see that $(c_\mu,f)=\sum_{\nu=1}^{L}\Phi^{\mu\nu}\mathcal{C}(\mathbf{T}[f],\mathbf{q}_\nu)$. A small calculation shows that the minimizing quadratic form is that of the integral operator with the kernel

$$k^{(L)}(\mathbf{x},\mathbf{y}) :== \mathcal{C}(\mathbf{t}(\cdot,\mathbf{x}),\mathbf{t}(\cdot,\mathbf{y})) - \sum_{\mu,\nu=1}^{L}\Phi^{\mu\nu}\mathcal{C}(\mathbf{t}(\cdot,\mathbf{x}),\mathbf{q}_\mu(\cdot))\mathcal{C}(\mathbf{t}(\cdot,\mathbf{y}),\mathbf{q}_\nu(\cdot)). \tag{3.19}$$

Since we have chosen the integral operator with this kernel so that it has the smallest quadratic form, its square also has the smallest trace. By construction, the kernel which is being subtracted from the kernel in (3.16) is nonnegative semidefinite and nondecreasing in L. If the sequence of $\mathbf{q}_\nu$ satisfies a suitable completeness condition, the trace of the square of the operator whose kernel is the right-hand side of (3.19) will converge to the trace of G^2 as L approaches infinity, so that one obtains an arbitrarily close upper bound for the trace of G^2 in this way.

We observe that the evaluation of the kernel $k^{(L)}$ involves several two-dimensional integrals at each point. The evaluation of the trace of the square of the operator involves a four-fold integral. Moreover, the integrands contain unbounded terms. Thus finding and implementing a numerical method which gives an upper bound for this integral appears to be a formidable task. However, the application of Fichera's original method to problems for which it can be applied requires essentially the same task.

Bibliography

[1] L. Bassotti, *Fichera's orthogonal invariants method and numerical applications.* in Homage to Gaetano Fichera, ed. A. Cialdea, Quaderni di Matematica [Mathematics Series] 2, Aracne Editrice, Rome, 2001.

[2] J. B. Diaz and A. Weinstein, *Schwarz's inequality and the methods of Rayleigh-Ritz and Trefftz.* J. of Math. and Phys. 46 (1947), 133-136.

[3] G. Fichera, *Linear elliptic differential systems and eigenvalue problems.* Lecture Notes in Mathematics #8, Springer-Verlag, Berlin-New York, 1965.

[4] G. Fichera, *The Neumann eigenvalue problem.* Applicable Anal. 3 (1973), 213-240.

[5] H. Poincaré, Sur les équations aux dérivées partielles de la physique mathématique. American Journal of Mathematics 12 (1890), 211-294.

[6] W. Prager and J. L. Synge, *Approximations in elasticity based on the concept of function space.* Quarterly of Appl. Math. 5 (1947), 241-269.

[7] J. L. Synge, *The Hypercircle in Mathematical Physics.* Cambridge University Press, 1957.

[8] E. Trefftz, *Über Fehlerabschätzung bei Berechnen von Eigenwerten.* Math. Ann. 108 (1933), 595-604.

[9] H. F. Weinberger, *Variational Methods for Eigenvalue Problems.* CBMS Regional Conference Series in Applied Mathematics 15, SIAM 1974.

[10] H. F. Weinberger, *Fichera's method for bounding eigenvalues.* Convegno Int. Interactions Between Mechanics and Analysis: The Legacy of Gaetano Fichera. Atti dei Convegni Lincei 148 (1999), pp. 51-65.

[11] H. F. Weinberger, *A user-friendlier form of Fichera's method for two sided eigenvalue bounds.* In preparation.

[12] A. Weinstein and W. Stenger, *Methods of Intermediate Problems for Eigenvalues.* Acad. Press, New York, 1972.