Root system chip-firing

University of Minnesota Combinatorics Seminar

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Joint work with Pavel Galashin, Thomas McConville, and Alexander Postnikov (and, earlier, with James Propp)
Section 1

Motivation: labeled chip-firing
Classical chip-firing

Classical chip-firing (as introduced by Björner-Lovász-Shor, 1991) is a discrete dynamical system that takes place on a graph. The states are configurations of chips on the vertices. We may fire a vertex that has at least as many chips as neighbors, sending one chip to each neighbor:

A key property of this system is that it is confluent: from a given initial configuration, either all sequences of firings go on forever, or they all terminate at the same stable configuration (called the stabilization).
Chip-firing on an infinite path

One of the first articles to discuss chip-firing was “Disks, Balls, and Walls” by Anderson et al., in the *American Math Monthly*, 1989. They studied chip firing on $\mathbb{Z}$ (the infinite path graph):

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Labeled chip-firing is not confluent in general

What if we started with three chips at the origin:

\[ \begin{array}{c}
& 3 \\
2 &
\end{array} \]

Not sorted and not confluent!
Labeled chip-firing is not confluent in general

What if we started with three chips at the origin:

\[ \begin{array}{ccc} 2 & 1 & 3 \\ \end{array} \]

Not sorted and not confluent!
Motivation: labeled chip-firing

Sorting an even number of chips

Theorem (Hopkins-McConville-Propp, 2017)

Suppose $n = 2m$ is even. Then starting from $n$ labeled chips at the origin, the chip-firing process “sorts” the chips to a unique stable configuration:

$$-m \quad 0 \quad m$$
A “Type B” version of labeled chip-firing

Consider a modified version of labeled chip-firing on $\mathbb{Z}$ where we allow the following three kinds of moves:

1. for $i < j$, if $i$ and $j$ occupy the same vertex, move $i$ leftwards one vertex and $j$ rightwards one vertex (this is the same as before);

2. for any $i, j$, if $i$ is at vertex $a$ and $j$ is at vertex $-a$, move both $i$ and $j$ rightwards one vertex;

3. for any $i$, if $i$ is at the origin, move $i$ rightwards one vertex.
“Type B” labeled chip-firing example

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\[ \begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
1 & 3 & 2 & 0 & 1 & 3 & 2 & \ldots
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“Type B” sorting

**Theorem (Hopkins-McConville-Propp, 2017)**

For any \( n \), starting from \( n \) labeled chips at the origin, the “Type B” labeled chip-firing process (with moves (I), (II), and (III)) “sorts” the chips to the following unique stable configuration:
Proof of “Type B” sorting via symmetry

Use positive chips $1, 2, ..., n$ and negative chips $-n, -2, ..., -1$. Start with all of them at the origin, and carry out “symmetrical” labeled chip-firing moves (whenever we fire $i$ and $j$ we also fire $-i$ and $-j$). The way the positive chips evolve corresponds exactly to the “Type B” moves (I), (II), and (III) above:
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![Diagram showing chip-firing moves](image-url)
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Section 2

Central-firing
For any configuration of $n$ labeled chips, if we set $c := (c_1, \ldots, c_n) \in \mathbb{Z}^n$ where $c_i := \text{the position of the chip } i$, then, for $1 \leq i < j \leq n$, we are allowed to fire chips $i$ and $j$ in this configuration as long as $c$ is orthogonal to $e_j - e_i$; and doing so replaces the vector $c$ by $c + (e_j - e_i)$.

**Observation:** the vectors $e_j - e_i$ for $1 \leq i < j \leq n$ are exactly the positive roots $\Phi^+$ of the root system $\Phi$ of Type $A_{n-1}$. 
Let $V$ be a Euclidean vector space with standard inner product $\langle \cdot, \cdot \rangle$. For any $v \in V$, define the co-vector $v^\vee := \frac{2}{\langle v, v \rangle} v$, and the (orthogonal) reflection $s_v : V \to V$ by $s_v(w) := w - \langle w, v^\vee \rangle v$.

**Definition**

A (crystallographic) root system in $V$ is a finite subset $\Phi \subseteq V$ such that:

1. $\Phi$ spans $V$;
2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$;
3. $\mathbb{R}\alpha \cap \Phi = \{ \pm \alpha \}$ for all $\alpha \in \Phi$;
4. $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The elements of $\Phi$ are called roots. The rank of $\Phi$ is $r = \dim(V)$.

The Weyl group, denoted $W$, of $\Phi$ is the group generated by the reflections $s_\alpha$ for $\alpha \in \Phi$. By definition, it is a finite reflection group.
Positive roots and lattices

We choose a generic linear functional to separate $\Phi$ into positive $\Phi^+$ and negative $\Phi^-$ roots. This also defines a basis $\alpha_1, \alpha_2, \ldots, \alpha_r$ of simple roots with the property that every positive root is a nonnegative integral combination of simple roots. The length $\ell(w)$ of $w \in W$ is the minimal length of an expression of $w$ as a product of simple reflections $s_i := s_{\alpha_i}$.

Two important lattices attached to $\Phi$ are the root lattice $Q := \mathbb{Z}\Phi$, and the weight lattice $P := \{ \lambda \in V : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi \}$. By the assumption of crystallography, we have $Q \subseteq P$.

The fundamental weights $\omega_1, \ldots, \omega_r$ are defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ and these generate $P$. A weight is dominant if it is a nonnegative sum of fundamental weights. An important dominant weight is the so-called Weyl vector $\rho := \sum_{i=1}^r \omega_i = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Minusculer weights are certain distinguished fundamental weights. The minuscule weights together with zero give coset representatives for $P/Q$. 

Classification of root systems

Attached to each root system is a (decorated) graph called the *Dynkin diagram* that records inner products between the simple roots. These lead to a classification of irreducible root systems by the *(Cartan-Killing)* types, which include the *classical types*

\[
A_{n-1} := \{ \pm (e_i - e_j) : 1 \leq i < j \leq n \},
\]

\[
D_n := A_{n-1} \cup \{ \pm (e_i + e_j) : 1 \leq i < j \leq n \},
\]

\[
B_n := D_n \cup \{ \pm e_i : 1 \leq i \leq n \},
\]

\[
C_n := D_n \cup \{ \pm 2e_i : 1 \leq i \leq n \},
\]

as well as the *exceptional types* \( G_2, F_4, E_6, E_7, E_8 \).
Rank 2 root systems

The following are the positive roots and fundamental weights of the irreducible rank 2 root systems:

$A_2$

$B_2$

$G_2$
Central-firing for root systems

The description of labeled chip-firing in terms of positive roots of $A_{n-1}$ generalizes naturally to any root system $\Phi$: for a weight $\lambda \in P$, we allow the firing moves $\lambda \rightarrow \lambda + \alpha$ for a positive root $\alpha \in \Phi^+$ whenever $\lambda$ is orthogonal to $\alpha$.

We call the resulting system the *central-firing* process for $\Phi$ (because we allow firing from a weight $\lambda$ when $\lambda$ belongs to the Coxeter hyperplane arrangement, which is a central arrangement).

You can check that the previously described “Type B” labeled chip-firing really is central-firing for $\Phi = B_n$. Other classical types have similar description of central-firing in terms of chips.
Confluence of central-firing

**Question**

For any root system $\Phi$ and weight $\lambda \in P$, when is central-firing confluent from $\lambda$?

**Answer:** it’s complicated.
Classification of confluence for origin/fundamental weights

Conjecture

Confluence of central-firing from $\lambda$ for $\lambda = 0$ or $\lambda$ a fundamental weight is determined by the table on the right. To first order, central-firing is confluent from $\lambda$ iff $\lambda \neq \rho$ modulo $Q$. Exceptions to this are in red or green.

This conjecture is proved in some but not all cases (e.g. for $\lambda = 0$ and $\Phi = A_n$ or $B_n$, it follows from H.-M.-P. theorems above).
Confluence of central-firing modulo the Weyl group

**Theorem**

*For any root system* \( \Phi \), *and from any initial weight* \( \lambda \), *central-firing is confluent modulo the action of the Weyl group* \( W \).

In Type A the Weyl group is the symmetric group, so modding out by the Weyl group is the same as forgetting the labels of chips. Thus this theorem gives a generalization of *unlabeled* chip-firing on a line to any root system.

**Note:** this is very different from the Cartan matrix chip-firing studied by Benkart-Klivans-Reiner, 2016 (e.g., for \( \Phi = A_{n-1} \), ours corresponds to chip-firing of \( n \) chips on the infinite path, whereas B.-K.-R. corresponds to chip-firing of any number of chips on the \( n \)-cycle).
Unlabeled central-firing for simply laced root systems

Suppose \( \Phi \) is simply laced, i.e., its Dynkin diagram \( \Gamma \) is just an undirected graph with nodes 1, 2, \ldots, \( r \). Consider the following process on the set of labelings \( \gamma: \Gamma \to \mathbb{N} \) of the nodes of \( \Gamma \) by nonnegative integers:

1. choose any connected component \( X \) of \( \Gamma \setminus \{ i : \gamma(i) = 0 \} \);
2. extend \( X \) to an affine Dynkin diagram \( \tilde{X} \) in a unique way;
3. for each edge \((0, i)\), where 0 is the “affine node” of \( \tilde{X} \), add 1 to the label of \( i \);
4. for each \( j \in \Gamma \setminus X \) with \( j \) adjacent to \( i \) for some \( i \in X \), decrease the label of \( j \) by 1.

**Theorem**

Central-firing modulo the Weyl group is the same process as the one defined by the above moves, where we represent an orbit \( W.\lambda \) for a dominant weight \( \lambda = \sum_{i=1}^{r} c_i \omega_i \) by the function \( \gamma(i) = c_i \).
Here is an example of a few unlabeled central-firing moves...
Unlabeled central-firing example

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Another unlabeled central-firing example

Let’s try that same example with some other choices...

(We see the “abelian property” here.)
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Unlabeled central-firing versus chip-firing
Section 3

Interval-firing: confluence
Central-firing allows the firing move $\lambda \rightarrow \lambda + \alpha$ whenever $\langle \lambda, \alpha^\vee \rangle = 0$ for $\lambda \in P$ and $\alpha \in \Phi^+$. We found remarkable “deformations” of this process.

For any $k \in \mathbb{N}$, define the *symmetric interval-firing process* by

$$\lambda \rightarrow \lambda + \alpha \quad \text{if} \quad \langle \lambda, \alpha^\vee \rangle \in \{-k-1, -k, \ldots, k-1\}$$

and the *truncated interval-firing process* by

$$\lambda \rightarrow \lambda + \alpha \quad \text{if} \quad \langle \lambda, \alpha^\vee \rangle \in \{-k, -k+1, \ldots, k-1\}.$$

(These are analogous to the (extended) $\Phi^\vee$-Catalan and $\Phi^\vee$-Shi hyperplane arrangements, respectively. The symmetric closure of the symmetric process is $W$-invariant, explaining its name.)
Pictures of interval-firing for $A_2$

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Interval-firing in Type A via chips

When $\Phi = A_{n-1}$, we can interpret interval-firing in terms of chips. The smallest nontrivial case is symmetric $k = 0$ interval-firing, which has $\lambda \rightarrow \lambda + \alpha$ for $\lambda \in P, \alpha \in \Phi^+$ when $\langle \lambda, \alpha^\vee \rangle = -1$. This corresponds to allowing (adjacent) transpositions of $i$ and $j$ if they’re out of order:

Here confluence is obvious. The next smallest case is truncated $k = 1$ interval-firing, which has $\lambda \rightarrow \lambda + \alpha$ when $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0\}$. This corresponds to allowing transpositions as well as the usual labeled chip-firing moves:
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Here confluence is obvious. The next smallest case is truncated $k = 1$ interval-firing, which has $\lambda \rightarrow \lambda + \alpha$ when $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0\}$. This corresponds to allowing transpositions as well as the usual labeled chip-firing moves:
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Interval-firing is confluent

**Theorem**

For any root system $\Phi$, and any $k \geq 0$, the symmetric and truncated interval-firing processes are confluent (from all initial weights).

I’ll now go over some (geometric) ideas that go into the proof. The main ingredient is a formula for traverse lengths of permutohedra.
Traverse lengths of permutohedra

For $\lambda \in P$, we define the $(W-)permutohedron$ $\Pi(\lambda) := \text{ConvexHull} W(\lambda)$. We use $\Pi^Q(\lambda) := \Pi(\lambda) \cap (Q + \lambda)$ to denote the lattice points in $\Pi(\lambda)$.

An $\alpha$-string of length $k$ is a collection $\{\mu, \mu - \alpha, \ldots, \mu - k\alpha\} \subseteq P$. An $\alpha$-traverse in $\Pi(\lambda)$ is a maximal (by containment) $\alpha$-string inside $\Pi^Q(\lambda)$. We define $\ell(\alpha)$, the traverse length of $\Pi(\lambda)$ in direction $\alpha$, to be the minimum length of an $\alpha$-traverse in $\Pi(\lambda)$.

Examples for $\Phi = A_2$:

\[
\ell(\omega_1) = 0 \quad \forall \alpha \in \Phi \\
\ell(\omega_1 + \omega_2) = 1 \quad \forall \alpha \in \Phi
\]

Intuition: the minimal length $\alpha$-traverse should be an edge of $\Pi(\lambda)$. 

Sam Hopkins (2017)
“Funny weights” and traverse length formula

Counterexample to
(almost correct) intuition:
for $\Phi = B_2$, $\ell_{\omega_1}(\alpha_1) = 0$

**Definition**

If $\Phi$ is not simply laced, then there are $l$ and $s$ such that the long simple root $\alpha_l$ and short simple root $\alpha_s$ are adjacent in the Dynkin diagram. We say the dominant weight $\lambda = \sum_{i=1}^{r} c_i \omega_i$ is funny if $c_s = 0$, $c_l \geq 1$, and $c_i \geq c_l$ for all long $\alpha_i$. (No weight is funny for simply laced $\Phi$.)

**Theorem**

For a dominant weight $\lambda = \sum_{i=1}^{r} c_i \omega_i$, set $m_\lambda(\alpha) := \min \{ c_i : \alpha_i \in W(\alpha) \}$.

$$\ell_\lambda(\alpha) = \begin{cases} 
    m_\lambda(\alpha) - 1 & \text{if } \lambda \text{ is funny and } \alpha \text{ is long,} \\
    m_\lambda(\alpha) & \text{otherwise.}
\end{cases}$$
Lemma

\[ \ell_\lambda(\alpha) = \min\{\langle \mu, \alpha^\vee \rangle : \mu \in \Pi^Q(\lambda), \mu + \alpha \notin \Pi^Q(\lambda)\} \]

Proof.

Let \( \{\mu, \mu - \alpha, \ldots, \mu - k\alpha\} \) be an \( \alpha \)-traverse in \( \Pi^Q(\lambda) \). By the \( W \)-invariance of \( \Pi(\lambda) \) we have \( s_\alpha(\mu - i\alpha) = \mu - (k - i)\alpha \) for \( i = 0, \ldots, k \). Thus in particular

\[ \mu - \langle \mu, \alpha^\vee \rangle \alpha = s_\alpha(\mu) = \mu - k\alpha, \]

so \( \langle \mu, \alpha^\vee \rangle = k \). By definition \( \ell_\lambda(\alpha) \) is the minimal such \( k \).

So the formula for traverse lengths says that interval-firing processes get “trapped” inside certain permutohedra, leading to a proof of confluence.
Section 4

Interval-firing: stabilizations
The map $\eta_k$

Define $\eta_k : P \to P$ by $\eta_k(\lambda) = \lambda + w_\lambda(k\rho)$, where $w_\lambda \in W$ is of minimal length such that $w_\lambda^{-1}(\lambda)$ is dominant.
The stable points of interval-firing

Lemma

The stable points of symmetric interval-firing are

\[ \{ \eta_k(\lambda): \lambda \in P, \langle \lambda, \alpha^\vee \rangle \neq -1 \text{ for all } \alpha \in \Phi^+ \} , \]

and the stable points of truncated interval-firing are

\[ \{ \eta_k(\lambda): \lambda \in P \} . \]
Stabilization maps and Ehrhart-like polynomials

For \( k \geq 0 \), define the stabilization maps \( s_k^{\text{sym}}, s_k^{\text{tr}} : P \rightarrow P \) by

\[
\begin{align*}
  s_k^{\text{sym}}(\mu) = \lambda & \iff \text{the symmetric interval-firing stabilization of } \mu \text{ is } \eta_k(\lambda); \\
  s_k^{\text{tr}}(\mu) = \lambda & \iff \text{the truncated interval-firing stabilization of } \mu \text{ is } \eta_k(\lambda).
\end{align*}
\]

We want to show that there exists (Ehrhart-like) polynomials \( L^\text{sym}_\lambda(k) \), \( L^\text{tr}_\lambda(k) \) such that for all \( k \geq 0 \),

\[
\begin{align*}
  #(s_k^{\text{sym}})^{-1}(\lambda) &= L^\text{sym}_\lambda(k); \\
  #(s_k^{\text{tr}})^{-1}(\lambda) &= L^\text{tr}_\lambda(k).
\end{align*}
\]

**Theorem**

For all \( \Phi \) and all \( \lambda \in P \), the symmetric polynomial \( L^\text{sym}_\lambda(k) \) exists.

**Theorem**

For simply laced \( \Phi \) and all \( \lambda \in P \), the truncated polynomial \( L^\text{tr}_\lambda(k) \) exists.
Lattice points in dilated zonotope plus fixed polytope

**Theorem**

For any lattice polytope $\mathcal{P}$ and lattice zonotope $\mathcal{Z}$, the number of lattice points in $\mathcal{P} + k\mathcal{Z}$ is given by a polynomial (with $\mathbb{Z}_{\geq 0}$ coefficients) in $k$.

**Corollary**

For any dominant $\lambda \in \mathcal{P}$, $\# \Pi^Q(\lambda + k\rho)$ is given by a polynomial (with $\mathbb{Z}_{\geq 0}$ coefficients) in $k$.

The previous corollary leads to the existence of the $L^\text{sym}_\lambda(k)$. 
Decomposing connected components of interval firing

For fixed $k$, the firing moves in truncated interval-firing are a subset of the moves in symmetric interval-firing, so the symmetric “connected components” break into truncated components. Similarly, the $k - 1$ symmetric moves are a subset of the $k$ truncated moves, so the truncated components break into $k - 1$ symmetric components. Ideally the way that these break up would be consistent with $\eta_k$. This is indeed the case.

**Lemma**

For all $\Phi$, $\mu \in P$, and $k \geq 0$, $s_k^{\text{sym}}(\mu) = s_0^{\text{sym}}(s_k^{\text{tr}}(\mu))$.

**Lemma**

For simply laced $\Phi$, $\mu \in P$, and $k \geq 1$, $s_k^{\text{tr}}(\mu) = s_1^{\text{tr}}(s_k^{\text{sym}}(\mu))$.

The previous lemma leads to the existence of the $L_\lambda^{\text{tr}}(k)$. The simply laced assumption is technical and we expect it can be dropped.
Sizes of fibers of iterates of a function

By iterating the previous two lemmas we obtain (for simply laced $\Phi$) that

$$s_{k}^{\text{sym}}(\mu) = (s_{1}^{\text{sym}})^k(\mu).$$

But we know that $\#(s_{k}^{\text{sym}})^{-1}(\mu) = L_{\lambda}^{\text{sym}}(k)$ is given by a polynomial. So we conclude that $s_{1}^{\text{sym}} : P \rightarrow P$ is a function for which the sizes of fibers of iterates are all given by polynomials.

Example for $\Phi = A_1$: 

$$\cdots 1 \ 1 \ 0 \ k+1 \ k+2 \ 1 \ 1 \ \cdots$$

\[
\begin{align*}
0 &\quad \omega_1 &\quad \alpha_1 &\quad = 2\omega_1
\end{align*}
\]
Another iteration example

For $\Phi = A_2$, it turns out that $\rho \in Q$ and hence $s^{\text{sym}}_1$ descends to a map $s^{\text{sym}}_1 : Q \to Q$. Here is that map:
Conjecture

For all $\Phi$ and all $\lambda \in P$, the polynomials $L^\text{sym}_\lambda (k)$ and $L^\text{tr}_\lambda (k)$ exist and have coefficients in $\mathbb{Z}_{\geq 0}$.

Our proofs show the coefficients are in $\mathbb{Z}$. We can prove positivity of coefficients only when $\lambda$ is zero or a minuscule weight. A reasonable amount of computational evidence backs up this conjecture.
Future directions

- Prove the Ehrhart-like polynomial positivity conjecture. (Hard?)
- Is there a connection between interval-firing and the quasi-invariants of $W$? (For simplicity I didn’t define this but for non-simply laced $\Phi$ there is a “two parameter” version of interval-firing.)
- Is there a connection between interval-firing and the extended $\Phi^\vee$-Catalan and $\Phi^\vee$-Shi arrangements (known to be free, affirming conjecture of Edelman-Reiner, by work of Yoshinaga & Terao)?
- Is there a more conceptual proof that central-firing modulo the Weyl group is confluent (our proof uses Newman’s lemma in an unilluminating way)?
- Further understand the pattern of central-firing confluence.
Thank you!

References: