Inward linear perturbation can produce unbounded solutions

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Abstract

One may expect that if one adds a vector field which points toward the origin to the right-hand side of a first-order system of ordinary differential equations for which the origin is globally stable, then the origin is globally stable for the new system as well. The purpose of this note is to present an example which shows that this apparently obvious statement is, in fact, false.

1 Introduction.

It seems intuitively obvious that if one adds a vector field which points toward the origin to the right-hand side of a first-order system of ordinary differential equations for which the origin is globally stable, then the origin is globally stable for the new system as well. In this paper we show by example that perturbation by a linear vector field which points towards the origin can, in fact, change an autonomous system with a global attractor at the origin to one which has unbounded

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solutions. Our main result is that the family of systems of autonomous ordinary differential equations

$$\begin{aligned} u_t &= (u+2v)[u\sin(3\ln\sqrt{u^2+v^2}) + v\cos(3\ln\sqrt{u^2+v^2})] - 3v - \alpha u \\ v_t &= (v-2u)[u\sin(3\ln\sqrt{u^2+v^2}) + v\cos(3\ln\sqrt{u^2+v^2})] + 3u - \alpha v \end{aligned} \tag{1.1}$$

with the constant parameter α has the following property:

Theorem 1 *When $1 \leq \alpha < \frac{3}{2}$, all solutions $(u(t), v(t))$ of the autonomous system* (1.1) *converge to* $(0,0)$ *as t goes to infinity.*

When $\alpha > \frac{3}{2}$, any solution of the system (1.1) *for which the initial value of u^2+v^2 is sufficiently large has the property that u^2+v^2 approaches infinity.*

When $\alpha = \frac{3}{2}$, all solutions of the system (1.1) *are bounded, and all solutions with sufficiently large initial values of u^2+v^2 are bounded away from the origin.*

This Theorem shows, in particular, that the origin is a global attractor of the system (1.1) with $\alpha = 1$, but that adding the linear inward radial vector field $(-u, -v)$ to the right-hand sides of the equations produces unbounded solutions. Thus the "intuitively obvious" statement above is incorrect.

Another apparently paradoxical corollary of the Theorem is the following: Perturbing the right-hand side of the system (1.1) with $\alpha = 3/2$ with an arbitrarily small linear inward vector field creates unbounded solutions, while perturbing the same system with an arbitrarily small outward linear vector field makes all solutions approach the origin.

Section 2 presents the proof of the above Theorem. Section 3 presents a variant of the example which shows that adding a nonlinear inward radial vector field to the right-hand side of a system all of whose solutions converge to the origin can produce solutions which blow up in finite time.

In another paper [1] the authors will present an example of an ecological system with a global attractor at the origin for which adding an arbitrarily small linear inward vector field to the right-hand side produces solutions which blow up in finite time.

2 Proof of the Theorem.

Proof. We begin by introducing the usual polar coordinates defined by

$$u = r\cos\theta, \qquad v = r\sin\theta$$

into the system (1.1). An exercise in using the chain rule shows that

$$\begin{aligned}(\ln r)_t &= r\sin(3\ln r + \theta) - \alpha \\ \theta_t &= -2r\sin(3\ln r + \theta) + 3.\end{aligned} \tag{2.1}$$

By taking linear combinations of the two equations, we see that

$$(2\ln r + \theta)_t = 3 - 2\alpha \tag{2.2}$$

and

$$(3\ln r + \theta)_t = r\sin(3\ln r + \theta) + 3(1-\alpha). \tag{2.3}$$

Suppose that $1 \leq \alpha < 3/2$. Then the right-hand side of (2.2) is positive, so that the function $2\ln r + \theta$ increases to infinity. Moreover, the right-hand side of (2.3) is nonpositive whenever $3\ln r + \theta$ is an integral multiple of π, so that the quantity $3\ln r + \theta$ is bounded above for any solution. Therefore

$$\ln r = (3\ln r + \theta) - (2\ln r + \theta) \tag{2.4}$$

converges to $-\infty$. This means that r approaches zero, and hence that (u, v) approaches zero, which is the first statement of the Theorem.

Next we suppose that $\alpha > 3/2$. Then the right-hand side of (2.2) is negative, so that $2\ln r + \theta$ decreases to $-\infty$. In particular, if r_0 and θ_0 are the initial values of r and θ,

$$2\ln r + \theta \leq 2\ln r_0 + \theta_0. \tag{2.5}$$

Suppose that

$$\ln r_0 \geq 2\pi + \ln[3(\alpha - 1)]. \tag{2.6}$$

Then there is an integer n such that

$$3\ln r_0 + \theta_0 \geq (4n+1)\pi/2 \geq 2\ln r_0 + \theta_0 + \ln[3(\alpha-1)]. \tag{2.7}$$

The second inequality and (2.5) show that when $3\ln r + \theta = (4n+1)\pi/2$, the right-hand side of (2.3) is nonnegative. It follows that the set

$$\{(r,\theta) : 2\ln r + \theta \leq 2\ln r_0 + \theta_0,\ 3\ln r + \theta \geq (4n+1)\pi/2\}$$

is a positive invariant set of the system (2.1). Since the first inequality of (2.7) shows that the initial state lies in this set, (r, θ) lies in the set for all positive t. In particular, $3\ln r + \theta$ is bounded below. Since $2\ln r + \theta$ approaches $-\infty$, the identity (2.4) shows that $\ln r$ goes to infinity whenever the initial value r_0 satisfies the inequality (2.6). This is the second statement of the Theorem.

We now consider the remaining case $\alpha = 3/2$. In this case the right-hand side of (2.2) is zero, so that $2\ln r + \theta$ remains constant. The proof of the first part of the Theorem still shows that $3\ln r + \theta$ is bounded above. The representation (2.4) then shows that $\ln r$ is bounded above for any solution. On the other hand, the proof of the second statement of the Theorem shows that if r_0 satisfies the inequality (2.6), then $3\ln r + \theta$ is bounded below, and hence $\ln r$ is bounded below. That is, the solution is bounded away from the origin. Thus all the statements of the Theorem have been established.

3 Finite-time blowup.

In order to show that an added inward radial vector field can produce finite-time blowup, we consider the system

$$\begin{aligned} u_t = {}& (u+2v)[u\sin(3\ln\sqrt{u^2+v^2}) + v\cos(3\ln\sqrt{u^2+v^2})] \\ & -3[1+(\ln\sqrt{u^2+v^2})^2]v - \alpha[1+(\ln\sqrt{u^2+v^2})^2]u \\ v_t = {}& (v-2u)[u\sin(3\ln\sqrt{u^2+v^2}) + v\cos(3\ln\sqrt{u^2+v^2})] \\ & +3[1+(\ln\sqrt{u^2+v^2})^2]u - \alpha[1+(\ln\sqrt{u^2+v^2})^2]v. \end{aligned} \tag{3.1}$$

In polar coordinates this system is

$$\begin{aligned} (\ln r)_t &= r\sin(3\ln r + \theta) - \alpha[1+(\ln r)^2] \\ \theta_t &= -2r\sin(3\ln r + \theta) + 3[1+(\ln r)^2]. \end{aligned} \tag{3.2}$$

The proofs used to establish the Theorem still work to establish the same facts for this system, with one slight change: In the proof of the case $\alpha > 3/2$ one must replace the quantity $\ln[3(\alpha-1)]$ in (2.6) and (2.7) by the unique root q of the equation

$$e^q/[1+q^2] = 3(\alpha-1).$$

More importantly, this system has a stronger property. Since

$$(-2\ln r - \theta)_t = (2\alpha-3)[1+(\ln r)^2],$$

we see from (2.4) and elementary algebra that

$$(-2\ln r - \theta)_t \geq (2\alpha-3)[\tfrac{1}{2}(-2\ln r - \theta)^2 - (3\ln r + \theta)^2 + 1]. \tag{3.3}$$

We can establish as before that when the inequality

$$\ln r_0 \geq 2\pi + q$$

is satisfied, the function $3 \ln r + \theta$ is bounded both above and below. Therefore the right-hand side of (3.3) is bounded below by a constant plus a positive constant times the function $(-2 \ln r - \theta)$. This function is known to converge to infinity, and a well-known theorem of differential equations then shows that it reaches infinity in a finite time. Then (2.4) shows that $\ln r$ goes to infinity in a finite time. We have thus shown that when $\alpha > 3/2$, any solution with a sufficiently large initial value of $u^2 + v^2$ blows up in a finite time.

In particular, we see that when $\alpha = 1$, every solution of the system (3.1) converges to the origin, but that adding the inward radial vector field $(-[1 + \ln(u^2 + v^2)]u, -[1 + \ln(u^2 + v^2)]v)$ to the right-hand side produces solutions which blow up in a finite time.

In another work [1] we shall give an example of a system for which the origin is a global attractor such that adding an arbitrarily small positive multiple of the linear inward radial vector field $(-u, -v)$ produces solutions which blow up in finite time.

References

[1] Hirokazu Ninomiya and Hans F. Weinberger, Pest control may make the pest population explode, *Z. Angew. Math. Phys.* **54**, (2003), pp. 869-873.