The expected jaggedness of order ideals
Joint Mathematics Meetings, Seattle, 2016
AMS Special Session on Algebraic and Topological Methods in Combinatorics

Sam Hopkins
MIT
January 9th, 2016

Joint work with Melody Chan (Brown), Shahrzad Haddadan (Dartmouth), and Luca Moci (Paris 7)
Section 1

Motivation from algebraic geometry
Let $C$ be a curve of genus $g$. Brill-Noether theory studies the collection $G^r_d(C)$ of all maps $C \to \mathbb{P}^r$ of degree $d$. It turns out that $G^r_d(C)$ is some kind of “space” (e.g., algebraic variety or scheme). So we can study the geometry of $G^r_d(C)$. The Brill-Noether number is

$$
\rho := \rho(g, r, d) := g - (r + 1)(g - d + r).
$$

**Theorem (Griffiths-Harris, 1980)**

When $C$ is general:

- if $\rho$ is nonnegative then $G^r_d(C)$ has dimension $\rho$;
- if $\rho$ is negative then $G^r_d(C)$ is empty.
From now on $C$ is general. We can ask for more refined geometrical information about $G^r_d(C)$ than dimension. For instance, when $\rho = 1$, $G^r_d(C)$ is a smooth curve, so we can ask what its genus is.

**Theorem (Pirola 1985/Eisenbud-Harris 1987)**

When $\rho = 1$, the genus of $G^r_d(C)$ is

$$1 + \frac{g - d + 1}{g - d + 2r + 1} \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!} \cdot g!.$$
Degeneration and limit linear series

Chan-López Martín-Pfleuger-Teixidor I Bigas, following early work of Castorena-López Martín-Teixidor I Bigas, give a new “combinatorial” proof of the genus formula for $G_d^r(C)$ (in the spirit of other tropical approaches to B-N theory, e.g. Baker-Norine’s Riemann-Roch theorem for graphs).

They use \emph{limit linear series} of Eisenbud-Harris: degenerate the smooth moduli space to a singular curve consisting of elliptic curves glued nodally. Then they count vertices and edges of the \emph{dual graph} of the singular curve.

![Dual graph example](image)
Number of turns in lattice paths

Counting the number of vertices of the dual graph is easy (basically, it is the number of SYT of $a \times b$). Counting the number of edges amounts to finding the expected number of turns in a certain distribution on lattice paths. A lattice path from SW to NE corner of $a \times b$ is compatible with an SYT if entries above the path are less than entries below the path.

Define a distribution $\mu_{\text{lin}}$ on such lattice paths in $a \times b$ where the prob. of a path is proportional to the number of SYT with which it is compatible:
Key combinatorial theorem for genus formula

Theorem (Chan-López Martín-Pfleuger-Teixidor I Bigas 2015)

The expected number of turns of a lattice path drawn with distribution $\mu_{\text{lin}}$ is $2ab/(a + b)$, the harmonic mean of $a$ and $b$. Note that this expectation is the same as if we used the uniform distribution $\mu_{\text{uni}}$ instead.

This theorem was the starting point of our research: Melody Chan discussed this theorem at the March 2015 AIM workshop on Dynamical Algebraic Combinatorics. It begged for a combinatorial “explanation.”
Section 2

Toggle-symmetric distributions on order ideals
Let $P$ be a (finite) poset. Recall that an order ideal $I$ of $P$ is $I \subseteq P$ such that $y \in I$ and $x \leq y$ implies $x \in I$ (downward closed subset). $\mathcal{J}(P)$ is the set of order ideals.

Define “toggleability statistics” $\mathcal{T}^+_p, \mathcal{T}^-_p : \mathcal{J}(P) \to \mathbb{R}$ for all $p \in P$:

\[
\mathcal{T}^+_p(I) := \begin{cases} 
1 & \text{if } p \text{ is minimal in } P \setminus I, \\
0 & \text{otherwise}; 
\end{cases}
\]

\[
\mathcal{T}^-_p(I) := \begin{cases} 
1 & \text{if } p \text{ is maximal in } I, \\
0 & \text{otherwise.} 
\end{cases}
\]

A distribution $\mu$ on $\mathcal{J}(P)$ is toggle-symmetric if $\mathbb{E}_\mu(\mathcal{T}^+_p) = \mathbb{E}_\mu(\mathcal{T}^-_p)$ for all elements $p \in P$. 
Examples from $P$-partitions

The following distributions on $\mathcal{J}(P)$, all coming from $P$-partitions, are toggle-symmetric:

- $\mu_{\text{uni}}$: choose $l$ uniformly at random in $\mathcal{J}(P)$,
- $\mu_{m,\leq}$: choose a weakly order-preserving map $f : P \to \{0,1,\ldots,m\}$ and $k \in \{1,\ldots,m\}$ uniformly at random; let $l$ be $f^{-1}(\{0,\ldots,k\})$,
- $\mu_{\text{lin}}$: choose a linear extension of $P$ and $k \in \{0,1,\ldots,\#P\}$ uniformly at random; let $l$ be the first $k$ elements of the extension,
- $\mu_{m,<}$: choose a strictly order-preserving map $f : P \to \{0,\ldots,m\}$ and $k \in \{0,\ldots,m+1\}$ uniformly at random; let $l$ be $f^{-1}(\{0,\ldots,k-1\})$,
- (if $P$ is graded) $\mu_{\text{rk}}$: choose $k \in \{0,\ldots,\text{rk}(P)+1\}$ uniformly at random; let $l$ be the elements of rank less than $k$.

These distributions make up the following “spectrum”:

$$
\mu_{\text{uni}} \xrightarrow{\mu_{m,\leq}} \mu_{\text{lin}} \xleftarrow{\mu_{m,<}} \mu_{\text{rk}}
$$
Toggling

The basic tool used to show that these distributions are toggle-symmetric is toggling. For each $p \in P$ we have a “toggle” $\tau_p : \mathcal{J}(P) \to \mathcal{J}(P)$:

$$
\tau_p(I) :=
\begin{cases}
    I \cup p & \text{if } \mathcal{T}_p^+(I) = 1; \\
    I \setminus p & \text{if } \mathcal{T}_p^-(I) = 1; \\
    I & \text{otherwise}.
\end{cases}
$$

The toggle group (group generated by $\tau_p, p \in P$) was introduced by Cameron and Fon-Der-Flass (see also Striker and Williams). Observe that

$$
\mathcal{T}_p^+(I) = \mathcal{T}_p^- (\tau_p(I)) \quad \text{for all } I \in \mathcal{J}(P)
$$

so these toggles already show $\mu_{\text{uni}}$ is toggle-symmetric. Other $P$-partition distributions are similar.
Examples from dynamical algebraic combinatorics

Other toggle-symmetric distributions come from orbits of elements of the toggle group. Define *rowmotion* $\text{row}: \mathcal{J}(P) \to \mathcal{J}(P)$ by

$$\text{row} := \tau_{p_1} \circ \cdots \circ \tau_{p_{n-1}} \circ \tau_{p_n}$$

where $p_1 \prec p_2 \prec \cdots \prec p_n$ is any linear extension of $P$ (rowmotion = toggle top-to-bottom). If $P$ is ranked, define *gyrataion* $\text{gyr}: \mathcal{J}(P) \to \mathcal{J}(P)$ by

$$\text{gyr} := \tau_{o_1} \circ \tau_{o_2} \circ \cdots \circ \tau_{o_{n_1}} \circ \tau_{e_1} \circ \tau_{e_2} \circ \cdots \circ \tau_{e_{n_0}}$$

with $\{e_1, \ldots, e_{n_0}\} = \{p: \text{rk}(p) \text{ even}\}$ and $\{o_1, \ldots, o_{n_1}\} = \{p: \text{rk}(p) \text{ odd}\}$ (gyration = toggle odd then even).

**Theorem (Striker 2015)**

*If $\mu$ is a distribution that is uniform on a rowmotion orbit or a gyration orbit, then $\mu$ is toggle-symmetric.*
Section 3

Expected jaggedness of order ideals for skew shapes
Main result: expected jaggedness for skew shapes

A skew shape $\lambda/\nu$ defines a poset where northwest elements are less than southeast elements; order ideals correspond to lattice paths:

\[ \text{Instead of “turns”, we want to study jaggedness: } \text{jag} := \sum_{p \in P} T_p^+ + T_p^- \] (jaggedness = total number of boxes that can be added or removed).

Theorem (Chan-Haddadan-H.-Moci 2015)

Let $\lambda/\nu$ be a skew shape with height $a$ and width $b$. Let $\mu$ be any toggle-symmetric probability distribution on $\mathcal{J}(\lambda/\nu)$. Then

\[
\mathbb{E}_\mu(\text{jag}) = \frac{2ab}{a+b} \left( 1 + \sum_{c \in C(\lambda/\nu)} \delta(c) \mathbb{P}_\mu(c) \right).
\]
Outward corners and displacement

An outward corner $c$ of $\lambda/\nu$ is a corner on the “outside of the shape”; it is either a southeast corner (in red below) or a northwest corner (in blue below). The displacement $\delta(c)$ of a SE outward corner $c$ is the **signed** ratio of the vector connecting the main anti-diagonal to $c$ and the vector connecting the main anti-diagonal to $(a, b)$:

\[
\delta(c) = \frac{v}{u}
\]

For a NW corner $c$ we define $\delta(c)$ similarly but with $(0, 0)$ instead of $(a, b)$. 

Sam Hopkins (2016)
Let us look at the error term in the main theorem again:

\[ \sum_{c \in C(\lambda/\nu)} \delta(c) P_\mu(c). \]

Here \( C(\lambda/\nu) \) is the set of all outward corners of \( \lambda/\nu \), \( \delta(c) \) is the displacement of \( c \) as defined in the previous slide, and \( P_\mu(c) \) is the probability that \( c \) occurs when we think of \( \mu \) as a distribution on paths.

In particular, we see that if \( \delta(c) = 0 \) for all \( c \in C(\lambda/\nu) \), then

\[ \mathbb{E}_\mu(\text{jag}) = \frac{2ab}{a + b}. \]

We call shapes \( \lambda/\nu \) with \( \delta(c) = 0 \) for all \( c \in C(\lambda/\nu) \) balanced.
Balanced shapes

Here are some examples of balanced shapes:

There are $3^{\gcd(a,b)-1}$ balanced shapes of height $a$ and width $b$. In particular, rectangles $a \times b$ are balanced, so we recover (and explain?) the combinatorial theorem of Chan-López Martín-Pfleuger-Teixidor I Bigas.
Homomesy of antichain cardinality

Combining our main result with the aforementioned result of Striker that uniform distribution on rowmotion or gyration orbits are toggle-symmetric, we obtain the following unexpected corollary, generalizing a homomesy result of Propp-Roby:

**Corollary**

If $P$ a poset corresponding to a balanced skew shape, $\varphi \in \{\text{row, gyr}\}$, then the average value on $\mathcal{O}$ of $\sum_{p \in P} T_p^-$ (the antichain cardinality statistic) is

\[
\frac{ab}{a + b}
\]

for any $\varphi$-orbit $\mathcal{O}$.

So the problem was related to dynamical algebraic combinatorics!
Section 4

Open questions
Open questions

- What does the polytope of toggle-symmetric distributions for $P$ look like? It is an intersection of the standard $(\mathcal{J}(P) - 1)$-simplex with linear subspace of codimension $\# P$. Are row/gyr-orbits vertices?

- Are there other toggle-group elements $\varphi$ such that the uniform distribution on a $\varphi$-orbit is toggle-symmetric?

- Easy to see $0 < \mathbb{E}_\mu(\text{jag}) < \frac{4ab}{a+b}$ for any distribution $\mu$ on $\mathcal{J}(\lambda/\nu)$. Thus by main result, get $-1 < \sum_{c \in C(\lambda/\nu)} \delta(c) \mathbb{P}_\mu(c) < 1$ for $\mu$ toggle-symmetric. No a priori reason for this. What does it mean?

- Can we generalize the main result to posets other than skew shapes? In particular, what is a “balanced” poset in general?
Thank you!

our paper is on the arXiv
these slides are available on my website