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On Dualizing a Multivariable Poisson Summation Formula

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ABSTRACT. It is known [7] that dualizing a form of the Poisson summation formula yields a pair of linear transformations which map a function ϕ of one variable into a function and its cosine transform in a generalized sense. The present work presents conditions on ϕ for which the transform relation holds in the classical sense, and extends this result to a class of generalizations of the Poisson formula in any number of dimensions.

1. Introduction

Let the Fourier transform g of a function f of one variable be defined by the formula

$$g(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} f(s)\ ds.$$

By the cosine transform we mean the special case in which f is even, and by the sine transform we mean $-i$ times the special case in which f is odd.

It was shown by one of the authors [4] that if $\phi(s)$ is a "sufficiently nice" odd function, then the sums

$$\phi(x) - \frac{1}{3}\phi(x/3) + \frac{1}{5}\phi(x/5) - \cdots \tag{1.1}$$

and

$$\frac{1}{|x|}\phi(1/x) - \frac{1}{|x|}\phi(3/x) + \frac{1}{|x|}\phi(5/x) - \cdots \tag{1.2}$$

are sine transforms of each other in several generalized senses. This result was extended in [5]. A large class of other pairs of such sums which are either sine or cosine transforms of each other in some generalized senses were given in [10] and [6]. Boas [1] showed that the result about the pair (1.1), (1.2) could be obtained from a form of the Poisson summation formula.

A particular case of these results is the fact that for any sufficiently nice even function ϕ the sums

$$\frac{1}{|x|}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \phi\left(\frac{n}{x}\right) - \int_{-\infty}^{\infty}\phi(t)dt \tag{1.3}$$

and

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\phi\left(\frac{x}{n}\right) - \int_{-\infty}^{\infty}\frac{1}{t}\phi\left(\frac{1}{t}\right) dt. \tag{1.4}$$

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form a pair of cosine transforms in a generalized sense. It was shown in [7] that this fact can be considered as the dual of a form of the Poisson summation formula.

The purpose of the present work is threefold: In Section 2 we shall show that all the above statements about transform pairs can be obtained as the duals of a generalization of the Poisson summation formula due to Mordell. In Section 3 we present conditions on the arbitrary function ϕ which ensure that the sums form a transform pair in the distribution sense, as well as an additional condition which forces the limit of one of the sums to be an L_1 function, so that the other sum is its Fourier transform in the classical sense. In Section 4 we show how these ideas can be extended to obtain pairs of higher-dimensional transforms.

As we shall point out when we introduce the Poisson-Mordell summation formula in Section 2, this formula leads to a variant of the Shannon sampling theorem which may be of some interest in itself.

2. The Poisson-Mordell formula and its dual

The classical Poisson summation formula states that

$$\sum_{k=-\infty}^{\infty} g(k) = \sum_{k=-\infty}^{\infty} f(k). \tag{2.1}$$

This equation is just the statement that the value at $t = 0$ of the Fourier series of the 1-periodic function $\sum_{-\infty}^{\infty} f(t+k)$ is equal to the value of this function at $t = 0$. It is certainly valid for the Schwartz class of rapidly decaying functions, that is, infinitely differentiable functions with the property that the function and all its partial derivatives approach zero more rapidly than any negative power of $|x|$ as $|x|$ goes to infinity. It is also true for larger classes of functions. See, for example, Titchmarsh [9].

When the function $f(t)$ is replaced by $f(xt)$ where x is a parameter with a value different from zero, and when the terms with $m = 0$ are transposed, the Poisson summation formula takes the equivalent form

$$\frac{1}{|x|}\sum_{m\neq 0} g\left(\frac{m}{x}\right) - \int_{-\infty}^{\infty} g(t)dt = \sum_{m\neq 0} f(mx) - \frac{1}{|x|}\int_{-\infty}^{\infty} f(t)dt. \tag{2.2}$$

The formula (2.2) with x as a variable may be viewed as the statement that two linear transformations applied to f (the one which produces the left-hand side and that which produces the right side) are equal. In a previous work [7] we showed that the dual of this equation (that is, the equality of the adjoint transformations) leads to the statement that if $\phi(x)$ is a sufficiently nice even function, then the function (1.3) has, in a generalized sense, the Fourier cosine transform (1.4).

In order to include other results such as the fact that the sums (1.1) and (1.2) form a pair of sine transforms, we first let $\{c_m\}$ be any sequence of real numbers which is periodic of some period K:

$$c_{m+K} = c_m \quad \text{for } -\infty < m < \infty,$$

and define its discrete Fourier transform

$$d_m = \frac{1}{\sqrt{K}} \sum_{\ell=0}^{K-1} e^{2\pi i \ell m/K}\ c_\ell. \tag{2.3}$$

Note that $\{d_m\}$ is automatically K-periodic.

We recall that for any constants $\alpha \neq 0$ and β the Fourier transform of the function $f(\alpha t+\beta)$ is $e^{-2\pi i\beta t/\alpha} g(t/\alpha)/|\alpha|$, where $g(t)$ is the Fourier transform of $f(t)$. Therefore, if we follow Mordell [8] in replacing $f(t)$ in (2.1) by the finite linear combination $\sum_{\ell=0}^{K-1} d_\ell f([t+(\ell/K)]x)$ with the parameter x different from zero, we obtain the equation

$$\frac{1}{|x|} \sum_{0<|m|<\infty} c_m g\left(\frac{m}{x}\right) - \frac{d_0}{\sqrt{K}} \int_{-\infty}^{\infty} g(t)dt = \frac{1}{\sqrt{K}} \sum_{0<|m|\infty} d_m f\left(\frac{mx}{K}\right) - \frac{c_0}{|x|} \int_{-\infty}^{\infty} f(t)dt. \tag{2.4}$$

(We have used the well-known fact that the sequence $\{c_m\}$ can be obtained from $\{d_m\}$ by interchanging the two sequences and replacing i by $-i$ in (2.3).) We shall refer to (2.4) as the **Poisson-Mordell summation formula**.

We note in passing that if this formula is used in Boas's derivation [2] of the Shannon sampling theorem, it gives the reconstruction formula

$$f(x) \sum_{m=0}^{K-1} c_m e^{2\pi i[2m-K+1]\Omega t} = \sum_{n=-\infty}^{\infty} d_n f\left(\frac{n}{2K\Omega}\right) e^{-\pi i(K-1)[n-2K\Omega x]/K}\ \frac{\sin \pi[n-2K\Omega x]}{\pi[n-2K\Omega x]}$$

for any function f whose Fourier transform $g(y)$ vanishes for $|y| > \Omega$.

We think of the two sides of (2.4) as linear transformations. For the given K-periodic sequence $\{c_j\}$ we define the **discrepancy** $\mathcal{D}[\{c_j\}; g]$ so that it gives the left side of this equation:

$$\mathcal{D}[\{c_j\}; g](x) := \frac{1}{|x|} \sum_{0<|m|<\infty} c_m g\left(\frac{m}{x}\right) - \frac{d_0}{\sqrt{K}} \int_{-\infty}^{\infty} g(t)dt. \tag{2.5}$$

(Note that for large $|x|$ the sum can be viewed as a quadrature rule for approximating $d_0/\sqrt{K}$ times the integral, and the discrepancy is the error of this approximation.)

We observe that if we define the **reversion** $\mathcal{R}_K$ by

$$\mathcal{R}_K[\phi](x) := \frac{\sqrt{K}}{|x|} \phi\left(\frac{K}{x}\right) \tag{2.6}$$

for $x \neq 0$,we can write the equation (2.4) in the form

$$\mathcal{D}[\{c_j\}; g] = \mathcal{R}_K[\mathcal{D}[\{d_j\}; f]]. \tag{2.7}$$

We now consider the the dual of the equation (2.7) with respect to the pairing

$$\langle \phi, \psi \rangle := \int_{-\infty}^{\infty} \phi(x)\psi(x)dx.$$

To obtain the formal adjoint of the transformation $\mathcal{D}$, we multiply its definition (2.5) by a "sufficiently nice" function $\phi(x)$ and integrate over the real line. We assume that we may replace the integral of the sum by a sum of integrals, make the change of variable $s = m/x$ in each of these integrals, and then interchange summation with integration again. This leads to the formula

$$\int_{-\infty}^{\infty} \phi(x)\mathcal{D}[\{c_j\};g](x)dx = \int_{-\infty}^{\infty} \mathcal{D}[\{c_j\};\phi](s)g(s)ds. \tag{2.8}$$

That is, $\mathcal{D}$ is its own formal adjoint.

In order to treat the right-hand side of (2.7), we also need the adjoint of $\mathcal{R}_K$. The change of variable $y = K/x$ shows that

$$\int_{-\infty}^{\infty} \phi(x)\mathcal{R}_K[\psi](x)dx = \int_{-\infty}^{\infty} \mathcal{R}_K[\phi](y)\psi(y)dy.$$

That is, the transformation $\mathcal{R}_K$ is also its own formal adjoint. Thus we see that if we multiply the Mordell formula (2.7) by a function ϕ and integrate, we formally obtain the equation

$$\int_{-\infty}^{\infty} F(s)g(s)ds = \int_{-\infty}^{\infty} G(s)f(s)ds, \tag{2.9}$$

where

$$F(s) = \mathcal{D}[\{c_j\};\phi](s) = \sum_{0<|m|<\infty} c_m \frac{1}{|s|}\phi\left(\frac{m}{s}\right) - \frac{d_0}{\sqrt{K}} \int_{-\infty}^{\infty} \phi(x)dx \tag{2.10}$$

and

$$G(s) = \mathcal{D}[\{d_j\};\mathcal{R}_K[\phi]] = \sum_{0<|m|<\infty} d_m \frac{\sqrt{K}}{|m|}\phi\left(\frac{Ks}{m}\right) - c_0 \int_{-\infty}^{\infty} \frac{\phi(x)}{|x|}dx. \tag{2.11}$$

If the identity (2.9) is valid for all rapidly decaying functions f and their (rapidly decaying) Fourier transforms g, then, by definition, G is the Fourier transform of F in the sense of distributions.

The sequence $c_m = \sin m\pi/2$ is periodic of period $K = 4$, and its finite Fourier transform (2.3) is $d_m = ic_m$. The fact that $-i$ times the series (2.11) with this sequence d_m is the Fourier transform of (2.10) is just the statement that the sum (1.2) is the sine transform of the sum (1.1). This was proved in various generalized senses in [4] and [5]. The results were extended in [10] and [6] to K-periodic sequences c_m which are eigenvectors of the finite Fourier transform F_K defined by (1.2). Because $(F_K)^4 = I$, the only possible eigenvalues are ± 1 and $\pm i$. When the eigenvalue is real, the corresponding sequence c_m is even, so that the even function (2.11) with $d_m = \pm c_m$ is the cosine transform of the even function (2.10). If the eigenvalue is $\pm i$, then (2.11) with $d_m = \pm c_m$ is the sine transform of the odd function (2.10).

We have shown that, at least formally, dualizing the Poisson-Mordell formula gives an extension of all the previously known results. Of course, the formal derivation needs to be justified.

3. Convergence in the classical sense

We note that if ϕ has a limit other than 0 at $x = 0$, the sequence in (2.11) diverges. Thus some conditions on ϕ are certainly needed to verify the equation (2.9). In this Section we shall prove the following result.

Theorem A.

Assume that $\phi(x)$ and $\phi'(x)$ are absolutely continuous, and that

$$\int_{-\infty}^{\infty} \{[1 + (1/|x|)][|\phi(x)| + |x||\phi'(x)|] + |\phi''(x)|\}\, dx < \infty. \tag{3.1}$$

Then the functions F and G defined by the equations (2.10) and (2.11) are continuous, F is absolutely integrable on the real line, and G is the Fourier transform of F in the classical sense.

Proof. Our results will be based on three lemmas.

Lemma 1.

If $\phi(x)$ is of bounded variation and

$$\begin{aligned} &\int_{-\infty}^{\infty} [1 + (1/|x|)]|\phi(x)|dx + \int_{-\infty}^{\infty} [1 + |x|]|d\phi(x)| < \infty, \\ &\phi(x) = \frac{1}{2}[\phi(x+0) + \phi(x-0)] \qquad \text{for all } x, \end{aligned} \tag{3.2}$$

then the series in (2.10) converges uniformly on every bounded s-set to a function which is zero at $s = 0$. Moreover, the partial sums of the series are bounded uniformly in s.

Proof of Lemma 1. It is useful to introduce the piecewise linear function p with jumps only at the integers which is defined by the properties

(i.) $p'(r) = d_0/\sqrt{K}$ for r not an integer

(ii.) $p(m+0) - p(m-0) = -c_m$ and $p(m) = \frac{1}{2}\{p(m+0) + p(m-0)\}$ for $m = 0, \pm1, \cdots$

(iii.) $p(0) = 0$

We observe that if r is increased by K, the function p increases by $d_0\sqrt{K}$ because of its slope, and decreases by the sum of K consecutive jumps c_m. By (2.3) these quantities are equal, and we conclude that p **is periodic of period** K. In particular, it is bounded.

Integration by parts and the properties **i-iii** show that for any integers M and $N > M$ and any s different from zero

$$\begin{aligned} &\sum_{m=M}^{N} c_m \frac{1}{|s|}\phi(m/s) - (d_0/\sqrt{K})\frac{s}{|s|}\int_{M/s}^{N/s} \phi(x)dx = -\int_{M-0}^{N+0} \frac{1}{|s|}\phi(y/s)dp(y) \\ &= -\frac{p(N+0)\phi((N+0)/s)}{|s|} + \frac{p(M-0)\phi((M-0)/s)}{|s|} + \int_{(M-0)/s}^{(N+0)/s} \frac{1}{|s|}p(sx)d\phi(x). \end{aligned} \tag{3.3}$$

The condition (3.2) implies that $|\phi(x)|$ and $|x\phi(x)|$ are uniformly bounded, and that $\phi(0) = 0$. If we take $N > M > 0$, we observe that $1/|s| \leq |x|/M$ on the interval of integration. Therefore we see from (3.3) that

$$\left|\sum_{m=M}^{N} c_m \frac{1}{|s|}\phi(m/s)\right| \leq |d_0/\sqrt{K}| \left|\int_{M/s}^{N/s} |\phi(x)|dx\right| + \frac{C}{N} + \frac{C}{M} + \frac{C}{M}\left|\int_{M/s}^{N/s} |x||\phi'(x)|dx\right|, \tag{3.4}$$

where C is a bound for $|p|$ and for $|p(sx)x\phi(x)|$. When M goes to infinity, the right-hand side approaches zero uniformly in s on any bounded set. The same argument also works when $M < N < 0$ and N goes to $-\infty$.

Because $|\phi(x)|$ and $|x\phi(x)|$ are uniformly bounded, each term of the series is uniformly bounded, and the uniform convergence on bounded sets shows that the partial sums of the series are uniformly bounded for $|s| \leq 1$. On the other hand, it is easily seen from the equation (3.3) that the partial sums are also uniformly bounded for $|s| \geq 1$, so that the last statement of Lemma 1 is proved.

We can now find conditions under which the equation (2.9) is valid.

Lemma 2.

Suppose that the function f and its Fourier transform g as well as ϕ satisfy the condition (3.2) of Lemma 1. Then the equation (2.9) is valid. In particular, G is the Fourier transform of F in the sense of distributions.

Proof of Lemma 2. By Lemma 1, the series on the left of (2.4) converges everywhere and has bounded partial sums. Hence by the Lebesgue dominated convergence theorem the integral of ϕ times the sum on the left is equal to the sum of the integrals. That is, the first interchange of integration and summation in the derivation of the left-hand side of (2.9) is justified. The other interchange is justified by interchanging the roles of g and ϕ in this argument.

To justify the interchanges on the right-hand side, we first observe that the change of variables $y = K/x$ takes the integral of ϕ times the right-hand-side of (2.4) into the integral of $\mathcal{R}_K[\phi]$ times $\mathcal{D}[\{d_j\}; f]$, while the right-hand side of (2.8) is $\mathcal{D}[\{d_j\}; \mathcal{R}_K[\phi]]$. It is easily verified that if ϕ satisfies the conditions (3.2), the same is true of $\mathcal{R}_K[\phi]$. Therefore, the right-hand side of (2.9) is established by the arguments which were used to establish the left-hand side.

Any rapidly decaying function f and its (rapidly decaying) Fourier transform g satisfy the conditions (3.2), and this proves the last sentence of the Lemma.

Remark.

If one chooses f to be the characteristic function of the interval between 0 and a real number a, Lemma 2 yields the generalized transform formula (see [4, 7])

$$G(a) = \frac{d}{da} \int_{-\infty}^{\infty} F(x) \frac{e^{2\pi i a x} - 1}{2\pi i x} dx.$$

In order to obtain a classical result, we give a condition which forces F to be absolutely integrable.

Lemma 3.

Suppose that ϕ and ϕ' are absolutely continuous, that the conditions (3.2) of Lemma 1 are satisfied, and that

$$\int_{-\infty}^{\infty} |\phi''(x)| dx < \infty. \tag{3.5}$$

Then the function F is absolutely integrable on the real line.

Proof of Lemma 3. The conditions (3.2) imply that $\phi(0) = 0$, so that the term with $m = 0$ is absent from the sum in (3.3). We let $M \to -\infty$ and $N \to \infty$ in (3.3) to obtain the representation

$$F(s) = \frac{1}{s} \int_{-\infty}^{\infty} p(sx)\phi'(x)dx. \tag{3.6}$$

Because p is periodic of period K, the same is true of the function

$$P(r) := \int_0^r p(s)ds - K^{-1} r \int_0^K p(s)ds.$$

The derivative of this function differs from p by a constant. Because ϕ vanishes at $\pm\infty$, the integral of ϕ' over the real line is zero. Thus we may replace p by P' in (3.6). We integrate this new equation by parts to obtain the formula

$$F(s) = \frac{-1}{s^2} \int_{-\infty}^{\infty} P(sx)\phi''(x)dx.$$

By construction, the function P is periodic, and hence bounded. Because ϕ'' is absolutely integrable, it follows that $s^2|F(s)|$ is uniformly bounded. Because ϕ is continuously differentiable and $x\phi(x)$ has the limit zero at infinity, the terms of the series in (2.10), defined as 0 at $s = 0$, are continuous. By the uniform convergence, F is continuous. Thus $(1 + s^2)F(s)$ is bounded, so that F is absolutely integrable. This completes the proof of Lemma 3.

Proof of Theorem A. Lemma 2 states that G is the Fourier transform of F in the distribution sense, and Lemma 3 states that $F \in L_1(-\infty, \infty)$. Therefore G is the transform of F in the classical sense, which proves Theorem A.

Remark.

If, in addition, $|x||\phi(x)| + |x|^2|\phi'(x)| + |x|^3|\phi''(x)| \in L_1$ *so that* $R_K[\phi]'' \in L_1$*, then the proof of Lemma 3 shows that* G *is also in* L_1*, so that* F *is its inverse Fourier transform in the classical sense.*

We observe that if $d_0 = 0$ so that the first integral on the right of (3.3) is absent, the inequality (3.4) shows that the series for F converges uniformly on the whole real line. A similar comment applies to the series for G if $c_0 = 0$.

We shall say that the periodic sequence $\{c_m\}$ is **oscillatory** if $c_0 = d_0 = 0$, or, equivalently, if $c_0 = \sum_{j=0}^{K-1} c_j = 0$. Note that when $\{c_m\}$ is oscillatory, the function $p(r)$ is piecewise constant and vanishes for $|r| < 1$.

The absence of the integral terms in the definitions of F and G when an oscillatory sequence is used will be very helpful in extending the results to higher dimensions, which we proceed to do. For the sake of simplicity, we shall only present the results and proofs for the case of two dimensions.

4. The two-dimensional formula.

If $f(x, y)$ is a function of two variables, we define the Fourier transform

$$g(x, y) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i(x\xi + y\eta)} f(\xi, \eta) d\xi d\eta.$$

The Poisson summation formula in two dimensions is (see [3])

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(m,n) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n). \tag{4.1}$$

As before, it states that the value at the origin of the doubly periodic function $\sum f(m+s, n+t)$ is equal to the value of its Fourier series there. This is certainly true when f is rapidly decaying.

Let $\{c_{mn}\}$ be a double sequence which is periodic of some period K_1 in m and periodic of period K_2 in n. We define its discrete Fourier transform

$$d_{mn} := \frac{1}{\sqrt{K_1 K_2}} \sum_{k=0}^{K_1}\sum_{\ell=0}^{K_2} e^{2\pi i(mk+n\ell)} c_{k\ell}.$$

Let x and y be two nonzero parameters. If we replace the function $f(s,t)$ in **4.1** by the finite linear combination

$$\sum_{k=0}^{K_1}\sum_{\ell=0}^{K_2} d_{k\ell} f([s+(k/K_1)]x, [t+(\ell/K_2)]y),$$

we obtain the two-dimensional extension

$$\frac{1}{|xy|}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_{mn} g\left(\frac{m}{x}, \frac{n}{y}\right) = \frac{1}{\sqrt{K_1 K_2}}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} d_{mn} f\left(\frac{mx}{K_1}, \frac{ny}{K_2}\right) \tag{4.2}$$

of the Poisson-Mordell summation formula.

In order to write this equation in a form whose dual one can find, one would need to transpose those terms one of whose indices is zero, and then rewrite these terms as integrals. Because there are, in general, infinitely many such terms, this gives a very complicated analog of (2.4). We prefer to assume this difficulty away by means of the following definition.

DEFINITION. *The doubly periodic sequence $\{c_{mn}\}$ is said to be* **oscillatory** *if*

$$c_{mn} = d_{mn} = 0 \quad \text{whenever } m \text{ or } n \text{ is zero.}$$

It is easily verified that $\{c_{mn}\}$ is oscillatory if and only if the one-dimensional sequences obtained by fixing m at any integer and those obtained by fixing n at any integer are oscillatory.

We assume from now on that the sequence $\{c_{mn}\}$ is oscillatory. Then the sums in (4.2) do not contain terms in which either m or n is zero, so that we can easily compute the formal adjoints of the linear transformations on the two sides of the identity (4.2) with respect to the pairing

$$\langle \phi, \psi \rangle := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\psi(x,y)\,dxdy.$$

If we define the two-dimensional discrepancy $\mathcal{D}[\{c_{mn}\};g](x,y)$ to be the left-hand side of (4.2), and the two-dimensional reversion

$$\mathcal{R}_{K_1K_2}[\phi](x,y) := \frac{\sqrt{K_1K_2}}{|xy|}\phi\left(\frac{K_1}{x},\frac{K_2}{y}\right),$$

we can write the equation (4.2) in the form $\mathcal{D}[\{c_{mn}\};g] = \mathcal{R}_{K_1,K_2}[\mathcal{D}[\{d_{mn}\};f]]$. It is easily seen that the transformations $\mathcal{D}$ and $\mathcal{R}_{K_1,K_2}$ are again their own formal adjoints. Therefore, if we multiply both sides of (4.2) by a nice function $\phi(x,y)$, integrate over the plane, and perform manipulations like those used to derive (2.9), we obtain the formal dual relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(s,t)g(s,t)dsdt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(s,t)f(s,t)dsdt, \tag{4.3}$$

where

$$F(s,t) = \mathcal{D}[\{c_{mn}\};\phi](s,t) = \sum_{0<|m|,|n|<\infty} \frac{c_{mn}}{|st|}\phi\left(\frac{m}{s},\frac{n}{t}\right) \tag{4.4}$$

and

$$G(s,t) = \mathcal{D}[\{d_{mn}\};\mathcal{R}_{K_1,K_2}[\phi]] = \sum_{0<|m|,|n|<\infty} \frac{\sqrt{K_1K_2}d_{mn}}{|mn|}\phi\left(\frac{K_1s}{m},\frac{K_2t}{n}\right). \tag{4.5}$$

We shall again obtain a condition under which G is the classical Fourier transform of F. We use subscripts to denote partial derivatives.

Theorem A$'$.

Suppose that the sequence $\{c_{mn}\}$ is periodic and oscillatory. Assume that the function ϕ satisfies the conditions

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{&[1+(1/|x|)][1+(1/|y|)][|\phi(x,y)|+|x||\phi_x|+|y||\phi_y|+|x||y||\phi_{xy}|] \\ &+(1+|y|)|\phi_{xxy}|+(1+|x|)|\phi_{xyy}|+|\phi_{xxyy}|\}dxdy<\infty,\end{aligned} \tag{4.6}$$

and that ϕ and all the derivatives which appear in this integral are continuous. Then the series in the definitions (4.4) and (4.5) of F and G converge uniformly to functions, F is absolutely integrable, and G is the classical Fourier transform of F.

Proof. We shall prove this result by adapting the proof of Theorem A. We begin with the analog of the version of Lemma 1 which is obtained when the sequence is oscillatory.

Lemma 1$'$.

If $\phi(x,y)$, ϕ_x, ϕ_y, and ϕ_{xy} are continuous and if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+(1/|x|)][1+(1/|y|)][|\phi(x,y)|+|x||\phi_x|+|y||\phi_y|+|x||y||\phi_{xy}|]dxdy<\infty, \tag{4.7}$$

then the series in (4.4) converges uniformly in the plane.

Proof of Lemma $1'$. We define the two-dimensional version of the periodic function p in the proof of Lemma 1 for the case of an oscillatory sequence by the following properties.

(i.) For every pair of integers (m, n), the function $p(q, r)$ is a constant a_{mn} in the square $m-1 < q < m$, $n-1 < r < n$.

(ii.) $a_{m+1,n+1} - a_{m+1,n} - a_{m,n+1} + a_{mn} = c_{mn}$.

(iii.) $a_{mn} = 0$ when either m or n is 0.

The function p is defined as the average of its limits on the integer lines and their intersections.

Because the sequence $\{c_{mn}\}$ is oscillatory, the sequence $\{a_{mn}\}$, and hence the function p, is doubly periodic with periods K_1 and K_2. In particular, p is bounded. The fact that $\{c_{mn}\}$ is oscillatory also permits us to conclude from (ii) and (iii) that $a_{mn} = 0$ when either m or n is 1, so that $p(r, s) = 0$ when either $|r| < 1$ or $|s| < 1$.

Because p is piecewise constant, a sequence of integrations shows that the analog

$$\begin{aligned}
\sum_{m=M_1}^{M_2} \sum_{n=N_1}^{N_2} \frac{1}{|st|} c_{mn} \phi\left(\frac{m}{s}, \frac{n}{t}\right) = &\int_{M_1/s}^{M_2/s} \int_{N_1/t}^{N_2/t} \frac{1}{|st|} p(sx, ty) \phi_{xy}(x, y)\, dx dy \\
&- \int_{M_1/s}^{M_2/s} \frac{1}{|st|} [p(xs, N_2+0)\phi_x(x, N_2/t) - p(xs, N_1-0)\phi_x(x, N_1/t)]\, dx \\
&- \int_{N_1/t}^{N_2/t} \frac{1}{|st|} [p(M_2+0, ty)\phi_y(M_2/s, y) - p(M_1-0, ty)\phi_y(M_1/s, y)]\, dy \\
&+ \frac{1}{|st|} \{p(M_2+0, N_2+0)\phi(M_2/s, N_2/t) \\
&\qquad + p(M_1-0, N_1-0)\phi(M_1/s, N_1/t) \\
&\qquad - p(M_1-0, N_2+0)\phi(M_1/s, N_2/t) \\
&\qquad - p(M_2+0, N_1-0)\phi(M_2/s, N_1/t)\}
\end{aligned} \tag{4.8}$$

of (3.3) is valid for any integers M_1, M_2, N_1, and N_2.

The obvious extensions of the arguments which led from (3.3) to the statement of Lemma 1 now serve to show that (4.8) implies the statement of Lemma $1'$.

The argument by which we derived Lemma 2 from Lemma 1 shows that Lemma $1'$ implies the analogous Lemma:

Lemma $2'$.

If the function $f(x, y)$ and its Fourier transform g as well as the function ϕ satisfy the conditions of Lemma $1'$, then the formula (4.3) is valid. In particular, G is the Fourier transform of F in the distribution sense.

We finish the proof of Theorem A$'$ by proving the analog of Lemma 3.

Lemma $3'$.

If ϕ satisfies the conditions of Lemma $1'$, if ϕ_{xxy}, ϕ_{xyy}, and ϕ_{xxyy} are continuous, and if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{(1+|y|)|\phi_{xxy}| + (1+|x|)|\phi_{xyy}| + |\phi_{xxyy}|\}\, dx dy < \infty, \tag{4.9}$$

then F is absolutely integrable over the plane.

Proof of Lemma $3'$. We let $-M_1,\ M_2,\ -N_1$, and N_2 approach infinity in (4.8) to obtain the representation

$$F(s,t) = \frac{1}{st}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(sx,ty)\phi_{xy}(x,y)dxdy. \tag{4.10}$$

In analogy with the proof of Lemma 3, we define the function

$$\begin{aligned} P(q,r) = &\int_0^q\int_0^r p(\xi,\eta)d\xi d\eta - \frac{q}{K_1}\int_0^{K_1}\int_0^r p(\xi,\eta)d\xi d\eta \\ &- \frac{r}{K_2}\int_0^q\int_0^{K_2} p(\xi,\eta)d\xi d\eta + \frac{qr}{K_1K_2}\int_0^{K_1}\int_0^{K_2} p(\xi,\eta)d\xi d\eta, \end{aligned}$$

which, as is easily verified, is again periodic with periods K_1 in q and K_2 in r. Its mixed partial derivative P_{rs} differs from $p(r,s)$ by the sum of a function of r only and a function of s only. Because under our hypotheses the function $\int|\phi_y(x,y)|dy$ goes to zero as $|x|$ goes to infinity and $\int|\phi_x(x,y)|dx$ goes to zero as $|y|$ goes to infinity, we may replace p by P_{rs} in (4.10).

An easily justified integration by parts shows that

$$F(s,t) = -\frac{1}{s^2t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_r(sx,ty)\phi_{xxy}(x,y)dxdy.$$

By construction, the function $P_r(q,r)$ is doubly periodic and hence bounded. It also vanishes when $|r|<1$, so that $1/|t| \le |y|$ where $P_r(sx,yt)\neq 0$. Therefore, the equation (4.10) shows that $s^2|F|$ is bounded by a multiple of the integral of $|y\phi_{xxy}|$.

A similar argument shows that $t^2|F|$ is bounded.

A second integration by parts gives the representation

$$F(s,t) = \frac{1}{s^2t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(sx,ty)\phi_{xxyy}(x,y)dxdy,$$

from which we can conclude that $s^2t^2|F(s,t)|$ is also bounded.

Since we know from Lemma $1'$ that F is continuous, we conclude that $(1+s^2)(1+t^2)|F|$ is bounded, which proves the statement of Lemma $3'$.

Proof of Theorem A'. We have shown that G is the Fourier transform in the sense of distributions of the L_1 function F. Hence it is also the classical Fourier transform, which is the statement of Theorem A$'$.

The above Lemmas and Theorem are easily extended to any number of dimensions. The analogs of Lemmas 1 and 2 require assumptions about integrals which involve polynomials times mixed derivatives of ϕ in which the order of differentiation with respect to any one variable is at most one. The analog of Lemma 3 requires such assumptions about derivatives in which the order with respect to any one variable is at most 2. All these conditions are certainly satisfied if both ϕ and $R_K[\phi]$ are rapidly decaying.

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