Necessary conditions for the optimality of an extrusion die for a rigid-plastic material

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To my friend Piero Villagio

Abstract

Conditions which must be satisfied by the steady flow of a rigid-plastic material through an extrusion die which minimizes dissipation power are derived.

1 Introduction

This work deals with the question of designing an extrusion die which minimizes the power required to convert a bar of given cross section to one or more new bars of prescribed cross section. It has been thought for a long time (see Hill [1], Nadai [5], pp. 74-81) that a sufficient condition for this optimality is that at each point where the rate of strain is not zero the flow through this die has the property that the eigenvalue of largest magnitude of the rate-of strain matrix D is positive, and the velocity $\mathbf{v}$ is a corresponding eigenvector. Richmond and Devenpeck [7] called such a flow a streamlined flow, and showed that one can construct a die which produces such a flow for the case when a plate is to be converted to a thinner plate. Hill [2] generalized this result. Richmond and Morrison [8] showed how to construct a die with streamlined flow for converting a circular wire to a thinner circular wire. This result was extended by Hill [3], who called the corresponding flow an ideal forming operation. A proof of the conjecture in the general three-dimensional case was given by Hill [4]. However, this work did not discuss the existence of optimal dies with given billet and product cross-sections.

In [10] the author used variational methods to find a sufficient condition for a die with general billet and product cross sections to be optimal for a material which obeys the Tresca stress-rate-of-strain condition. This condition is to be satisfied on the die boundary rather than in the interior of the deforming material. It was shown that for plate thinning and wire drawing this boundary condition implies streamlining. However, it was also shown that for general billet and product cross sections the assumption of streamlining leads to an overdetermined system, so that one cannot usually expect to find a die with such a flow. It was shown in [9] that for a streamlined flow, the perimeter of the product cross section must be no larger than that of the input cross section. It follows in particular that there are product cross sections which cannot be produced from a given billet by a streamlined flow, so that the optimal die cannot satisfy the streamlining condition. Thus the extra boundary condition of [10] is the only known necessary condition for optimality in a general problem with the Tresca yield condition.

From the point of view of partial differential equations, all rigid-plastic laws lead to difficulties at points where the rate-of-strain becomes zero. However, the Tresca law also produces a serious difficulty at points where the determinant of D changes sign, because the corresponding stress σ is discontinuous there. For this reason it may well be better to use other more regular stress-rate-of-strain laws such as the von Mises law. Thus it is useful to derive necessary conditions for optimality for more general yield conditions. We shall do just that in this paper. In addition to an extra boundary condition on the die boundary, we shall derive two conditions on the boundaries of regions where deformation occurs. For the sake of generality, we shall include dies which simultaneously extrude more than one product.

2 Formulation of the problem

We shall investigate the concept of optimal deformation for an incompressible isotropic rigid-plastic material. We consider the family of rigid-plastic yield laws which satisfy the so-called normality condition. Namely, the stress corresponding to a given rate of strain matrix D is that member of a prescribed closed convex set K of 3×3 symmetric stress matrices S which minimizes the rate of dissipation $\operatorname{tr}(SD)$. K is assumed to have the following properties:

a. **Indifference to pure pressure:** If a matrix S is in K, then for any number γ, $S+\gamma I$, where I is the identity matrix, is also in K.

b. **Isotropy:** If S is in K and U is an orthogonal matrix with the transpose U^*, then USU^* is also in K.

c. K is closed and convex, and $S=0$ is an interior point of K.

d. The set $\hat{K}=\{S\in K : \operatorname{tr}(S)=0\}$ is bounded.

e. The set $\hat{K}$ as a subset of matrices with trace 0 is strictly convex in the sense that its boundary does not contain a line segment.

The condition (e) means that the for any trace-free nonzero matrix D the member of K which maximizes $\operatorname{tr}(SD)$ is uniquely defined to within a an additive scalar matrix. This maximizer clearly lies on the boundary of K. This boundary is called the yield surface. The Tresca yield condition satisfies all the conditions other than (e).

We define the rate-of-dissipation function

$$H(D)=\max_{S\in K}\operatorname{tr}(SD). \tag{2.1}$$

Because of the properties (a), (c), and (d) of the set K, this maximum is positive when $D\neq 0$, and it is finite when $\operatorname{tr}(D)=0$. When $\operatorname{tr}(D)\neq 0$, the property (a) shows that $H(D)=+\infty$.

Our purpose is to find conditions which characterize an extrusion die which converts a billet, which is a half-cylinder of given cross section, to a given product, which consists of one or more half-cylinders with prescribed cross sections, which minimizes the power

$$\mathcal{P}(\Omega,\mathbf{v}) := \int_\Omega H\left(\frac{1}{2}[v^i_{,x^j}+v^j_{,x^i}]\right) d\mathbf{x} \tag{2.2}$$

when the speed of the billet is normalized at 1. Because $H(D) = 0$ where the motion is a rigid body motion such as a motion of constant velocity, the integrand in (2.2) is only nonzero on a bounded set. Since $H(D) = +\infty$ when $\mathrm{tr}(D) \neq 0$, this minimization forces the condition $\mathrm{tr}(D) = 0$, so that the deformation is incompressible.

To be more precise, we introduce the following definitions.

Definition 2.1 *A domain Ω is* **admissible** *if*

1. *Ω has a smooth boundary $\partial\Omega$ with a continuous outward normal vector* $\mathbf{n}$*;*

2. *Ω is the union of*

   a. *an open half cylinder of prescribed cross section called the* **billet***;*

   b. *one or more open half-cylinders of prescribed cross section called the* **product***;*

      and

   c. *a bounded open set which contains the ends of all the half-cylinders.*

Definition 2.2 *A velocity field* $\mathbf{v}$ *on the closure $\bar{\Omega}$ of an admissible domain Ω is said to be* **admissible on Ω** *if*

1. $\mathbf{v}$ *is continuously differentiable on $\bar{\Omega}$;*

2. $\nabla \cdot \mathbf{v} = 0$*;*

3. $\mathbf{v} \cdot \mathbf{n} = 0$ *on $\partial\Omega$;*

4. *everywhere on the billet with the exception of a bounded subset* $\mathbf{v}$ *is constant and axial in the direction toward the product, and the speed* $|\mathbf{v}|$ *is 1;*

5. *everywhere on each component of the product with the exception of a bounded subset* $\mathbf{v}$ *is constant and axial in the direction away from the billet.*

Our problem is to find an admissible domain and an admissible velocity $\mathbf{v}$ on this domain which make the power (2.2) no larger than that of any other admissible Ω and $\mathbf{v}$.

3 Conditions for an optimal die

We recall a well known variational property of the flow on a fixed domain Ω. We shall use the summation convention in the rest of this work.

Proposition 1 *Suppose that* $\mathbf{v}$ *minimizes the power functional (2.2) among all* $\mathbf{v}$ *which are admissible on a given domain* Ω*. Then there is a stress field* σ^{ij} *defined on the set of points where the rate-of-strain tensor*

$$D_{ij} := \frac{1}{2}[v^i_{,x^j} + v^j_{,x^i}]$$

is different from zero with the properties

$$\sigma^{ij} D_{ij} = H(D) \tag{3.1}$$

and

$$\sigma^{ij}_{,x^j} = 0 \text{ in } \{\mathbf{x} \in \Omega : D(\mathbf{x}) \neq 0\}, \tag{3.2}$$

and there is a scalar-valued function $\mu(\mathbf{x})$ *such that*

$$\sigma^{ij} n_j = \mu(\mathbf{x}) n_i \text{ on } \{\mathbf{x} \in \partial\Omega : D(\mathbf{x}) \neq 0\}. \tag{3.3}$$

Proof. Let $\mathbf{v}(\mathbf{x}, \epsilon)$ be a smooth one-parameter family of flows which are admissible for Ω such that $\mathbf{v}(\mathbf{x}, 0) = \hat{\mathbf{v}}(\mathbf{x})$, where $\hat{\mathbf{v}}$ maximizes $\mathcal{P}(\Omega, \mathbf{v})$ among admissible flows. Then $\mathcal{P}(\Omega, \mathbf{v}(\mathbf{x}, \epsilon))$ takes on its minimum at $\epsilon = 0$. Because the function $H(D)$ is homogeneous of degree 1, it is not differentiable at $D = 0$. On the other hand, it follows from the definition of $H(D)$ as a maximum that if E is any matrix with trace 0 and $D \neq 0$, then

$$\frac{\partial H(D)}{\partial D_{ij}} E_{ij} = S^{ij} E_{ij}, \tag{3.4}$$

where S is any matrix in K such that

$$\mathrm{tr}(SD) = H(D). \tag{3.5}$$

We restrict our attention to those $\mathbf{v}(\mathbf{x}, \epsilon)$ which agree with $\hat{\mathbf{v}}$ outside a fixed set where $D \neq 0$. Define

$$\mathbf{w}(\mathbf{x}) := \mathbf{v}_{,\epsilon}(\mathbf{x}, 0),$$

and note that $H(D)$ is differentiable where $\mathbf{w} \neq \mathbf{0}$. The facts that the derivative of $\mathcal{P}(\Omega, \mathbf{v})(\mathbf{x}, \epsilon))$ must vanish at $\epsilon = 0$ and that the matrices of K are symmetric imply that

$$\int_\Omega S^{ij} w^i_{,x^j} d\mathbf{x} = 0, \tag{3.6}$$

where S is any matrix of K which satisfies (3.5). Because $\mathbf{w}$ is an arbitrary divergence-free vector field on a fixed set where $D \neq 0$, and vanishes outside this set, a standard argument implies that the matrix function S^{ij} must satisfy the equation

$$S^{ij}_{,x^j} = p_{,x^i}, \tag{3.7}$$

where p is a scalar-valued function defined at those points where $D \neq 0$.

We now define $\sigma^{ij} := S^{ij} - p\delta_{ij}$, and observe that (3.7) is just (3.2). Because $\mathrm{tr}(D) = 0$, (3.5) implies (3.1). To obtain the boundary condition (3.3), we first note that because $\mathbf{w}$ is divergence free, we can replace S^{ij} by σ^{ij} in (3.6). We then apply the divergence theorem and (3.2) to find that

$$\int_{\partial\Omega} \sigma^{ij} w^i n_j dS = 0$$

for any vector field $\mathbf{w}$ which is tangent on the boundary, is divergence-free, and vanishes where $D = 0$. We easily conclude that the tangential component of the vector field $\sigma^{ij} n_j$ is zero, which is the statement of (3.3). Thus the Proposition is proved.

Our principal result is the following theorem, which gives additional conditions which are satisfied when one minimizes $\mathcal{P}$ among all dies as well as all flows.

Theorem 1 *Suppose that the pair $(\Omega, \mathbf{v})$ gives the smallest value to the functional $\mathcal{P}$ in (2.2) among all admissible domains and flows. Then the boundary traction μ in (3.3) satisfies the additional condition*

$$\mathbf{v} \cdot \nabla\mu = H(D) \text{ on } \{\mathbf{x} \in \partial\Omega : D(\mathbf{x}) \neq 0\}. \tag{3.8}$$

Let P be a component of the subset of Ω where $D \neq 0$, and let Γ be a component of the intersection of the boundary of P with Ω. Let $\boldsymbol{\nu}$ be the unit normal vector on Γ, directed outward from P. If Γ is on the boundary of an *open set where $D = 0$, then $\mathbf{v}$ has the rigid-flow form $\mathbf{a} + \boldsymbol{\omega} \times \mathbf{x}$ on Γ, and the equations*

$$\int_{\partial\Omega\cap\Gamma} \mu\mathbf{n}[\mathbf{n} \times (\mathbf{a} + \boldsymbol{\omega} \times \mathbf{x})] \cdot d\mathbf{s} + \boldsymbol{\omega} \times \int_{\Gamma} \sigma\boldsymbol{\nu} dS = 0. \tag{3.9}$$

and

$$\int_{\partial\Omega\cap\Gamma} \mathbf{x} \times \mu\mathbf{n}[\mathbf{n} \times (\mathbf{a} + \boldsymbol{\omega} \times \mathbf{x})] \cdot d\mathbf{s} + \boldsymbol{\omega} \times \int_{\Gamma} \mathbf{x} \times \sigma\boldsymbol{\nu} dS = 0 \tag{3.10}$$

are satisfied.

If Γ is on the boundary of an unbounded open set where $D = 0$, then $\boldsymbol{\omega} = \mathbf{0}$, so that the last two equations are simplified.

Proof. As in [9], we shall use the method of transplantation introduced by Polya and Schiffer [6]. Let $\mathbf{y} = \mathbf{Y}(\mathbf{x}, \epsilon)$ be a one-parameter family of smooth one-to-one mappings defined on the closure $\bar{\Omega}$ of an optimal Ω. Suppose that the inverse mapping $\mathbf{X}(\mathbf{y}, \epsilon)$ is also smooth, and that $Y(\mathbf{x}, 0) \equiv \mathbf{x}$. We define the new velocity field $\mathbf{V}(\mathbf{y}, \epsilon)$ on the set $\bar{\Omega}_\epsilon := \mathbf{Y}(\bar{\Omega}, \epsilon)$ by the equation

$$V^i(\mathbf{y}, \epsilon) := \det\left[(\partial\mathbf{Y}/\partial\mathbf{x})\right]^{-1} Y^i_{,x_k} v^k \Big|_{\mathbf{x}=\mathbf{X}(\mathbf{y},\epsilon)}. \tag{3.11}$$

A calculation shows that

$$V^i_{,y^j} = [\det(\partial\mathbf{Y}/\partial\mathbf{x})]^{-1} \{Y^i_{,x^k} v^k_{,x^\ell} + Y^i_{,x^k x^\ell} v^k - Y^m_{,x^\ell x^n} X^n_{,y^m} Y^i_{,x^k} v^k\} X^\ell_{,y^j}, \tag{3.12}$$

so that

$$V^i_{,y^i} = [\det(\partial\mathbf{Y}/\partial\mathbf{x})]^{-1} v^k_{,x^k} = 0$$

because $v^k_{,x^k} = 0$. Since $\mathbf{v}$ is tangent to the boundary of Ω, $\mathbf{V}$ is tangent to the boundary of Ω_ϵ. Finally, we observe that if the mapping $\mathbf{y} = \mathbf{Y}(\mathbf{x}, \epsilon)$ is an isometry on the all but a bounded subset of the billet half-cylinder and is a (possibly different) isometry on all but a bounded part of each of the product half-cylinders, then Ω_ϵ is admissible, and $\mathbf{V}$ is admissible for Ω_ϵ. Therefore, if Ω is optimal, the functional

$$\int_{\Omega_\epsilon} H\left(\frac{1}{2}[V^i_{,y^j} + V^j_{,y^i}]\right) d\mathbf{y}$$

has its minimum at $\epsilon = 0$. Thus if the left-hand side is differentiable at $\epsilon = 0$, we find that

$$\frac{d}{d\epsilon}\left\{\int_{\Omega_\epsilon} H\left(\frac{1}{2}[V^i_{,y^j} + V^j_{,y^i}]\right) d\mathbf{y}\right\}_{\epsilon=0} = 0. \tag{3.13}$$

We observe that

$$\begin{aligned}\int_{\Omega_\epsilon} H\left(\frac{1}{2}[V^i_{,y^j} + V^j_{,y^i}]\right) d\mathbf{y} \\ = \int_\Omega H\left(\frac{1}{2}[V^i_{,y^j}(\mathbf{Y}(\mathbf{x},\epsilon),\epsilon) + V^j_{,y^i}(\mathbf{Y}(\mathbf{x},\epsilon),\epsilon)]\right) \det\left(\frac{\partial \mathbf{Y}}{\partial \mathbf{x}}\right) d\mathbf{x}.\end{aligned} \tag{3.14}$$

In order to make the integrand be differentiable with respect to ϵ, we require the map $\mathbf{Y}(\mathbf{x},\epsilon)$ to be an isometry outside a closed set where $D \neq 0$. This implies that outside this set the matrix $V^i_{,y^j} + V^j_{,y^i}$ is unitarily equivalent to D_{ij} so that the integrand is independent of ϵ. The formula (3.4) now shows that

$$\begin{aligned}\frac{\partial H(\frac{1}{2}[V^i_{,j}(\mathbf{Y}(\mathbf{x},\epsilon),\epsilon) + V^j_{,y^i}(\mathbf{Y}(\mathbf{x},\epsilon),\epsilon)])}{\partial\epsilon} &= \sigma^{ij}\frac{1}{2}\frac{\partial}{\partial\epsilon}[V^i_{,y^j} + V^j_{,y^i}] \\ &= \sigma^{ij}\frac{\partial}{\partial\epsilon}V^i_{,y^j}(\mathbf{Y}(\mathbf{x},\epsilon),\epsilon)),\end{aligned}$$

where σ^{ij} is the stress field with the properties (3.1), (3.2), and (3.3).

It thus follows from (3.12) and (3.14) that

$$\begin{aligned}&\frac{d}{d\epsilon}\left\{\int_{\Omega_\epsilon} H\left(\frac{1}{2}[V^i_{,y^j} + V^j_{,y^i}]\right) d\mathbf{y}\right\} \\ &\quad = \int_\Omega \sigma^{ij}\left\{\frac{\partial}{\partial\epsilon}[V^i_{,y^j}]\det(\partial\mathbf{Y}/\partial\mathbf{x}) + V^i_{,y^j}\frac{\partial}{\partial\epsilon}[\det(\partial\mathbf{Y}/\partial\mathbf{x})]\right\} \\ &\quad = \int_\Omega \sigma^{ij}\frac{\partial}{\partial\epsilon}[V^i_{,y^j}\det(\partial\mathbf{Y}/\partial\mathbf{x})].\end{aligned} \tag{3.15}$$

We define

$$\eta^i := \left.\frac{\partial Y^i}{\partial\epsilon}\right|_{\epsilon=0},$$

and recall that at $\epsilon = 0$ $\mathbf{Y}$ is the identity map so that the Jacobian matrices $Y^i_{,x^j}$ and $X^i_{,y^j}$ are the unit matrix. Moreover, the definition of an inverse map shows that

$$\left.\frac{\partial X^i}{\partial\epsilon}\right|_{\epsilon=0} = -\eta^i.$$

By applying these facts and the formula (3.12) to the identity (3.15), we find that

$$\begin{aligned}
&\frac{d}{d\epsilon}\left\{\int_{\Omega_\epsilon} H(\frac{1}{2}[V^i_{,y^j}+V^j_{,y^i}])d\mathbf{y}\right\}_{\epsilon=0}\\
&=\int_\Omega \sigma^{ij}[\{\eta^i_{,x^k}v^k\}_{,x^j}-\eta^m_{,x^m x^j}v^i-v^i_{,x^\ell}\eta^\ell_{,y^j}]d\mathbf{x}\\
&=\int_\Omega \{\sigma^{ij}(\eta^i v^k-\eta^k v^i)_{,x^k x^j}+[\eta^k H(D)]_{,x^k}\}d\mathbf{x}.
\end{aligned}$$

We first assume that for every ϵ the mapping $\mathbf{Y}$ is the identity $\mathbf{Y}=\mathbf{x}$ outside a certain bounded closed set where $D\neq 0$. Then $\boldsymbol{\eta}=\mathbf{0}$ outside this set. Integration by parts together with the already established conditions (3.2) and (3.3) shows that the condition (3.13) becomes

$$\int_{\partial\Omega}[\mu n_i(\eta^i v^k-\eta^k v^i)_{,x^k}+H(D)\boldsymbol{\eta}\cdot\mathbf{n}]dS=0. \tag{3.16}$$

We observe that

$$(\eta^i v^k-\eta^k v^i)_{,x^k}=\{\mathrm{curl}(\boldsymbol{\eta}\times\mathbf{v})\}_i. \tag{3.17}$$

Integration by parts using the Stokes theorem shows that (3.16) can be written as

$$\int_{\partial\Omega}[-\mathbf{v}\cdot\nabla\mu+H(D)]\boldsymbol{\eta}\cdot\mathbf{n}dS=0. \tag{3.18}$$

Since $\boldsymbol{\eta}\cdot\mathbf{n}$ is arbitrary, we conclude that the equation (3.8) holds on the boundary.

Now let P be a component of the set where $D\neq 0$, and let Γ be a component of the intersection of the boundary of P with the open set Ω. Instead of requiring the map $\mathbf{Y}(\mathbf{x})$ to be the identity outside a closed subset of the interior of P, we require $\mathbf{Y}(\mathbf{x},\epsilon)$ to be an isometry near and outside Γ, and to be the identity on and outside the remainder of the intersection of the boundary of P with Ω. Then the equation (3.16) is replaced by the equation

$$\int_{\partial\Omega}[\mu n_i(\eta^i v^k-\eta^k v^i)_{,x^k}+H(D)\boldsymbol{\eta}\cdot\mathbf{n}]dS+\int_\Gamma \sigma_{ij}\nu_j(\eta^i v^k-\eta^k v^i)_{,x^k}dS=0. \tag{3.19}$$

(Note that $H(D)=0$ on Γ. In fact, one derives this result by first integrating over a set with smooth boundary inside P and letting this boundary approach Γ.)

As in the derivation of (3.8), we can apply the Stokes Theorem to the first term in (3.19) to find that because (3.8) is valid, it is equal to

$$\int_{\partial\Omega\cap\Gamma} \mu\boldsymbol{\eta}\cdot\mathbf{v}\times d\mathbf{s}. \tag{3.20}$$

Since both $\mathbf{v}$ and $d\mathbf{s}$ are tangent to the die boundary, their vector product is a scalar multiple of $\mathbf{n}$. Thus by (3.2) we may rewrite this integral as

$$\int_{\partial\Omega\cap\Gamma} \boldsymbol{\eta}\cdot(\mu\mathbf{n})\mathbf{n}\cdot\mathbf{v}\times d\mathbf{s}. \tag{3.21}$$

Because $\boldsymbol{\eta}$ is the derivative of an isometry, it must be a rigid-body velocity of the form

$$\boldsymbol{\eta} = \mathbf{b} + \mathbf{c}\times\mathbf{x} \tag{3.22}$$

on Γ, where the constant vectors $\mathbf{b}$ and $\mathbf{c}$ can be chosen arbitrarily. We first take $\mathbf{c} = 0$, so that $\boldsymbol{\eta}$ is equal to the constant vector $\mathbf{b}$. By substituting this $\boldsymbol{\eta}$ in (3.19) and using the form (3.21) for the first term, we see that

$$b^k\left\{\int_{\partial\Omega\cap\Gamma} \mu n_k\mathbf{n}\times\mathbf{v}\cdot d\mathbf{x} - \int_\Gamma \sigma^{ij}n_j v^i_{,x^k} dS\right\} = 0. \tag{3.23}$$

Since the vector $\mathbf{b}$ is arbitrary, we conclude that the vector equation

$$\int_{\partial\Omega\cap\Gamma} \mu n_k\mathbf{n}\times\mathbf{v}\cdot d\mathbf{x} - \int_\Gamma \sigma^{ij}\nu_j v^i_{,x^k} dS = 0 \tag{3.24}$$

for $k = 1, 2, 3$ is satisfied.

Because $D = 0$ on the other side of Γ, the velocity $\mathbf{v}$ must also have the form

$$\mathbf{v} = \mathbf{a} + \boldsymbol{\omega}\times\mathbf{x} \tag{3.25}$$

with constant vectors $\mathbf{a}$ and $\mathbf{d}$ on Γ. Then (3.24) takes the form (3.9). By applying the same reasoning with $\boldsymbol{\eta} = \mathbf{c}\times\mathbf{x}$ on Γ, we find (3.10).

If Γ lies on the boundary of an unbounded open set where $D = 0$, then the flow at Γ must agree with the flow on either the billet or a product, and hence it must be constant. This means that $\boldsymbol{\omega} = \mathbf{0}$, and the Theorem is established.

Remarks.

1. If one wishes to include boundary friction, the condition (3.3) is to be replaced by $\sigma^{ij} n_j + \gamma v^i = \mu n_i$, where γ is the coefficient of friction. This can be brought about by adding the dissipation term $\frac{1}{2}\gamma \int_{\partial\Omega} |\mathbf{v}|^2 d\mathbf{x}$ to the right-hand side of (2.2). Similarly, gravity can be included by adding the term $\int_\Omega \rho \mathbf{g} \cdot \mathbf{v} d\mathbf{x}$. The proof of the Theorem still yields the same conditions.

2. The fact that the left-hand side of (3.18) is the first variation of the power when the die is varied can be used to produce a Newton scheme for improving a non-optimal die.

3. Any variational scheme will at best produce the die of lowest dissipation in the class of dies whose complements have the same connectivity. It may be possible to reduce the dissipation by adding or deleting one or more internal bridges.

4. We observe that the second integrals in (3.9) and (3.10) are the total force and total moment applied by the flow in P to the boundary Γ. Because there are no other forces, one might expect these integrals to be zero, but we have not been able to establish this.

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