Root System Chip-Firing

by

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Abstract

This thesis investigates an extension of the classical chip-firing process to “other Cartan-Killing types.” In Chapter 1 we review the classical chip-firing game: the states of this process are configurations of chips on the vertices of a graph; the transition moves are firings whereby a vertex with at least as many chips as neighbors may send one chip to each neighbor. A fundamental property of chip-firing is that it is confluent: from any initial configuration, all sequences of firings lead to the same terminal configuration. In Chapter 2 we discuss Propp’s labeled chip-firing process on the infinite path, for which confluence becomes a subtler question. We prove that labeled chip-firing is confluent starting from an even number of chips at the origin (but not from an odd number). In Chapter 3 we reinterpret labeled chip-firing as a process on the weight lattice of a root system, where the firing moves consist of adding a positive root whenever the weight we are at is orthogonal to that root. We call this the central-firing process. We give conjectures about certain initial weights from which central-firing is confluent. We also prove that central-firing is always confluent from all initial weights if we mod out by the action of the Weyl group, thereby giving a generalization of unlabeled chip firing on the infinite path to other types. In Chapter 4 we introduce some remarkable deformations of the central-firing process which we call the symmetric and truncated interval-firing processes. These are analogous to the Catalan and Shi hyperplane arrangements. We prove that these interval-firing processes are always confluent from all initial weights. In Chapter 5 we study the set of weights with given interval-firing stabilization. We show that the number of weights with given stabilization is a polynomial in our deformation parameter. We call these polynomials the symmetric and truncated Ehrhart-like polynomials, because they are analogous to the Ehrhart polynomial of a polytope. We conjecture that the Ehrhart-like polynomials have nonnegative integer coefficients. In Chapter 6 we prove “half” of this positivity conjecture by providing an explicit, positive formula for the symmetric Ehrhart-like polynomials.
Figure 0-1: Above: the Coxeter plane projection of the root system $E_8$ (thanks to John Stembridge). Below: the stabilization of the chip configuration with four million chips at the origin of $\mathbb{Z}^2$ (thanks to David Perkinson and Seth Terashima).
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Chapter 1

Review of classical chip-firing

The *Abelian Sandpile Model* is a discrete dynamical system whose states are configurations of grains of sand on the vertices of some graph. When a vertex has at least as many grains of sand as it has neighbors, it is said to be *unstable*. Any unstable vertex may *fire*, sending one grain of sand to each of its neighbors. The system evolves according to some nondeterministic sequence of firings, and either the system eventually reaches a *stable* state (that is, a state in which every vertex is stable) or the sequence of firings goes on forever. The Abelian Sandpile Model was originally studied, in the case of the two-dimensional square lattice, by the physicists Bak, Tang, and Wiesenfeld [10] as the first model of *self-organized criticality*; much of the general graphical theory was subsequently developed by Dhar [24, 25]. “Self-organized criticality” means that the system enjoys spatial and temporal scale-invariance, irrespective of the fine-tuning of its underlying parameters. Some proponents, such as Bak [9], posit that self-organized criticality is a principal means by which complexity arises in natural systems.

Independently of its introduction in the statistical mechanics community, the same model was introduced and studied from a combinatorial perspective by Björner, Lovász, and Shor [17] under the name of *chip-firing* (see also the earlier work of Engel [28, 29] in the context of math pedagogy). Rather than grains of sand, we imagine that chips are placed on the vertices of the graph; otherwise, the mechanics of the two systems are the same. A fundamental result of Björner-Lovász-Shor is that the
chip-firing process is confluent: from a given initial configuration of chips, either every sequence of firings goes on forever, or all sequences of firings terminate at the same stable configuration, independent of the order in which vertices were fired. This confluence property underpins all study of the chip-firing process and, in particular, it explains the adjective “abelian” in “Abelian Sandpile Model.”

A closely related chip-firing model to that of Björner-Lovász-Shor, studied for instance by Biggs [14] and by Dhar [24, 25], is where a certain distinguished vertex is chosen to be a sink. This sink will never become unstable and is allowed to accumulate any number of chips; hence, any initial configuration of chips will eventually stabilize via a sequence of firings to a unique (thanks to confluence) stable configuration. The theory of chip-firing with a sink turns out to be highly algebraic because this stabilization map is related to certain canonical choices of representatives (such as the recurrent or superstable configurations) for the cokernel of the reduced Laplacian matrix of the graph.

In this introductory chapter we review this well-known story. None of the material in this chapter is original.

Chip-firing is a very active area of research in mathematics, studied in many distinct parts of both physics and pure mathematics (see e.g. [52]). For instance, following the celebrated Riemann-Roch theorem for graphs of Baker and Norine [11], it is known that chip-firing is intimately related to a discrete version of divisor theory for curves (i.e., divisor theory for tropical algebraic curves [38, 57]). Moreover, the sandpile model is studied in mathematical probability and statistical mechanics because of its remarkable scaling limit behavior [62, 50, 51, 47] (this scaling-limit behavior is on display in the bottom image in Figure 0-1). And there are also interesting complexity-theoretic questions one can ask about chip-firing, such as, what is the complexity of determining whether an initial chip configuration will stabilize [76, 30, 58, 48, 32].

But in fact throughout this thesis the aspect of chip-firing with which will be chiefly concerned is one of the most “trivial” properties the system possesses: namely, the aforementioned confluence property. Hence, we will not discuss further any of the connections with other parts of math listed above. Moreover, we will for the most
part not provide any proofs for the results stated in this chapter. Consult [23] for a much more detailed treatment of all of the material presented here.

Since the work of Dhar and of Björner-Lovász-Shor, various generalizations of chip-firing have also been studied. A straightforward generalization is to directed graphs [16]. Other generalizations include M-matrix chip-firing [34, 64, 40], chip-firing for group representations [13, 35] and chip-firing for Hopf algebra modules [39]. Because it will make an appearance later in the thesis when we start discussing root systems, in the last section in this chapter we review M-matrix chip-firing as well.

Throughout this thesis we use $\mathbb{R}$ for the set of real numbers, $\mathbb{Z}$ for the set of integers, and $\mathbb{N}$ for the set of natural numbers (which includes zero). For $a, b \in \mathbb{Z}$ we define $[a, b] := \{a, a + 1, \ldots, b\}$ and for $n \in \mathbb{N}$ we define $[n] := [1, n]$.

1.1 The basic chip-firing model

Let $G = (V, E)$ be a finite directed graph (from now on, “digraph”) with vertex set $V$ and edge set $E$. We allow $G$ to have multiples of the same directed edge, but we do not allow $G$ to have self-loops. In other words, $G$ is a “directed multigraph.” (If we want to consider chip-firing for an undirected graph $G$, then we can treat $G$ as a digraph by viewing each undirected edge $\{u, v\}$ as constituting the pair of directed edges $(u, v)$ and $(v, u)$.) A chip configuration on $G$, or just configuration for short, is some assignment of a nonnegative number of indistinguishable chips to the vertices of $G$. For example, the following is a chip configuration on a digraph with four vertices, where we write the number of chips that each vertex has next to that vertex:
The outdegree, denoted outdeg($v$), of a vertex $v \in V$ is the number of directed edges in $E$ which start at $v$, i.e., the number of edges of the form $(v, u) \in E$. A vertex with at least as many chips as its outdegree is said to be unstable. At any moment, we may fire an unstable vertex: firing $v \in V$ causes the vertex $v$ to lose outdeg($v$) many chips, and also causes each other vertex $u \in V$ to gain one chip for each edge of the form $(v, u) \in E$. We think of $v$ as sending one chip along each outgoing edge when it fires. For instance, the following depicts a sequence of firings starting from the above example, where at each step we highlight in red the vertex at which we are firing:

Observe that after these two firings we have reached a stable configuration, i.e., a configuration in which every vertex is stable. We might wonder whether we could have arrived at a different stable configuration via a different sequence of firings (for example, by first firing the vertex which started with two chip). In fact, this is impossible: if the chip-firing process starting from some initial configuration ever terminates at a stable configuration, then all sequences of firings from that initial configuration must eventually terminate at that same stable configuration. This is the key confluence property of chip-firing, which we will state formally as Theorem 1.1.3.

Of course, it is also possible that the sequence of firings from some initial configuration could go on forever. For instance, consider the following configuration on the same digraph:
The total number of chips present in this configuration is 7. However, observe that for this digraph we have \( \sum_{v \in V} (\text{outdeg}(v) - 1) = 6 \). So by the pigeonhole principle there is no way to distribute 7 chips on this graph to form a stable configuration. And hence the chip-firing process (which preserves the total number of chips in the system) can never terminate starting from the above initial configuration.

Here we have vaguely talked about chip-firing as a “process” or a “system.” It will be helpful to adopt a formal language to describe chip-firing more precisely. Following conventions from the study of abstract rewriting systems, we will adopt the formalism of viewing “processes” as binary relations. This formalism will be used over-and-over throughout the thesis, so we describe it in some detail now.

Let \( X \) be a set and \( \to \) a binary relation on \( X \). We use \( \Gamma \to \) to denote the digraph with vertex set \( X \) and with a directed edge \((x, y)\) whenever \( x \to y \). (Do not confuse this digraph \( \Gamma \to \), whose whose vertices represent the states of a system and which therefore often has an infinite number of vertices, with the finite digraph \( G \) above.) Clearly \( \Gamma \to \) contains exactly the same information as \( \to \) and we will often implicitly identify binary relations and digraphs (specifically, digraphs without multiple edges in the same direction) in this way. We use \( \to^{*} \) to denote the reflexive, transitive closure of \( \to \); that is, we write \( x \to^{*} y \) to mean that \( x = x_0 \to x_1 \to \cdots \to x_k = y \) for some \( k \in \mathbb{N} \). In other words, \( x \to^{*} y \) means there is a path from \( x \) to \( y \) in \( \Gamma \to \). We use \( \leftrightarrow \) to denote the symmetric closure of \( \to \): \( x \leftrightarrow y \) means that \( x \to y \) or \( y \to x \). For any digraph \( \Gamma \), we use \( \Gamma^{\text{un}} \) to denote the underlying undirected graph of \( \Gamma \); in fact, we view \( \Gamma^{\text{un}} \) as a digraph: it has edges \((x, y)\) and \((y, x)\) whenever \((x, y)\) is an edge of \( \Gamma \). Hence \( \Gamma \leftrightarrow = \Gamma^{\text{un}} \). Finally, we use \( \leftrightarrow^{*} \) to denote the reflexive, transitive, symmetric closure of \( \to \): \( x \leftrightarrow^{*} y \) means that \( x = x_0 \leftrightarrow x_1 \leftrightarrow \cdots \leftrightarrow x_k = y \) for some \( k \in \mathbb{N} \). In other words, \( x \leftrightarrow^{*} y \) means there is a path from \( x \) to \( y \) in \( \Gamma^{\text{un}} \).

Now let us review some notions of confluence for binary relations. Here we generally follow standard terminology in the theory of abstract rewriting systems, as laid out for instance in [45]; however, instead following chip-firing terminology, we use “stable” in place of what would normally be called “irreducible,” and rather than “normal forms” we refer to “stabilizations.” We say that \( \to \) is \textit{terminating} (also some-
Figure 1-1: Examples of various relations: (I) is confluent from $x$ but not from $y$; (II) and (III) are confluent but not terminating; (IV) and (VI) are locally confluent but not confluent; (V) is confluent and terminating.

times called *noetherian*) if there is no infinite sequence of relations $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$; i.e., $\rightarrow$ is terminating means that $\Gamma \rightarrow$ has no infinite walks (which implies in particular that this digraph has no directed cycles). Generally speaking, throughout this thesis the relations we are most interested in will all be terminating and it will be easy for us to establish that they are terminating (although the chip-firing process without a sink is *not* terminating). For $x \in X$, we say that $\rightarrow$ is *confluent from* $x$ if whenever $x \overset{*}{\rightarrow} y_1$ and $x \overset{*}{\rightarrow} y_2$, there is $y_3$ such that $y_1 \overset{*}{\rightarrow} y_3$ and $y_2 \overset{*}{\rightarrow} y_3$. We say $x \in X$ is $\rightarrow$-stable (or just *stable* if the context is clear) if there is no $y \in X$ with $x \rightarrow y$. In graph-theoretic language, $x$ is $\rightarrow$-stable means that $x$ is a sink (vertex of outdegree zero) of $\Gamma \rightarrow$. If $\rightarrow$ is terminating, then for every $x \in X$ there must be at least one stable $y \in X$ with $x \overset{*}{\rightarrow} y$. On the other hand, if $\rightarrow$ is confluent from $x \in X$, then there can be at most one stable $y \in X$ with $x \rightarrow y$. Hence if $\rightarrow$ is terminating and is confluent from $x$, then there exists a unique stable $y$ with $x \overset{*}{\rightarrow} y$; we call this $y$ the *$\rightarrow$-stabilization* (or just *stabilization* if the context is clear) of $x$. We say that $\rightarrow$ is *confluent* if it is confluent from every $x \in X$. As we just explained, if $\rightarrow$ is confluent and terminating then a unique stabilization of $x$ exists for all $x \in X$. A weaker notion than confluence is that of local confluence: we say that $\rightarrow$ is *locally confluent* if for any $x \in X$, if $x \rightarrow y_1$ and $x \rightarrow y_2$, then there is some $y_3$ with $y_1 \overset{*}{\rightarrow} y_3$ and $y_2 \overset{*}{\rightarrow} y_3$. 

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Figure 1-2: A graphical depiction of the proof of the diamond lemma, Lemma 1.1.1.

and \( y_2 \rightarrow y_3 \). Figure 1-1 gives some examples of relations comparing these various notions of confluence and termination. Observe that there is no example in this figure of a relation that is locally confluent and terminating but not confluent. That is no coincidence: Newman’s lemma, a.k.a. the diamond lemma, says that local confluence plus termination implies confluence. We prove the diamond lemma now.

**Lemma 1.1.1** (Diamond lemma, see [60, Theorem 3] or [45, Lemma 2.4]). *Suppose that \( \rightarrow \) is terminating. Then \( \rightarrow \) is confluent if and only if it is locally confluent.*

**Proof.** We reproduce the elegant proof given in [68, Theorem 1.7.10]. One direction (confluence implies local confluence) of this lemma is immediate. So suppose that \( \rightarrow \) is locally confluent and terminating. And assume to the contrary that \( \rightarrow \) is not confluent. Thus we can find some \( x \) such that \( \rightarrow \) is not confluent from \( x \). Because \( \rightarrow \) is terminating, in fact we are free to assume that \( x \) is a “minimal” counterexample in the sense that \( \rightarrow \) is confluent from \( x' \) for all \( x \rightarrow x' \) with \( x \neq x' \) (indeed, otherwise we would be able to find an infinite sequence \( x_0 \rightarrow x_1 \rightarrow \cdots \) with \( x_i \neq x_{i+1} \) for all \( i \in \mathbb{N} \), which would contradict that \( \rightarrow \) is terminating). Now suppose that \( x \rightarrow y \) and \( x \rightarrow z \).

To contradict our assumption that \( \rightarrow \) is not confluent from \( x \), we want to show that there is some \( w \) with \( y \rightarrow w \) and \( z \rightarrow w \). If we have either \( x = y \) or \( x = z \) this is clearly the case; so suppose \( x \neq y \) and \( x \neq z \). Thus \( x \rightarrow y' \rightarrow y \) and \( x \rightarrow z' \rightarrow z \) for some \( x \neq y' \) and \( x \neq z' \). By the local confluence of \( \rightarrow \) there exists \( x' \) with \( y' \rightarrow x' \) and \( z' \rightarrow x' \). Since by assumption that \( x \) is a “minimal” counterexample we have that \( \rightarrow \) is confluent
Figure 1-3: A chip-firing “diamond,” explaining the local confluence of the chip-firing process. The color of the arrow corresponds to the color of the vertex that was fired. From $y'$ and $z'$, there exist $y''$ and $z''$ with $y\rightarrow y''$, $x'\rightarrow y''$, $z\rightarrow z''$, and $x'\rightarrow z''$. Finally, again because $x$ is a “minimal” counterexample, we know also that $\rightarrow$ is confluent from $x'$, so there exists a $w$ with $y''\rightarrow w$ and $z''\rightarrow w$. This $w$ satisfies $y\rightarrow y''\rightarrow w$ and $z\rightarrow z''\rightarrow w$, as required. See Figure 1-2 for a graphical depiction of this proof. \hfill $\square$

Although easily proved, the diamond lemma is a very powerful result. It will be a key tool in establishing confluence at many points throughout this thesis.

We are now prepared to define chip-firing formally in terms of binary relations. Let us write $\mathbb{N}[V]$ for the set of chip configurations on $G$. As this notation suggests, we view a configuration $c \in \mathbb{N}[V]$ as a function $c : V \rightarrow \mathbb{N}$ where $c(v)$ is the number of chips at $v \in V$; we also write the configuration as the formal sum $c = \sum_{v \in V} c(v) \cdot v$. Similarly, we use $\mathbb{Z}[V]$ to denote the larger set of “virtual chip configurations” which may possibly have negative numbers of chips assigned to some vertices. For $a, a' \in \mathbb{Z}$ and $c, c' \in \mathbb{Z}[V]$ we ascribe the obvious meaning to $a \cdot c + a' \cdot c' \in \mathbb{Z}[V]$. The chip-firing process for the digraph $G$ is the binary relation $\rightarrow_G$ on $\mathbb{N}[V]$ defined by

$$c \rightarrow_G c - \text{outdeg}(v) \cdot v + \sum_{(v,u) \in E} u,$$

for $c \in \mathbb{N}[V]$ and $v \in V$ with $c(v) \geq \text{outdeg}(v)$.  

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Proposition 1.1.2. The relation $\rightarrow_G$ is locally confluent.

Proof. The key point is that firing at one vertex can only make another vertex more able to fire. In other words, if we have $c \rightarrow_G c_1$ and $c \rightarrow_G c_2$ for $c_1 \neq c_2$, where $c_1$ is obtained from $c$ by firing at $v_1$ and $c_2$ is obtained from $c$ by firing at $v_2$, then in $c_1$ we can still fire at $v_2$ to obtain some $c_3$, and in $c_2$ we can still fire at $v_1$ to obtain that same configuration $c_3$. This chip-firing “diamond” is depicted in Figure 1-3.

In light of Proposition 1.1.2 and the diamond lemma (Lemma 1.1.1) we would be able to conclude that the chip-firing process is confluent if we knew that it is terminating. However, as we have already seen above, chip-firing need not be terminating. We can at least say that chip-firing is confluent when restricted to those configurations from which it necessarily terminates. But in fact, more than this is true: as Björner-Lovász-Shor demonstrated, there is a strict dichotomy between those configurations which stabilize and those which go on firing forever.

Theorem 1.1.3 (Björner-Lovász-Shor [17, Theorem 1.1]). For any $c \in \mathbb{N}[V]$, either there is no stable $c'$ with $c^* \rightarrow_G c'$, or there is a unique stable $c'$ with $c^* \rightarrow_G c'$ and all firing sequences $c = c_0 \rightarrow_G c_1 \rightarrow_G c_2 \rightarrow_G \cdots$ starting from $c$ eventually terminate at this $c'$ after the same (finite) number of steps.

Observe how Theorem 1.1.3 says that in the case of stabilization, not only is the final stable configuration predetermined, but in fact the number of firings needed to stabilize is also predetermined as well. Actually, as Björner-Lovász-Shor showed, the number of times that each individual vertex fires during the stabilization is also predetermined.

Strictly speaking, Theorem 1.1.3 does not imply that $\rightarrow_G$ is confluent (because in the case of non-termination, the various firing sequences might irreparably diverge). In a sequel paper, Björner and Lovász showed that in fact $\rightarrow_G$ is confluent:

Theorem 1.1.4 (Björner-Lovász [16, Proposition 1.2]). The relation $\rightarrow_G$ is confluent.
So we see that we did not need to appeal to any kind of termination property to establish the confluence of chip-firing. Moreover, as mentioned in the introduction to this chapter, there are many interesting questions concerning the termination of the chip-firing process that we can pose, such as: can we efficiently decide if a configuration will terminate; how few chips do we need in order to guarantee termination; how long can it take for a configuration to terminate; et cetera.

Nevertheless, it would be convenient for various purposes (such as computing stabilizations) if we could modify the chip-firing process to make it necessarily terminating from every initial configuration. This is precisely what adding a sink does.

1.2 Chip-firing with a sink

Continue to fix a digraph $G = (V, E)$ as in the previous section. We now designate a distinguished vertex $q \in V$ to be a sink. Note that we do not require $q$ to be a “sink” in the graph-theoretic sense of a “vertex with outdegree zero,” although in fact it is the case that outgoing edges from $q$ will be irrelevant from the perspective of the “chip-firing with sink” process. However, we do impose a connectivity requirement on the pair $(G, q)$: we require that $q$ be a globally accessible vertex, which means that for every vertex $v \in V^q := V \setminus \{q\}$, there is a directed path in $G$ from $v$ to $q$.

We ignore all chips that accumulate at the sink $q$ and hence the sink $q$ will never become unstable. When we depict a chip configuration on a graph with a sink we therefore just write “$q$” next to the sink vertex, like this:
Let’s see what a sequence of firings looks like for the above configuration:

The fact that the sink $q$ is globally accessible means that every firing move either transports some chips to $q$ and hence removes those chips from the system, or at least brings chips “closer” to $q$. Thus, although it may take quite a while to stabilize, every step brings the system closer to stability. This intuitive heuristic can be made precise and used to show that chip-firing with a globally accessible sink is always terminating.

Now let us define chip-firing with a sink formally in terms of relations. We use $\mathbb{N}[V^q]$ to denote the set of configurations of (nonnegative numbers of) chips on the non-sink vertices of $G$, and $\mathbb{Z}[V^q]$ to denote the set of “virtual configurations” which might have negative numbers of chips. We can treat any element of $\mathbb{Z}[V]$ as an element of $\mathbb{Z}[V^q]$ by ignoring the value at $q$. Chip-firing with respect to the sink $q$ is the binary relation on $\mathbb{N}[V^q]$ defined by

$$c \xrightarrow{(G,q)} c - \text{outdeg}(v) \cdot v + \sum_{(v,u) \in E, u \neq q} u,$$

for $c \in \mathbb{N}[V^q]$ and $v \in V^q$ with $c(v) \geq \text{outdeg}(v)$.

Exactly the same argument as in Proposition 1.1.2 implies that chip-firing with a sink is locally confluent:
Proposition 1.2.1. The relation $\rightarrow_{(G,q)}$ is locally confluent.

And, as mentioned, the fact that $q$ is globally accessible means that chip-firing with respect to the sink $q$ is terminating:

Lemma 1.2.2 (See [23, Theorem 6.12]). The relation $\rightarrow_{(G,q)}$ is terminating.

Hence by the diamond lemma (Lemma 1.1.1) we conclude:

Theorem 1.2.3. The relation $\rightarrow_{(G,q)}$ is confluent (and terminating).

Theorem 1.2.3 tells us that in the chip-firing process with a sink, unique stabilizations exist for all configurations. So for any configuration $c \in \mathbb{N}[V^q]$ we define $\tilde{c}$ to be the stabilization of $c$.

We might want to understand further this stabilization operation. There are various perspectives from which we could approach the problem of “understanding” stabilizations. One such perspective is probabilistic: we could introduce some randomness into the system and then study the system’s long-run behavior.

Here is a natural way to study chip-firing stabilizations “randomly.” We start from some stable configuration $c \in \mathbb{N}[V^q]$. We then choose a vertex $v \in V$ at random, add one chip at $v$ to $c$, and then re-stabilize the resulting configuration: i.e., we replace the configuration $c$ by $\tilde{(c + v)}$ (if we choose the sink $q$ as our $v$, then this leaves $c$ unchanged). Suppose furthermore that we choose each vertex $v \in V$ with some constant nonzero probability. This random process, which we call the Abelian Sandpile Model, defines a finite, aperiodic Markov chain on the set of stable configurations. As mentioned in the introduction to this chapter, this model goes back to Bak-Tang-Wiesenfeld [10] and Dhar [24, 25].

In order to understand the long-run behavior of the Abelian Sandpile Model we need to introduce the notion of recurrent configurations.

1.3 Recurrent configurations

For two stable configurations $c, c' \in \mathbb{N}[V^q]$, we define their stable addition to be the configuration $c \oplus c' := \tilde{(c + c')}$. With this notation, the Abelian Sandpile Model can
be described as the Markov chain which repeatedly replaces a stable configuration $c$ by $c \oplus v$ for a random vertex $v \in V$. This $\oplus$ operation equips the set of stable configurations the structure of an abelian monoid (where the associativity is implied by confluence).

In order to understand the stationary distribution of the Abelian Sandpile Model we will identify a special subset of stable configurations which, as it turns out, forms a group under the operation $\oplus$.

**Definition 1.3.1.** A configuration $c \in \mathbb{N}[V^q]$ is called *recurrent* if for any $c' \in \mathbb{N}[V^q]$, there exists $c'' \in \mathbb{N}[V^q]$ such that $c = (c' + c'')$.

Note that a recurrent configuration must be stable.

The *maximal stable configuration* is $c_{\text{max}} := \sum_{v \in V^q} (\text{outdeg}(v) - 1) \cdot v \in \mathbb{N}[V^q]$. It is a simple exercise to show that a configuration $c \in \mathbb{N}[V^q]$ is recurrent if and only if there exists some $c' \in \mathbb{N}[V^q]$ such that $c = (c_{\text{max}} + c')$. (Thus in particular we have that $c_{\text{max}}$ is always recurrent.) For example, the following is $c_{\text{max}}$ for our running example:

![Diagram](image)

Earlier we saw that:

![Diagram](image)

Hence we conclude that the stable configuration on the right is in fact recurrent.

As their name suggests, the recurrent configurations are those configurations which
will be visited infinitely often in the Abelian Sandpile Model; in fact, in the long run we will spend an equal amount of time in each recurrent configuration:

Theorem 1.3.2 (See [23, Corollary 8.28]). The unique stationary distribution of the Abelian Sandpile Model is the uniform distribution on the set of recurrent configurations.

Theorem 1.3.2 explains the dynamical significance of the recurrent configurations. The algebraic significance was hinted at already: the set of recurrent configurations forms an abelian group with respect to the operation $\oplus$ of stable addition. More precisely, the set of recurrent configurations is the minimal ideal of the monoid of stable configurations (see [8]). The group of recurrent configurations was studied for instance by Dhar [24, 25] and by Biggs [14].

Actually, we can say exactly what the structure of the group of recurrent configurations is in terms of the Laplacian matrix of the graph $G$. The Laplacian matrix $L_G$ of $G$ is the square matrix with rows and columns indexed by the vertices $v \in V$ whose entry at position $(v, u)$ is

$$L_G(v, u) = \begin{cases} \text{outdeg}(v) & \text{if } v = u; \\ -\#\{(v, u) \in E\} & \text{otherwise.} \end{cases}$$

The Laplacian is a singular matrix of corank one. The reduced Laplacian matrix $L_{(G,q)}$ of $G$ (with respect to $q$) is the matrix $L_{(G,q)}$ obtained from $L_G$ by deleting the row and column corresponding to $q$. Because we have assumed that $q$ is globally accessible, the matrix $L_{(G,q)}$ is nonsingular. In particular $\det(L_{(G,q)})$ is equal to the number of rooted arborescences at $q$; if $G$ is an undirected graph this is the same as the number of spanning trees of $G$. (This is the celebrated Matrix Tree Theorem of Kirchhoff.)

The reduced Laplacian $L_{(G,q)}$ encodes the dynamics of chip-firing: we can represent in the obvious way a virtual configuration $c \in \mathbb{Z}[V^q]$ as a column vector, and then firing at some $v \in V$ amounts to subtracting the column of $L'_{(G,q)}$ corresponding to $v$. Hence two elements $c, c' \in \mathbb{Z}[V^q]$ are said to be chip-firing equivalent if they lie in the same equivalence class modulo the image of transposed matrix $L'_{(G,q)}$.
There is a unique recurrent configuration in each chip-firing equivalence class. That is to say, the recurrent configurations serve as canonical representatives for the quotient $\text{coker}(L^t_{(G,q)}) := \mathbb{Z}[V^q]/\text{Im}(L^t_{(G,q)})$. Thus we can give the following description of the group of recurrent configurations.

**Theorem 1.3.3** (See [23, Theorem 6.28]). The set of recurrent configurations with the operation $\oplus$ of stable addition is isomorphic to the cokernel $\text{coker}(L^t_{(G,q)})$, where we view the reduced Laplacian as a ring homomorphism $L^t_{(G,q)} : \mathbb{Z}[V^q] \rightarrow \mathbb{Z}[V^q]$.

The group of recurrent configurations with stable addition is called the *sandpile group*. Theorem 1.3.3 says that the structure of the sandpile group is determined by the invariant factors of the matrix $L^t_{(G,q)}$, which can be obtained by computing the Smith Normal Form of this matrix. The order of this group is $\det(L_{(G,q)})$ (i.e., the number of spanning trees in the case that $G$ is undirected).

### 1.4 Superstable configurations

There is another set of representatives for $\text{coker}(L^t_{(G,q)})$ which is dual to the set of recurrent configurations. These are the superstable configurations.

**Definition 1.4.1.** A configuration $c \in \mathbb{N}[V^q]$ is *superstable* if there does not exist any $\sigma \in \mathbb{N}[V^q]$ with $\sigma \neq 0$ for which $c - L^t_{(G,q)}\sigma \in \mathbb{N}[V^q]$.

In other words, $c$ being superstable means that there is no sequence of “firings” of both unstable and stable vertices we can carry out starting from $c$ which, although the intermediary steps may have negative numbers of chips on some vertices, eventually results in a honest configuration with only nonnegative numbers of chips at every vertex. For example, the following stable configuration is not superstable:

![Diagram of a graph with vertices labeled 0, 2, and q, and arrows indicating chip movements.](image-url)
This is because we can carry out a sequence of “firings” as follows:

Note in particular that $c$ being superstable implies that $c$ is stable.

If $G$ is an undirected graph, then it is enough to take $\sigma$ with coefficients in $\{0, 1\}$ in Definition 1.4.1 (these $\sigma$ represent “set firings”), but for arbitrary digraphs $G$ this is not the case.

As mentioned, the superstable configurations also serve as canonical representatives for $\text{coker}(L^t(G, q))$. And there is a natural notion of “superstable addition” which turns the set of superstable configurations into an abelian group isomorphic to the sandpile group. (See [23, §3.3 and §7.3] for more details on superstable configurations.)

Moreover, we have the following intriguing duality between recurrent and superstable configurations:

**Theorem 1.4.2 (See [23, Theorem 7.12]).** The configuration $c \in \mathbb{N}[V^q]$ is superstable if and only if $c_{\text{max}} - c$ is recurrent.

Note that the duality map in Theorem 1.4.2 does not preserve the group structure: for instance, 0 is always the identity for the group of superstables, but $c_{\text{max}}$ is usually not the identity for the group of recurrences.

One last remark about recurrent and superstable chip configurations: there is an important algorithm, called Dhar’s burning algorithm, which can decide if a given configuration $c \in \mathbb{N}[V^q]$ is recurrent (or equivalently, by Theorem 1.4.2, if it superstable). The burning algorithm was originally introduced by Dhar in the context of undirected graphs [24] and later extended to directed graphs by Speer [70]. Describing the algorithm here would take us too far afield; see for instance [23, Chapter 7]...
for a complete account.

1.5 M-matrix chip-firing

Above we saw how the investigation of a discrete dynamical system, coming from statistical mechanics, lead us to uncover a beautiful algebraic theory. Ultimately underlying all of this was the Laplacian matrix. It is reasonable to ask whether we can generalize the theory of chip-firing to "other matrices"; in other words, it is reasonable to ask what properties of the reduced Laplacian $L_{(G,q)}$ are required to make the above theory work. The most general setting which naturally extends the classical theory of chip-firing is what is called $M$-matrix chip-firing.

Let $C = (C_{ij}) \in \mathbb{Z}^{n \times n}$ be any $n \times n$ matrix with integer entries. We can define chip-firing with respect to the matrix $C$ in the obvious way: it is the binary relation on $\mathbb{Z}^n$ defined by

$$c \xrightarrow{c} c - C^i e_i \quad \text{for} \quad c = (c_1, \ldots, c_n) \in \mathbb{Z}^n \quad \text{and} \quad i \in [n] \quad \text{with} \quad c_i \geq C_{ii},$$

where $e_i$ is the $i$th standard basis vector. (From a physical perspective it might make more sense to define this as a relation on $\mathbb{N}^n$, but from an algebraic perspective $\mathbb{Z}^n$ is more convenient and the differences are at any rate minimal.)

At this level of generality there is not much that one can say about $\xrightarrow{c}$. To even get started with the parts of chip-firing theory that deal with canonical representatives for $\text{coker}(C^i)$ we need stabilizations to exist, so we need to understand when $\xrightarrow{c}$ is confluent and terminating.

There is a simple condition which at least guarantees that $\xrightarrow{c}$ is locally confluent. **Definition 1.5.1.** The matrix $C = (C_{ij}) \in \mathbb{Z}^{n \times n}$ is called a $Z$-matrix if $C_{ij} \leq 0$ for all $1 \leq i \neq j \leq n$.

Exactly the same simple "diamond" proof as in Proposition 1.1.2 then gives us: **Proposition 1.5.2.** Suppose that $C \in \mathbb{Z}^{n \times n}$ is a $Z$-matrix. Then $\xrightarrow{C}$ is locally confluent.
Now we can ask: what additionally do we need to assume about $C$ to make sure that $\rightarrow_C$ is terminating? This was answered by Gabrielov [34].

**Definition 1.5.3.** Let $C = (C_{ij}) \in \mathbb{Z}^{n \times n}$ be a Z-matrix. We say that that $C$ is a (nonsingular) M-matrix if any one of the following equivalent conditions holds:

1. the real parts of all eigenvalues of $C$ are strictly positive;

2. $C^{-1}$ exists and has all nonnegative entries;

3. there exists $x \in \mathbb{R}^n$ with $x \geq 0$ such that $Cx$ has all positive entries.

See for instance Plemmons [63] for the equivalence between these conditions.

These families of matrices (Z-matrices and M-matrices) commonly arise in various areas of science and hence have been heavily investigated. The term “M-matrix” goes back to Ostrowski [61], who coined it in honor of Hermann Minkowski because of a result of Minkowski which says that a Z-matrix with all positive row sums has a positive determinant. An M-matrix which has all nonnegative row sums is the reduced Laplacian of some directed graph with respect to a globally accessible sink. But M-matrices are really more general than reduced Laplacians; for example the following matrix $C$ is an M-matrix (as can be seen from its inverse), but $C$ is not a reduced Laplacian of any graph:

$$C = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}; \quad C^{-1} = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

Gabrielov established the following:

**Theorem 1.5.4** (Gabrielov [34] Theorem 1.5). Let $C = (C_{ij}) \in \mathbb{Z}^{n \times n}$ be a Z-matrix. Then $\rightarrow_C$ is terminating if and only if $C$ is an M-matrix.

Therefore, Theorem 1.5.4 together with Proposition 1.5.2 and the diamond lemma (Lemma 1.1.1) says that the relation $\rightarrow_C$ is confluent and terminating when $C$ is an M-matrix. Hence we can define chip-firing stabilizations for an M-matrix $C$. And
indeed, the entire theory of chip-firing with a sink outlined above goes through for M-matrices: there are recurrent configurations and superstable configurations; these both serve as canonical representatives for $\text{coker}(C^t)$; the recurrences and superstables are dual to one another; et cetera. The theory of chip-firing for M-matrices has been worked out carefully by Guzmán and Klivans [40] (see also the appendix of [64]).

One interesting class of M-matrices are the Cartan matrices of irreducible, crystallographic root systems. Chip-firing with respect to Cartan matrices was extensively investigated by Benkart, Klivans, and Reiner [13]. Since the title of this thesis is “Root System Chip-Firing,” one might reasonably assume at this point that we would focus on Cartan matrix chip-firing for the remainder of the thesis.

We will indeed discuss Cartan matrix chip-firing in more detail later on. However, Cartan matrix chip-firing is not the primary concern of this thesis. We will connect root systems and chip-firing in another way. Actually, the connection between root systems and chip-firing we are interested in has its origins in labeled chip-firing, which is a certain “non-commutative” variant of chip-firing that was recently introduced by James Propp. The next chapter is about labeled chip-firing.
Chapter 2

Labeled chip-firing

This chapter discusses confluence for James Propp’s labeled chip-firing process. The material in this chapter is joint work with Thomas McConville and James Propp and appears in [43].

Before describing labeled chip-firing, let us take a moment to explain how Propp came up with this chip-firing variant. Levine and his coauthors [18, 19, 20, 42, 22], building on the original program of Dhar [25], have developed a general framework for studying networks of processors that are able to compute some predictable output despite a lack of global coordination. They call these abelian networks. The abelian sandpile model is a prototypical example of an abelian network (as is the related rotor-router model [41]). Inspired by this conception of chip-firing as a computational process, Propp got interested in the question of whether sorting is possible in such a network and this lead him to consider a variant of chip-firing where the chips are labeled. There is something inherently “non-commutative” about sorting, so one initial hope (as yet unrealized) was that this line of inquiry could possibly lead to a “non-abelian sandpile model.”

Thus in one sense Propp’s introduction of labeled chip-firing was an attempt to extend the frontiers of chip-firing research. But in another sense, it was a return to basics. This is because the setting for the labeled chip-firing process, i.e., the graph on which the chips are fired, is the infinite path graph. And chip-firing on the infinite path was in fact studied in two papers from the 1980s, of Spencer [71]
and Anderson et al. [4], which served as direct inspiration for the seminal work of Björner-Lovász-Shor [17].

We now define the labeled chip-firing process. As mentioned, labeled chip-firing takes place on the infinite path. The infinite path is the infinite undirected graph $G = (V, E)$ with vertex set $V = \mathbb{Z}$ and with edge set $E = \{(i, i + 1): i \in \mathbb{Z}\}$. By abuse of notation we write $G = \mathbb{Z}$. Although $\mathbb{Z}$ is an infinite graph, each vertex in this graph has only finitely many neighbors, so as long as we only consider configurations which have a finite total number of chips the chip-firing process (without a sink) described in Chapter [1] still makes sense for $\mathbb{Z}$.

In the classical chip-firing process all the chips are indistinguishable. In contrast, the labeled chip-firing process considers configurations of $N$ distinguishable labeled chips $\overline{1}, \overline{2}, ..., \overline{N}$ to the vertices of $\mathbb{Z}$. A firing move for the labeled chip-firing consists of selecting two chips $\overline{i}$ and $\overline{j}$, with $i < j$, which occupy the same vertex, and moving the lesser-labeled chip $\overline{i}$ one vertex to the left while moving the greater-labeled chip $\overline{j}$ one vertex to the right. Note crucially that lesser-labeled chips always move to the left and greater-labeled chips always move to the right: this gives the system a propensity towards sorting the chips, and without this assumption there is clearly no hope for any kind of confluence.

For example, suppose $N = 4$, and that we start from the following configuration (where we draw $\mathbb{Z}$ in the plane as a number line in the usual way, with vertex $i - 1$ to the left of $i$):

1
3
2
1

(The height of a chip within a pile is immaterial.) By firing chips $\overline{1}$ and $\overline{2}$ together we reach the following configuration:

1
4
3
2

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Then by firing 3 and 4 we reach this configuration:

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
\]

After firing 1 and 3, and then firing 2 and 4, in two more steps we reach the following configuration:

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
\]

Finally, by firing 2 and 3 we reach the following stable configuration, which has no more possible firings:

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
\]

Observe how all the chips have ended up in sorted order. There were several choices we made during this stabilization process, but one can verify that from the initial configuration with four chips at the origin, every sequence of firings will eventually terminate at this same stable configuration.

However, consider if we had started from the configuration with three chips at the origin instead:

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
\]

There are three possible firings we can carry out from this configuration: we can fire 1 and 2; we can fire 1 and 3; or we can fire 2 and 3. Each of these firings takes us to a different stable configuration. Hence, the labeled chip-firing process is not confluent starting from three chips at the origin.

These two examples indicate that questions about confluence for labeled chip-firing might be subtle. Based on computational evidence, Propp conjectured that starting
from an even number \( N = 2m \) of chips at the origin, the labeled chip-firing process is confluent and in particular always sorts the chips. Generalizing our observation above about three chips, it is easy to see that starting from an odd number \( N > 1 \) of chips at the origin, the labeled chip-firing process is not confluent. So we see that confluence is not at all a “local” property for the labeled chip-firing process: e.g., it may depend on the parity of the total number of chips, which is a “global” statistic.

Most of the remainder of this chapter is focused on proving Propp’s sorting conjecture. At the end of the chapter we discuss extensions of labeled chip-firing, which will ultimately segue us to the introduction of root systems in this thesis.

### 2.1 Unlabeled chip-firing on the infinite path

If we forget the labels of the chips, then the labeled chip-firing process on \( \mathbb{Z} \) becomes the same as the unlabeled chip-firing process on \( \mathbb{Z} \). Relating the labeled and unlabeled processes will be key in our proof of confluence for the labeled process. So before we study the labeled process in its own right, let us review unlabeled chip-firing on \( \mathbb{Z} \). As mentioned, classical (unlabeled) chip-firing on \( \mathbb{Z} \) was studied by Anderson et al. [1].

First let us establish some notation of unlabeled chip-firing on \( \mathbb{Z} \). As in Chapter [1], we denote the set of unlabeled configurations on \( \mathbb{Z} \) by \( \mathbb{N}[\mathbb{Z}] \). We will always use lowercase letters for unlabeled configurations. We view an element \( c \in \mathbb{N}[\mathbb{Z}] \) as a function \( c: \mathbb{Z} \to \mathbb{N} \) with \( \sum_{i \in \mathbb{Z}} c(i) < \infty \), where we think \( c \) as having \( c(i) \) chips at \( i \). We also sometimes write \( c \in \mathbb{N}[\mathbb{Z}] \) as the formal sum \( c = \sum_{i \in \mathbb{Z}} c(i) \cdot i \). We use \( \text{supp}(c) \) to denote the support of \( c \), i.e., \( \text{supp}(c) := \{ i \in \mathbb{Z} : c(i) \geq 1 \} \). We write \( \max(c) := \max(\text{supp}(c)) \) and \( \min(c) := \min(\text{supp}(c)) \). As is customary, we use the convention \( \max(\emptyset) := -\infty \) and \( \min(\emptyset) := \infty \).

If \( c \in \mathbb{N}[\mathbb{Z}] \) and \( i \in \mathbb{Z} \) is a vertex with \( c(i) \geq 2 \), then in the configuration \( c \) we may fire at \( i \), which sends one chip from vertex \( i \) to vertex \( i - 1 \) and sends one chip from vertex \( i \) to vertex \( i + 1 \); i.e., firing at \( i \) replaces \( c \) by \( c' = c - 2 \cdot i + (i - 1) + (i + 1) \). As in Chapter [1] we write \( c \to c' \) to denote that \( c' \) is obtained from \( c \) by firing at some vertex of \( \mathbb{Z} \) (thus \( \to \) is a relation on \( \mathbb{N}[\mathbb{Z}] \)).
The same simple “diamond” argument as in Proposition 1.1.2 implies that the relation $\rightarrow$ is locally confluent. However, there is no sink for the unlabeled chip-firing process on $\mathbb{Z}$, so we could be concerned that $\rightarrow$ might not be terminating. But in fact because the graph $\mathbb{Z}$ is infinite, any configuration of a finite number of chips will eventually spread out its chips enough to reach a stable configuration.

The following statistics of configurations will be used to establish termination:

\[
\varphi_\ell(c) := \sum_{i \leq \ell} (i - \ell - 1) \cdot c(i) \quad \text{for all } \ell \in \mathbb{Z};
\]

\[
\varphi_\infty(c) := \sum_{i \in \mathbb{Z}} i \cdot c(i);
\]

\[
\varphi_\infty^2(c) := \sum_{i \in \mathbb{Z}} i^2 \cdot c(i);
\]

\[
\gamma(c) := \# \{i \in \mathbb{Z} : \min(c) \leq i \leq \max(c) \text{ and } \{i, i+1\} \cap \text{supp}(c) = \emptyset\}.
\]

**Proposition 2.1.1.** Suppose that $c' \in \mathbb{N}[\mathbb{Z}]$ is obtained from $c \in \mathbb{N}[\mathbb{Z}]$ by firing at vertex $j \in \mathbb{Z}$; then:

1. $\varphi_\ell(c') = \begin{cases} 
\varphi_\ell(c) - 1 & \text{if } j = \ell + 1; \\
\varphi_\ell(c) & \text{otherwise}; 
\end{cases}$

2. $\varphi_\infty(c') = \varphi_\infty(c)$;

3. $\varphi_\infty^2(c') = \varphi_\infty^2(c) + 2$;

4. $\gamma(c') \leq \gamma(c)$.

**Proof.** These are routine to verify. \[ \square \]

**Lemma 2.1.2.** The relation $\rightarrow$ on $\mathbb{N}[\mathbb{Z}]$ is terminating.

**Proof.** To show $\rightarrow$ is terminating it suffices to show the following:

- $\rightarrow$ is *acyclic*, i.e., there is no cycle $c \rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_k = c$ for any $k \geq 1$;

- $\rightarrow$ is *globally finite*, i.e., for any $c$ there are only finitely many $d$ with $c^* \rightarrow d$. 

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Let $c \in \mathbb{N}[\mathbb{Z}]$ be a configuration of $N$ chips. If $N = 0$ there is nothing to show, so suppose that $N > 1$. To see that there cannot be a cycle of firings from $c$ to itself, recall from Proposition 2.1.1 that $c \rightarrow d$ means $\varphi_\infty^2(d) = \varphi_\infty^2(c) + 2$. So any purported cycle at configuration $c$ of length $k \geq 1$ would lead to $\varphi_\infty^2(c) = \varphi_\infty^2(c) + 2k$, a contradiction. Now let us show that there are only finitely many $d$ with $c \overset{\star}{\rightarrow} d$. Recall from Proposition 2.1.1 that $c \overset{\star}{\rightarrow} d$ implies $\varphi_\infty(d) = \varphi_\infty(c)$. This in particular implies that there is some $B_1 \in \mathbb{N}$ such that $\min(d) \leq B_1$ and $\max(d) \geq -B_1$ for any $c \overset{\star}{\rightarrow} d$. Also recall from Proposition 2.1.1 that $c \overset{\star}{\rightarrow} d$ implies $\gamma(d) \leq \gamma(c)$. This in particular implies that there is $B_2 \in \mathbb{N}$ such that $\max(d) - \min(d) \leq B_2$ for all $c \overset{\star}{\rightarrow} d$. Altogether, we can conclude that there is $B_3 \in \mathbb{N}$ such that $\max(d) \leq B_3$ and $\min(d) \geq -B_3$ for all $c \overset{\star}{\rightarrow} d$. Clearly there are only finitely many configurations $d$ of $N$ chips satisfying $\max(d) \leq B_3$ and $\min(d) \geq -B_3$ so indeed there can only be finitely many $d$ with $c \overset{\star}{\rightarrow} d$, and thus $\rightarrow$ is terminating.

We note that the termination of $\rightarrow$ was also established, essentially by the same means, by Anderson et al. [4] (at least starting from $N$ chips at the origin). Thanks to Lemma 2.1.2 together with the diamond lemma (Lemma 1.1.1), we conclude that $\rightarrow$ is confluent and terminating. Thus we use $\bar{c}$ to denote the stabilization of $c \in \mathbb{N}[\mathbb{Z}]$.

The algebraic theory of chip-firing discussed in Chapter 1 becomes trivial for $\mathbb{Z}$: for instance, recall that the order of the sandpile group of a graph is the number of spanning trees of the graph; but of course $\mathbb{Z}$ has only one spanning tree. Nevertheless, we can still ask about “computing” stabilizations. And in fact formulas for unlabeled stabilizations will be very useful for us in establishing confluence of the labeled process.

So let us introduce some notation for specific configurations of unlabeled chips. As in Chapter 1 for two configurations $c, d \in \mathbb{N}[\mathbb{Z}]$, we use $c + d$ to denote their sum, and for $N \in \mathbb{N}$ we use the shorthand $Nc := \underbrace{c + c + \cdots + c}_{N \text{ terms}}$. For $i \in \mathbb{Z}$ we let $\delta_i \in \mathbb{N}[\mathbb{Z}]$ denote the configuration that has a single chip at $i$ and no other chips; that is, $\delta_i$ is the unique stable configuration with $\text{supp}(\delta_i) = \{i\}$. For $i, j \in \mathbb{Z}$, we let $\delta_{[i,j]} \in \mathbb{N}[\mathbb{Z}]$ denote the configuration that has one chip at vertex $k$ for all $i \leq k \leq j$ and no other chips; in other words $\delta_{[i,j]}$ is the unique stable configuration with $\text{supp}(\delta_{[i,j]}) = [i,j]$.
Note in particular that $\delta_i = \delta_{[i,i]}$.

**Proposition 2.1.3.** Suppose that $c = \delta_{[a+1,b-1]} + \delta_i$, where $a, b, i \in \mathbb{Z}$ satisfy $a < b$, $a \leq i$, and $i \leq b$. Then we have $\tilde{c} = \delta_{[a,a+b-i-1]} + \delta_{[a+b-i+1,b]}$.

**Proof.** We prove this by induction on $b - a$. If $b - a = 1$ the proposition is clear because then $c = \tilde{c} = \delta_i$. So assume $b - a > 1$ and the result is known for smaller values of $b - a$. If $i = a$ or $i = b$ the proposition is also clear because then $c = \tilde{c}$. So assume that $a < i < b$. Set $\tilde{c} = \delta_{[a+1,i-1]} + \delta_{i-1}$ and $c' := \delta_{i+1} + \delta_{[i+1,b-1]}$. By firing at vertex $i$ we see that $c \rightarrow c' + c''$. Applying the inductive hypothesis gives

$$
\tilde{c'} = \delta_a + \delta_{[a+2,i]},
$$

$$
\tilde{c''} = \delta_{[i,b-2]} + \delta_b.
$$

So $c \rightarrow \delta_a + \delta_{[a+2,i]} + \delta_{[i,b-2]} + \delta_b = \delta_a + c'' + \delta_b$ where $c'' := \delta_{[a+2,b-2]} + \delta_i$. Applying the inductive hypothesis again gives

$$
\tilde{c''} = \delta_{[a+1,a+b-i-1]} + \delta_{[a+b-i+1,b-1]},
$$

so that $c \rightarrow \delta_{[a,a+b-i-1]} + \delta_{[a+b-i+1,b]}$. But $\delta_{[a,a+b-i-1]} + \delta_{[a+b-i+1,b]}$ is stable, which means we must have $\tilde{c} = \delta_{[a,a+b-i-1]} + \delta_{[a+b-i+1,b]}$. \(\square\)

The unlabeled configuration we are most interested in is $N\delta_0$, i.e., the configuration of $N$ chips at the origin. The following description of the stabilization of $N\delta_0$ appears in the original paper of Anderson et al. [4].

**Lemma 2.1.4 (Anderson et al. [4, Theorem 2]).** For all $N \geq 1$ we have,

$$
\tilde{N\delta_0} = \begin{cases} 
\delta_{[-m,-1]} + \delta_{[1,m]} & \text{if } N = 2m \text{ is even}; \\
\delta_{[-m,m]} & \text{if } N = 2m + 1 \text{ is odd}.
\end{cases}
$$

**Proof.** The proof is by induction on $N$. The case $N = 1$ is clear; so suppose $N > 1$ and the result is known for $N - 1$. Set $c := (N-1)\delta_0$. We have $\tilde{N\delta_0} = \tilde{c} + \delta_0$. 41
If $N = 2m$ is even then by induction $\tilde{c} = \delta_{[-(m-1),m-1]}$, so $\widehat{c + \delta_0} = \delta_{[-m,-1]} + \delta_{[1,m]}$ by Proposition 2.1.3. If $n = 2m + 1$ is odd, then by induction we have $\tilde{c} = \delta_{[-m,-1]} + \delta_{[1,m]}$ and so clearly we have $\widehat{c + \delta_0} = c = [-m,m]$. \hfill $\square$

Figure 2-1 depicts the stabilization of $N\delta_0$ described by Lemma 2.1.4. Note that the bottom image in Figure 0-1 depicts the two-dimensional analog of Figure 2-1 the stabilization of many chips at the origin of $\mathbb{Z}^2$. We see that the two-dimensional sandpile theory is much richer than this one-dimensional theory.

We need just a few more results about unlabeled chip-firing before we are ready to consider labeled chip-firing. Very roughly speaking, to prove confluence of the labeled chip-firing process we study how far we can move chips via chip-firing. Thus we now go over a few results concerning the movement of chips during stabilization. The following proposition is obvious but important.

**Proposition 2.1.5.** For $c, d \in \mathbb{N}[\mathbb{Z}]$, if $c \overset{*}{\to} d$ then $\min(\tilde{c}) \leq \min(d)$.

**Proof.** Each chip-firing move preserves or decreases the minimum occupied vertex, so we have $\min(d') \leq \min(d)$ for any $d \overset{*}{\to} d'$. Thus in particular we have $\min(\tilde{d}) \leq \min(d)$. But if $c \overset{*}{\to} d$, then $\tilde{d} = \tilde{c}$. \hfill $\square$
Proposition 2.1.5 says that the furthest left we can move chips via chip-firing is realized by stabilizing. This is useful, but what we would really like to be able to do is to compare the minimum occupied vertex of the stabilizations of two configurations \( c, d \in \mathbb{N}[\mathbb{Z}] \) which are not necessarily related via chip-firing. It is clearly not the case that \( \min(c) \leq \min(d) \) implies \( \min(\tilde{c}) \leq \min(\tilde{d}) \). But if we are more careful about which \( c \) and \( d \) we compare, something along these lines is true.

To that end, we define a partial order on unlabeled configurations of \( N \) chips that can informally be thought of as “\( c \leq d \) means that \( d \) is obtained from \( c \) by moving chips rightward”; it is defined formally as follows. For \( c, d \in \mathbb{N}[\mathbb{Z}] \) we write \( c \leq d \) if and only if \( \sum_{i \in \mathbb{Z}} c(i) = \sum_{i \in \mathbb{Z}} d(i) \) (i.e., \( c \) and \( d \) have the same number of chips) and \( \sum_{i \geq j} c(i) \leq \sum_{i \geq j} d(i) \) for all \( j \in \mathbb{Z} \). Observe that \( c \leq d \) implies that \( \max(c) \leq \max(d) \) and \( \min(c) \leq \min(d) \). We write \( c \prec d \) to mean that \( d \) covers \( c \) according to this partial order \( \leq \). In other words, \( c \prec d \) means that \( d \) is obtained from \( c \) by moving one chip rightward one vertex.

An important property of this partial order is that it is preserved under stabilization, as we establish right now. In fact, something even stronger is true: stabilization preserves the cover relations of this partial order. (Note that \( \varphi_\infty \) is a rank function for \( \leq \), where \( \varphi_\infty(c) := \sum_{i \in \mathbb{Z}} i \cdot c(i) \) is the statistic defined earlier in this section. By Proposition 2.1.1 chip-firing moves preserve \( \varphi_\infty \). So in fact stabilization being order-preserving is easily seen to be equivalent to it preserving cover relations.)

**Lemma 2.1.6.** For \( c, d \in \mathbb{N}[\mathbb{Z}] \), if \( c \prec d \) then \( \tilde{c} \prec \tilde{d} \).

**Proof.** That \( c \prec d \) means there is some \( c' \) and \( i \in \mathbb{Z} \) such that \( c = c' + \delta_i \) and \( d = c' + \delta_{i+1} \). Define \( a := \max\{j \leq i : j \notin \text{supp}(c')\} \) and \( b := \min\{j \geq i + 1 : j \notin \text{supp}(c')\} \). Thus there exists a configuration \( c'' \) such that \( \tilde{c}' = c'' + \delta_{[a+1,b-1]} \) and \( \text{supp}(c'') \cap [a,b] = \emptyset \). Proposition 2.1.3 then implies

\[
\begin{align*}
\tilde{c}' + \delta_i &= c'' + \delta_{[a,a+b-i-1]} + \delta_{[a+b-i+1,b]}, \\
\tilde{c}' + \delta_{i+1} &= c'' + \delta_{[a,a+b-i-2]} + \delta_{[a+b-i,b]}.
\end{align*}
\]
Thus, $\tilde{c}' + \delta_i < \tilde{c}' + \delta_{i+1}$. But $c = \tilde{c}' + \delta_i$ and $d = \tilde{c}' + \delta_{i+1}$, proving the lemma.

Lemma 2.1.6 will be key in our proof of the confluence of labeled chip-firing, but it is also an interesting result about classical chip-firing on the infinite path in its own right.

Example 2.1.7. The following depicts an example of Lemma 2.1.6:

![Diagram]

2.2 Confluence for an even number of chips

In this section we prove the confluence of labeled chip-firing starting from an even number of chips at the origin. As described in the introduction to this chapter, the labeled chip-firing process is not locally confluent. Thus in order to establish confluence from a particular labeled configuration we cannot apply the diamond lemma or any other “local” argument, making this confluence result very different from all the others we have described so far in this thesis.

Here is our formalism for labeled configurations. A labeled configuration of chips on $\mathbb{Z}$ is some assignment of a finite number of distinguishable chips, labeled by positive integers, to the vertices of $\mathbb{Z}$. We use uppercase calligraphic script for labeled configurations and use $\mathcal{C}$ to denote the chip labeled $i$. Formally, we treat a labeled configuration $\mathcal{C}$ as a function $\mathcal{C} : X \rightarrow \mathbb{Z}$ for some $X \subseteq \{1, 2, \ldots, N\}$, and we think of chip $\mathcal{C}(i)$ as being at the vertex $\mathcal{C}(i)$ in $\mathcal{C}$ for all $i \in X$. We write $\mathbb{Z}^X$ for the set of such configurations. Normally we will take $X = [N]$ and thus study labeled configurations of the $N$ chips $\mathcal{C}(1), \mathcal{C}(2), \ldots, \mathcal{C}(N)$. If $a < b$ and chips $\mathcal{C}(a)$ and $\mathcal{C}(b)$ are at the same vertex in $\mathcal{C}$, we may fire $\mathcal{C}(a)$ and $\mathcal{C}(b)$ together in $\mathcal{C}$ by moving $\mathcal{C}(a)$ leftward...
one vertex and \( \circ \) rightward one vertex. (The important point is that **chips with lesser labels move leftward**.) We write \( \mathcal{C} \rightarrow \mathcal{D} \) to mean that \( \mathcal{D} \) is obtained from \( \mathcal{C} \) by a labeled chip-firing moves of this form (thus \( \rightarrow \) is a relation on \( \mathbb{Z}^X \)). If \( \mathcal{C} \) is a labeled configuration we use \( [\mathcal{C}] \) to denote the underlying unlabeled configuration: thus \( [\mathcal{C}](i) := \#\mathcal{C}^{-1}(i) \) for all \( i \in \mathbb{Z} \). Observe that \( \mathcal{D} \) is stable exactly when \( [\mathcal{D}] \) is stable. As mentioned, our strategy in understanding labeled chip-firing will be to relate it to unlabeled chip-firing. To that end, here are some very basic facts relating labeled and unlabeled chip-firing which we will use freely from now on.

**Proposition 2.2.1.**

- For \( \mathcal{C}, \mathcal{D} \in \mathbb{Z}^X \), if \( \mathcal{C} \rightarrow \mathcal{D} \) then \( [\mathcal{C}] \rightarrow [\mathcal{D}] \).
- For \( \mathcal{C} \in \mathbb{Z}^X \), if \( c = [\mathcal{C}] \) and \( c \rightarrow d \), then there is \( \mathcal{D} \in \mathbb{Z}^X \) with \( \mathcal{C} \rightarrow \mathcal{D} \) and \( d = [\mathcal{D}] \).

So, thanks to Lemma 2.1.2, the relation \( \rightarrow \) on \( \mathbb{Z}^X \) is terminating. Furthermore, for any \( \mathcal{C} \in \mathbb{Z}^X \) there exists some stable \( \mathcal{D} \in \mathbb{Z}^X \) with \( \mathcal{C} \rightarrow \mathcal{D} \), and we have \( [\mathcal{D}] = [\widehat{\mathcal{C}}] \). There need not be a unique stable \( \mathcal{D} \) with \( \mathcal{C} \rightarrow \mathcal{D} \): the previous proposition only determines \( [\mathcal{D}] \) but not the way that the chips are labeled in \( \mathcal{D} \). Nevertheless we are interested in cases where we do have a unique labeled stabilization. In particular, we will consider the labeled analog of \( N\delta_0 \), which has chips \( \bigcirc \), \( \bigcirc \), \ldots, \( \bigcirc \), at vertex 0 and no other chips; we denote this configuration by \( \Delta^N \in \mathbb{Z}^{[N]} \). In other words, we have \( \Delta^N(i) := 0 \) for all \( i \in [N] \). Of course, \( [\Delta^N] = N\delta_0 \). Note, as mentioned in the introduction to this chapter, that \( \Delta^3 \) already does not have a unique stabilization. On the other hand, our main result is that when \( N \) is even, \( \Delta^N \) does have a unique stabilization.

First let us observe that there is a useful global symmetry in this labeled chip-firing process when we start from the configuration \( \Delta^N \). If \( \mathcal{C} \in \mathbb{Z}^{[N]} \), define its dual \( \mathcal{C}^* \) as follows: first reflect \( \mathcal{C} \) horizontally about the origin, then replace the label of each chip \( \bigcirc \) with the label \( (N + 1 - i) \) for all \( 1 \leq i \leq n \). Of course \((\mathcal{C}^*)^* = \mathcal{C} \).

**Lemma 2.2.2.** We have \( \Delta^N \rightarrow \mathcal{C} \) if and only if \( \Delta^N \rightarrow \mathcal{C}^* \).
Proof. It is easy to see that the duality operation respects labeled chip-firing moves, meaning that if \( \mathcal{D} \) is obtained from \( \mathcal{C} \) by a labeled chip-firing move then \( \mathcal{D}^* \) is obtained from \( \mathcal{C}^* \) by a labeled chip-firing move. The lemma then follows since \( (\Delta^N)^* = \Delta^N \).

We now take up the aforementioned strategy of tracking the movement of chips during stabilization. Applying Proposition 2.1.5 to our situation of interest tells us that if \( \Delta^N \rightarrow \mathcal{C} \) then \( \min[[\mathcal{C}]] \geq -\lfloor N/2 \rfloor \) and, by Lemma 2.2.2, \( \max[[\mathcal{C}]] \leq \lfloor N/2 \rfloor \). This puts some constraint on the movement of chips during the labeled chip-firing process, but it says nothing about the position of chips with particular labels. We want to strengthen this conclusion about how far chips can move to take into account chip labels.

Let us establish some notation for restricting labeled configurations to a subset of chips. For a labeled configuration \( \mathcal{C} \in \mathbb{Z}^X \) and \( Y \subseteq \{1, 2, \ldots\} \), we use \( \mathcal{C} \setminus Y \in \mathbb{Z}^{X \setminus Y} \) to denote the restriction of \( \mathcal{C} \) to the chips with labels in \( X \setminus Y \). For any labeled configuration \( \mathcal{C} \) and any \( k \in \mathbb{N} \), we use the shorthand \( \mathcal{C}|_{\geq k} := \mathcal{C} \setminus [k - 1] \). We want some way to describe how the largest-labeled chips evolve in the labeled chip-firing process. So for \( c, d \in \mathbb{Z}[\mathbb{N}] \) we write \( c \xrightarrow{R} d \) if \( d \) is obtained from \( c \) by either:

- performing a chip-firing move (i.e., we have \( c \rightarrow d \));
- moving one chip rightward one vertex (i.e., we have \( c \prec d \)).

If \( c \xrightarrow{R} d \) then we say that \( d \) is rightward-reachable from \( c \). This notion of rightward-reachability precisely captures the way the largest-labeled chips evolve under labeled chip-firing. Namely, we have the following.

**Proposition 2.2.3.** For \( \mathcal{C}, \mathcal{D} \in \mathbb{Z}^X \), if \( \mathcal{C} \xrightarrow{\Delta^N} \mathcal{D} \) then \( \mathcal{C}|_{\geq k} \xrightarrow{R} \mathcal{D}|_{\geq k} \).

*Proof.* Suppose we fire two chips \( \bigcirc \) and \( \bigcirc \) in \( \mathcal{C} \): if \( a, b < k \), that firing does not affect \( [\mathcal{C}|_{\geq k}] \); if \( k \leq a, b \), that firing corresponds to a firing in \( [\mathcal{C}|_{\geq k}] \); and if \( a < k \leq b \), then that firing corresponds to moving a chip rightward in \( [\mathcal{C}|_{\geq k}] \). \( \square \)

We now apply Lemma 2.1.6 to give a strengthening of Proposition 2.1.5 which applies to rightward-reachability.
Corollary 2.2.4. For $c, d \in \mathbb{N}[\mathbb{Z}]$, if $c \xrightarrow{R} d$ then $\tilde{c} \leq \tilde{d}$ and thus $\min(\tilde{c}) \leq \min(\tilde{d})$.

Proof. Suppose $c \xrightarrow{R} d$. Thus there is some sequence $c_0, c'_0, c_1, c'_1, \ldots, c_\ell, c'_\ell$ of configurations with $c = c_0$ and $c'_\ell = d$ such that:

- $c_i \xrightarrow{R} c'_i$ for all $0 \leq i \leq \ell$;

- $c'_{i-1} \leq c_i$ for all $1 \leq i \leq \ell$.

We claim that $\tilde{c} \leq \tilde{c}_i = \tilde{c}'_i$ for all $0 \leq i \leq \ell$. That $\tilde{c}_i = \tilde{c}'_i$ follows from $c_i \xrightarrow{R} c'_i$. So the crucial part of the claim is to show $\tilde{c} \leq \tilde{c}_i$. Clearly this holds for $i = 0$ since by definition $c_0 = c$. So assume $1 \leq i \leq \ell$ and $\tilde{c} \leq \tilde{c}_{i-1}$. Because $c'_{i-1} \leq c_i$, from Lemma 2.1.6 we get $\tilde{c}_{i-1} = \tilde{c}'_{i-1} \leq \tilde{c}_i$. Together with $\tilde{c} \leq \tilde{c}_{i-1}$ this implies $\tilde{c} \leq \tilde{c}_i$. So the claim is proved by induction. Taking $i = \ell$ in the claim gives $\tilde{c} \leq \tilde{c}'_\ell$, i.e. $\tilde{c} \leq \tilde{d}$. This implies $\min(\tilde{c}) \leq \min(\tilde{d})$. But $\min(\tilde{d}) \leq \min(d)$ by Proposition 2.1.5.

We can apply Corollary 2.2.4 in conjunction with Lemma 2.2.4 to restrict how far chips can move based on their labels when starting from $N$ chips at the origin.

Lemma 2.2.5. Suppose $\Delta^N \to \mathcal{C}$. Then $-\lfloor (N+1-k)/2 \rfloor \leq \mathcal{C}(k) \leq \lfloor k/2 \rfloor$ for all $1 \leq k \leq n$.

Proof. First we show $-\lfloor (N+1-k)/2 \rfloor \leq \mathcal{C}(k)$. By Proposition 2.2.3, $[\Delta^N|_{\geq k}] \xrightarrow{R}[\mathcal{C}|_{\geq k}]$. Thus by Lemma 2.2.4, $\min([\Delta^N|_{\geq k}]) \leq \min([\mathcal{C}|_{\geq k}])$. But $[\Delta^N|_{\geq k}] = (N+1-k)\delta_0$, and so Lemma 2.1.4 tells us that $\min([\Delta^N|_{\geq k}]) = -\lfloor (N+1-k)/2 \rfloor$. Thus indeed chip $\bigcirc^k$ must be at or to the left of the vertex $-\lfloor (N+1-k)/2 \rfloor$. That $\mathcal{C}(k) \leq \lfloor k/2 \rfloor$ then follows via Lemma 2.2.2.

We are now ready to prove the main theorem in this chapter, which says that when the number $n$ of chips is even, the labeled chip-firing process on $\mathbb{Z}$ necessarily sorts these chips.

Theorem 2.2.6. Suppose $N := 2m$ is even and $\Delta^N \to \mathcal{D}$ where $\mathcal{D}$ is stable. Then for all $1 \leq k \leq m$ we have that $\mathcal{D}(k) = -(m+1) + k$ and $\mathcal{D}(m+k) = k$. 47
Proof. Let $N = 2m$ be even and let $\Delta^N \rightarrow \mathcal{D}$ with $\mathcal{D}$ stable. For all $1 \leq k \leq m$, the assertion that $\mathcal{D}(m + k) = k$ follows from $\mathcal{D}(m + 1 - k) = -k$ by Lemma 2.2.2. Thus we prove only that $\mathcal{D}(k) = -(m + 1) + k$ for all $1 \leq k \leq m$.

The proof is by induction on $k$. So let us first address the base case $k = 1$. Lemma 2.2.5 says that $\mathcal{D}(i) > -m$ for all $2 \leq i \leq n$. (Here we use crucially the fact that $N = 2m$ is even.) But on the other hand, we know thanks to Lemma 2.1.4 that vertex $-m$ is occupied in $\mathcal{D}$. So in fact it must be occupied by chip $1$.

Now assume $k \geq 2$ and the result holds for all smaller values of $k$. We will use some internal lemmas in the proof (“internal” because they assume the inductive hypothesis).

**Lemma 2.2.7.** If $\mathcal{D}(k) > -(m + 1) + k$ then for all $1 \leq j \leq k - 1$, chip $\bigcirc$ never fired together with chip $\bullet$ in the labeled chip-firing process $\Delta^N \rightarrow \mathcal{D}$.

Proof. Suppose that $\mathcal{D}(k) > -(m + 1) + k$. And suppose to the contrary that chip $\bigcirc$ did fire together with chip $\bullet$ for some $1 \leq j \leq k - 1$ at some point in the labeled chip-firing process $\Delta^N \rightarrow \mathcal{D}$. Let us concentrate on the last moment when this happened: let $\mathcal{C}'$ be the step before chip $\bigcirc$ fired with chip $\bullet$ with $1 \leq j \leq k - 1$ for the last time (and thus define $j$ to be the label of this other chip). Let $\mathcal{C}$ be the result of firing $\bigcirc$ and $\bullet$ together in $\mathcal{C}'$. So $\Delta^N \rightarrow \mathcal{C}'$, $\mathcal{C}$ is obtained from $\mathcal{C}'$ by firing $\bigcirc$ and $\bullet$ together, and $\mathcal{D}$ is obtained from $\mathcal{C}$ by a sequence of firings that either do not involve $\bigcirc$, or fire $\bigcirc$ together with a chip with a greater label. This implies that $[\mathcal{C} \setminus \{k\}] \rightarrow \mathcal{D} \setminus \{k\}$ and Corollary 2.2.4 thus yields $[\mathcal{C} \setminus \{k\}] \leq [\mathcal{D} \setminus \{k\}]$. As a consequence of the assumptions that $\mathcal{D}(k) > -(m + 1) + k$ and that $k \leq m$, together with Lemma 2.1.4 we have $[-m, -(m + 1) + k] \subseteq \text{supp}(\mathcal{D} \setminus \{k\})$. Thus, since $\min([\mathcal{C} \setminus \{k\}]) \geq \min(\tilde{N} \delta_0) = -m$ and $[\mathcal{C} \setminus \{k\}]$ has at most one chip at each vertex, we have $[-m, -(m + 1) + k] \subseteq \text{supp}([\mathcal{C} \setminus \{k\}])$. Next, note that $[\mathcal{C} \setminus \{k\}] \preceq [\mathcal{C}' \setminus \{k\}]$. So by applying Lemma 2.1.6 we conclude that $[\mathcal{C} \setminus \{k\}] \preceq [\mathcal{C}' \setminus \{k\}]$, i.e., that $[\mathcal{C}' \setminus \{k\}]$ is obtained from $[\mathcal{C} \setminus \{k\}]$ by moving one chip rightward one vertex. In particular this means that we must have $[-m, -(m + 1) + k - 1] \subseteq \text{supp}([\mathcal{C}' \setminus \{k\}])$ (by the same reasoning as in the previous line about $\text{supp}([\mathcal{C} \setminus \{k\}])$. Now, chips $\bigcirc$ and $\bullet$
occupy the same vertex in \( C' \), which means \([C' \setminus \{k\}] = [C' \setminus \{j\}]\). So starting from \( C' \) and repeatedly firing all chips other than \( j \) until we stabilize these other chips, we can eventually reach some configuration \( D' \) with \([D' \setminus \{j\}] = [C' \setminus \{j\}] = [C' \setminus \{k\}]\).

The upshot of the previous paragraph is that if the lemma is false then we can find a configuration \( D' \) with \( \Delta^N \to D' \) and \([-m, -(m + 1) + k - 1] \subseteq \text{supp}([D' \setminus \{j\}]) \) for some \( 1 \leq j \leq k - 1 \). Let us show that this is impossible. For an unlabeled configuration \( c \) and \( \ell \in \mathbb{Z} \), recall the statistic \( \varphi_\ell(c) := \sum_{i \leq \ell} (i - \ell - 1) \cdot c(i) \) defined at the beginning of the last section. It follows from Proposition 2.1.1 that \( \varphi_\ell(c) \) weakly decreases with each chip-firing move, and so we always have \( \varphi_\ell(\tilde{c}) \leq \varphi_\ell(c) \); moreover, it follows from Proposition 2.1.1 that if \( \varphi_\ell(c) = \varphi_\ell(\tilde{c}) \) then vertex \( \ell + 1 \) never fires during the stabilization process \( c \to \tilde{c} \). Now, we claim that \( j \) is strictly to the right of vertex \( -(m + 1) + k - 1 \) in \( D' \): indeed, otherwise we would have

\[
\varphi_{-(m+1)+k-1}([D']) < \varphi_{-(m+1)+k-1}([\tilde{D}']) = \varphi_{-(m+1)+k-1}(\tilde{N} \delta_0)
\]

since \([-m, -(m + 1) + k - 1] \subseteq \text{supp}([D' \setminus \{j\}]\)). But if \( j \) is strictly to the right of vertex \( -(m + 1) + k - 1 \) in \( D' \) then \( \varphi_{-(m+1)+k-1}([D']) = \varphi_{-(m+1)+k-1}(\tilde{N} \delta_0) \). So if we continue to stabilize, that is, if we let \( D'' \) be such that \( D' \to D'' \) and \( D'' \) is stable, then the vertex \( -(m + 1) + k \) never fires during the labeled chip-firing process \( D' \to D'' \). Consequently, chip \( j \) always remains strictly to the right of \( -(m + 1) + k - 1 \) during the process \( D' \to D'' \). So chip \( j \) is strictly to the right of \( -(m + 1) + k - 1 \) in the stable configuration \( D'' \). But this contradicts our inductive hypothesis since we have \( 1 \leq j \leq k - 1 \).

**Lemma 2.2.8.** Set \( k' := k - 1 \). Then chip \( k \) must have fired together with chip \( k' \) at some point in the labeled chip-firing process \( \Delta^N \to D \).

**Proof.** Note that in the labeled chip-firing process, chips \( k \) and \( k' \) interact in the same way with all chips \( j \) for \( j \neq k, k - 1 \). So if chip \( k \) and chip \( k' \) never fire together in the labeled chip-firing process \( \Delta^N \to D \), we can swap the roles of \( k \) and \( k' \) to reach a stable configuration \( D' \) where \( k \) and \( k' \) have swapped places.
This contradicts our inductive hypothesis which says that there is only one vertex $k'$ could end up at in a stable configuration.

Lemmas 2.2.7 and 2.2.8 together imply that $D(k) \leq -(m+1)+k$. By our inductive hypothesis, we know that vertex $-(m+1)+j$ is occupied by $\square$ for all $1 \leq j \leq k-1$. Thus $D(k) = -(m+1) + k$. Therefore, the theorem is proved by induction.

\[\square\]

### 2.3 Extensions of labeled chip-firing

In this section we discuss some possible further directions for the study of labeled chip-firing, culminating with an example of a “Type B” variant of labeled chip-firing which will lead us to the next chapter in which we introduce a version chip-firing for arbitrary crystallographic root systems.

#### 2.3.1 Other configurations

For $C \in \mathbb{Z}^X$, we use the notation $\tilde{C} := \{D \in \mathbb{Z}^X : C \rightarrow D \text{ and } D \text{ is stable}\}$. A natural problem is to understand $\tilde{C}$ for other configurations $C \in \mathbb{Z}^X$ beyond $C = \Delta^{2m}$. For example, for $N = 1, 3, 5, 7, 9, \ldots$ we have $\#\tilde{\Delta}^N = 1, 3, 12, 54, 232, \ldots$ (a sequence which we submitted to the OEIS). We understand the even case, so let us concentrate on this odd case; thus set $N := 2m+1$. Since $|D| = \delta_{[-m,m]}$ for $D \in \tilde{\Delta}^N$, we may identify elements of $\tilde{\Delta}^N$ with permutations. Completely describing the permutations in $\tilde{\Delta}^N$ seems hard, but there are at least a few nontrivial things we can say. First of all, Lemma 2.2.5 applies equally when $N$ is odd and puts some restrictions on $\tilde{\Delta}^N$. We can also say the following: for any injective, order-preserving map $\iota : [N] \rightarrow [N+1]$, if we relabel a configuration $D \in \tilde{\Delta}^N$ according to $\iota$, add a new chip $\oplus$ to the origin where $\{j\} = [N+1] \setminus \text{im}(\iota)$, and then stabilize the resulting configuration, the chips have to appear in sorted order. Indeed, this is a consequence of our main theorem, Theorem 2.2.6, because one possible way to stabilize $\Delta^{N+1}$ is to ignore chip $\ominus$ for as long as possible and instead first stabilize the chips with labels in $\text{im}(\iota)$. Even these two conditions (Lemma 2.2.5 and the “add a chip and stabilize to sort”
condition) together fail to completely characterize \( \Delta^N \), however, because for instance the permutation 23154 satisfies both of these conditions but does not belong to \( \Delta^5 \).

In hopes of further understanding \( \Delta^N \), we offer the following attractive conjecture, which has been verified for \( N \leq 9 \) odd.

**Conjecture 2.3.1.** For \( N = 2m + 1 \), the maximum number of inversions among all permutations in \( \Delta^N \) is exactly \( m \).

Note in Conjecture 2.3.1 that a permutation with exactly \( m \) inversions can certainly be obtained: for instance, \( m \) inversions is obtained by any stabilization sequence where we never fire the chip \( 1 \) (so that it ends at the origin).

A different way to understand configurations for which there is no unique labeled stabilization would be probabilistically. There are several reasonable ways to carry out labeled chip-firing randomly. Here are three possible models: (1) at each step choose a chip-firing move uniformly at random among all possible moves; (2) at each step choose an unstable vertex uniformly at random and then choose a pair of chips at that vertex uniformly at random; or (3) choose a stabilization sequence uniformly at random among all (labeled) stabilization sequences. Based on some limited computer simulations, it appears that when \( m \) is large, random labeled chip-firing applied to \( \Delta^{2m+1} \) leads to all chips ending up sorted with probability around .33... under all three protocols. In fact, it is not hard to show that for any fixed \( m \), the probability that random firing will lead to sorting is at most \( 1/3 \); that is because the last move necessarily involves firing a vertex that has three chips on it, and so only one of the three possible labeled firings of these three chips will locally sort them. Let us define a penultimate configuration as one in which there is a vertex with three chips, with unoccupied vertices on either side, and with all other vertices containing at most one chip; call a penultimate configuration good if firing the lowest- and highest-labeled chips on the vertex with 3 chips causes all the chips to be sorted.

**Conjecture 2.3.2.** With respect to any of the three protocols for random labeled chip-firing described above, the probability that \( \Delta^{2m+1} \) sorts converges to \( 1/3 \) as \( m \rightarrow \infty \). Equivalently, the probability that the penultimate configuration is good converges to 1.
We call this the Odd Conjecture, and to get a sense of how odd it is, consider two variations on random labelled chip-firing: (A) every firing is chosen randomly, except the last firing, which is chosen purposefully (firing the lowest-labeled chip and the highest-labeled chip on the vertex with three chips); (B) every firing is chosen strategically, except the last firing, which is chosen randomly from among the three possibilities. It is easily seen that procedure (B) succeeds with probability at most $\frac{1}{3}$ for each finite $m$, yet our conjecture says that procedure (A) succeeds with probability approaching 1 as $m$ gets large. It seems that, in a certain sense, only the last move matters!

2.3.2 Other graphs

An obvious question is if the labeled chip-firing process can be extended to other graphs beyond $\mathbb{Z}$. Ideally any such extension would have unique labeled stabilizations for many of its initial configurations. While we are far from being able to propose an interesting extension of labeled chip-firing to arbitrary graphs, we have found that several minor variants of the infinite path (apparently) continue to exhibit confluence of certain initial configurations.

The graphs we consider here all have $\mathbb{Z}$ as their set of vertices. Hence we can consider the same configurations on these graphs (i.e., we will still talk about the configuration $\Delta^N$ of $N$ labeled chips at the origin). The edge structure of these graphs is slightly different from $\mathbb{Z}$: at any vertex $v$, there are $\ell_v$ directed edges from $v$ to the vertex immediately to its left, $m_v$ directed loops at $v$, and $r_v$ directed edges from $v$ to the vertex immediately to its right. That is, each vertex looks like:

Firing at $v$ consists of choosing $\ell_v + m_v + r_v$ distinct chips at $v$ and moving the $\ell_v$ lowest-labeled chips among these leftward one vertex, moving the $r_v$ highest-labeled chips rightward one vertex, and leaving the $m_v$ middle-labeled chips at $v$. 

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A modification of the argument from the previous section gives the following:

**Theorem 2.3.3.** The following variant of the infinite path, with \( m \) loops at the origin and no other modifications, sorts \( \Delta^N \) when \( N \equiv m \mod 2 \):

![Diagram of infinite path with \( m \) loops at the origin]

See [43] for a detailed proof of Theorem 2.3.3. We can also formulate some conjectures about other such variants, which we have been unable to prove:

**Conjecture 2.3.4.** The following variant of the infinite path, with one loop at every vertex, sorts \( \Delta^N \) when \( N \equiv 3 \mod 4 \):

![Diagram of infinite path with one loop at every vertex]

**Conjecture 2.3.5.** The following variant of the infinite path, with each edge replaced by \( r \) parallel edges, sorts \( \Delta^N \) when \( N \equiv 0 \mod 2r \):

![Diagram of infinite path with \( r \) parallel edges]

Looking at other modifications of the infinite path which involve parallel edges or loops, one experimentally observes similar confluence behavior for \( \Delta^N \) that depends on congruence conditions.

### 2.3.3 Other... types?

We now describe a “Type B” variant of labeled chip-firing on the infinite path \( \mathbb{Z} \). The states of this Type B process are the same configurations of labeled chips on \( \mathbb{Z} \), but the process has an expanded set of “firing moves”; namely, we are allowed to do any of the following kinds of moves to a configuration of labeled chips:

1. If \( \bigcirc \) and \( \bigcirc \) with \( a < b \) are both at vertex \( i \in \mathbb{Z} \), move \( \bigcirc \) leftward one vertex and \( \bigcirc \) rightward one vertex (this is the usual labeled chip-firing move);
(II) if \( a \) is at vertex \( i \in \mathbb{Z} \) and chip \( b \) is at vertex \(-i\), move both \( a \) and \( b \) rightward one vertex;

(III) if \( a \) is at the origin, move \( a \) rightward one vertex.

Observe how now the indices of the vertices \( i \in \mathbb{Z} \) now matter.

We will show that this system also exhibits confluence. Let’s consider applying a series of the Type B moves of the kinds (II)-(III) above to the initial configuration of 3 labeled chips at the origin:

Suppose we first carry out a move of kind (II); we fire 2 and 3 to arrive at:

Now, because 2 is at vertex \(-1\) while chip 3 is at vertex 1, we can carry out a move of kind (II), which moves both 2 and 3 to the right:

Now we do another move of kind (I), firing 1 and 2 to arrive at:
Another move of kind (II) yields:

Now, since \(1\) is at the origin, we can apply a move of kind (III) to it to get:

By firing \(2\) and \(3\) together, and then \(1\) and \(2\) together, with two moves of kind (I) we reach:

Finally, one more move of kind (III) applied to \(1\) brings us to the following stable configuration:

Observe how we ended with all the chips in sorted order.

This was no coincidence; in fact, the Type B process sorts any number of labeled chips at the origin:

**Theorem 2.3.6.** For all \(N \in \mathbb{N}\), starting from \(\Delta^N\) and carrying out moves of the kinds (I), (II), and (III) above in any order for as long as one can necessarily brings the chips into the sorted configuration with \(i\) at vertex \(i\) for all \(i \in [N]\).

**Proof.** In fact this follows from Theorem 2.2.6 via a simple “symmetry” argument which explains how we can embed the Type B process into the usual labeled chip-firing process. To realize this embedding, we start by placing \(2N\) chips at the origin:
chips $1, 2, \ldots, N$ with positive labels, as well as chips $-1, -2, \ldots, -N$ with negative labels. This looks like the following:

Now suppose that we carry out the usual labeled chip-firing moves to this configuration, subject to the following symmetry requirement: whenever we fire chips $a$ and $b$ with $a \neq -b$, we then immediately also fire $-a$ and $-b$. It is easy to see that doing labeled chip-firing moves in this way preserves the property that the configuration is invariant with respect to the operation of reflecting the configuration horizontally about vertex 0 and negating the labels of the chips. Moreover, we claim that the movement of the positively labeled chips exactly captures the Type B moves of kinds (I), (II), and (III). For example, a move of kind (I) corresponds to firing chips $a$ and $b$, and then firing chips $-a$ and $-b$, with $0 \leq a, b$:

Meanwhile, a move of kind (III) corresponds to firing chips $a$ and $-b$, and then firing chips $-a$ and $b$, with $0 \leq a, b$:  

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Finally, a move of kind (III) corresponds to firing $a$ and $-a$ (which can only happen at vertex 0 because of the symmetry property of these configurations):

$$\begin{array}{c}
\bullet a \\
\downarrow \\
\bullet -a
\end{array}$$

Since we started with an even number $2N$ of chips, thanks to Theorem 2.2.6 we know that the labeled chip-firing process sorts all the chips. Hence the Type B process indeed sorts the positively labeled chips.

In order to understand why we called this variant of labeled chip-firing a “Type B” variant, and in order to understand “where” this variant is coming from, in the next chapter we introduce root systems and generalize the labeled chip-firing process to any crystallographic root system.
Chapter 3

Central root-firing

In this chapter we generalize the labeled chip-firing process from Chapter 2 to any crystallographic root system. The material in this chapter is joint work with Pavel Galashin, Thomas McConville, and Alexander Poshtikov and appears in [37].

The fundamental observation that allows us to relate labeled chip-firing and root systems, which was first made by Pavel Galashin, is that we can reformulate the labeled chip-firing process on the infinite path as a certain process on the integer lattice $\mathbb{Z}^N$ as follows. Represent a configuration $\mathcal{C}$ of $N$ labeled chips on $\mathbb{Z}$ by the integer vector $v = (C(1), C(2), \ldots, C(N)) \in \mathbb{Z}^N$, where $C(i)$ is the position of chip $i$. Then for $1 \leq i < j \leq N$, we can fire $i$ and $j$ together in $\mathcal{C}$ if and only if $v$ is orthogonal to the vector $e_j - e_i$ (where $e_i \in \mathbb{Z}^N$, $i \in [N]$ are the standard basis vectors of $\mathbb{R}^N$). Furthermore, firing chips $i$ and $j$ corresponds to replacing the vector $v$ by $v + (e_j - e_i)$. But observe that the vectors $e_j - e_i$ for $1 \leq i < j \leq N$ are (one choice of) positive roots for the root system of Type $A_{N-1}$. Thus this reformulation very naturally suggests a possibly way to extend to labeled chip-firing to any root system: for $\Phi$ a root system in a vector space $V$, consider the process $v \rightarrow v + \alpha$ whenever the vector $v \in V$ is orthogonal to $\alpha \in \Phi^+$, where $\Phi^+$ is the set of positive root of $\Phi$.

Before going into formal definitions, let us take a minute to give a very broad overview of the study of root systems. Root systems are certain highly symmetrical, finite collections of vectors in a Euclidean vector space. They were first introduced in the context of Lie theory because there is a bijective correspondence between (ir-
reducible, crystallographic) root systems and finite dimensional simple Lie algebras over the complex numbers (see e.g. [46]). The fundamental work of the Lie theorists Wilhelm Killing and Élie Cartan in the late 19th century lead to the complete classification of root systems into what are now called the Cartan-Killing types, which include the classical infinite families of Types A, B, C, and D, as well as a number of exceptional types. The classification of simple Lie algebras is the prototype for all further classification results in algebra (such as the classification of finite simple groups). Dynkin diagrams, certain decorated graphs associated to root systems, are combinatorial gadgets which allows one to easily write down this classification. Remarkably, these same Dynkin diagrams appear in many parts of mathematics not obviously connected to discrete geometry or Lie theory: for instance, they have fundamental connections to singularity theory (see [6]), to the theory of quiver representations (see [33]), and to the finite subgroups of SU(2) (see [54]).

From a combinatorial point of view, the study of root systems clarifies and generalizes much of the classical combinatorics of permutations. This is because the Weyl group, i.e., the group generated by the reflections orthogonal to the elements of a root system, is the symmetric group in the case that our root system is of Type A. A major trend in modern combinatorics is to understand the combinatorics of root systems, ideally in a uniform way which does not invoke the classification of root systems. This program of understanding root system combinatorics uniformly sometimes goes under the moniker “Coxeter-Catalan combinatorics” [5].

In the next section section in this chapter we review the basic theory of root systems. We then introduce the root system generalization of labeled chip-firing described above. We call this process the central root-firing process for \( \Phi \) because it allows us to perform a firing move whenever our vector \( v \in V \) belongs to a certain central hyperplane arrangement: the Coxeter arrangement of \( \Phi \). (In later chapters of this thesis we will study “deformations” of central root-firing which allow firing under other conditions corresponding to affine arrangements.) We explain how in classical types this central-firing process can be interpreted as a “chip-firing process” on \( \mathbb{Z} \) with a certain expanded set of chip-firing moves. For example, we already saw at the end
Most of what we can be said about central-firing for arbitrary root systems remains conjectural at the moment. But there is one property of labeled chip-firing which does generalize directly to central-firing. Namely, observe that modding out by the action of the symmetric group $S_N$ on the set of configurations of $N$ labeled chips on $\mathbb{Z}$ corresponds to forgetting the labels of the chips. And after modding out in this way, the labeled chip-firing process reduces to the usual unlabeled (classical) chip-firing process, which is confluent from all initial configurations. The same is true for central-firing: after modding out by the action of the Weyl group, the central-firing process becomes confluent from all initial points. Moreover, for the simply laced types we can describe this unlabeled chip-firing process as a certain numbers game on the Dynkin diagrams of the root system.

As mentioned, a reason to generalize any combinatorial object from Type A to “arbitrary type” is that it can clarify the properties of the original Type A object. We saw in Chapter 2 that labeled chip-firing was confluent from $N$ labeled chips at the origin if and only if $N$ was even. This configuration corresponds to the origin of the vector space $V$. In other classical types, the confluence of central-firing from the origin can (apparently) be described in terms of congruence conditions: e.g., in Type $B_n$, we have confluence from the origin for all values of $n$; whereas in Type $C_n$ we conjecture confluence from the origin when $n \equiv 1, 2 \mod 4$. We see that confluence for central-firing is, like confluence for labeled chip-firing, a quite subtle question. In the last section in this chapter we put forward a complete conjectural classification of when central-firing is confluent starting from either zero or a fundamental weight (these are certain vectors which correspond to the vertices of the Dynkin diagram). To first order, confluence seems to have to do with whether the initial point is equivalent to the Weyl vector modulo the root lattice. So while generalizing to arbitrary type has not fully “explained” why confluence happens when it does for the original labeled chip-firing process, some discernible root theoretic patterns have become apparent.
3.1 Background on root systems

Here we review the basic facts about root systems we will need to define and study central-firing. We do not include proofs of the basic facts presented here. For details, consult [46], [21], or [15].

Fix $V$, an $n$-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$. For a nonzero vector $\alpha \in V \setminus \{0\}$ we define its covector to be $\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. Then we define the reflection across the hyperplane orthogonal to $\alpha$ to be the linear map $s_\alpha : V \to V$ given by $s_\alpha(v) := v - \langle v, \alpha^\vee \rangle \alpha$.

Definition 3.1.1. A root system is a finite collection $\Phi \subseteq V \setminus \{0\}$ of nonzero vectors such that:

1. $\text{Span}_R(\Phi) = V$;

2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$;

3. $\text{Span}_R(\{\alpha\}) \cap \Phi = \{\pm \alpha\}$ for all $\alpha \in \Phi$;

4. $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The key condition here is the second condition: reflecting across the hyperplane orthogonal to any element of $\Phi$ permutes the elements of $\Phi$. The first condition is really not a restriction at all because we can always project to $\text{Span}_R(V)$. Sometimes the third condition is omitted and those root systems satisfying the third condition are called reduced. On the other hand, sometimes the fourth condition is omitted and those root systems satisfying the fourth condition are called crystallographic. We will assume that all root systems under consideration are reduced and crystallographic and from now on will drop these adjectives.

From now on in this chapter we will fix a root system $\Phi$ in $V$. The vectors $\alpha \in \Phi$ are called roots. The dimension of $V$ (which is $n$) is called the rank of the root system. Figure 3-1 depicts all the root systems of rank 2.

The vectors $\alpha^\vee$ for $\alpha \in \Phi$ are called coroots and the set of coroots forms another root system, denoted $\Phi^\vee$, in $V$. 

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Figure 3-1: The root systems of rank 2: $A_1 \times A_1$, $A_2$, $B_2$, and $G_2$. The roots are the black vectors. The elements of $\Omega \cup \{0\}$ are shown as points in red.
We use $W$ to denote the Weyl group of $\Phi$, which is the subgroup of $GL(V)$ generated by the reflections $s_\alpha$ for $\alpha \in \Phi$. By the first and second conditions of the definition of a root system, $W$ is isomorphic as an abstract group to a subgroup of the symmetric group on $\Phi$, and hence is finite. Observe that the Weyl group of $\Phi^\vee$ is equal to the Weyl group of $\Phi$. Also note that all transformations in $W$ are orthogonal.

It is well-known that we can choose a set $\Delta \subseteq \Phi$ of simple roots which form a basis of $V$, and which divide the root system $\Phi = \Phi^+ \cup \Phi^-$ into positive roots $\Phi^+$ and negative roots $\Phi^- := -\Phi^+$ so that any positive root $\alpha \in \Phi^+$ is a nonnegative integer combination of simple roots. The choice of $\Delta$ is equivalent to the choice of $\Phi^+$; one way to choose $\Phi^+$ is to choose a generic linear form and let $\Phi^+$ be the set of roots which are positive according to this form. There are many choices for $\Delta$ but they are all conjugate under $W$. From now on we will fix a set of simple roots $\Delta$, and thus also a set of positive roots $\Phi^+$. It is known that any $\alpha \in \Phi$ appears in some choice of simple roots (in fact, every $\alpha \in \Phi$ is $W$-conjugate to a simple root appearing with nonzero coefficient in its expansion in terms of simple roots) and hence $W(\Delta) = \Phi$.

We use $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ to denote the simple roots with an arbitrary but fixed order. The coroots $\alpha_i^\vee$ for $i = 1, \ldots, n$ are called the simple coroots and they of course form a set of simple roots for $\Phi^\vee$. We will always make this choice of simple roots for the dual root system, unless stated otherwise. With this choice of simple roots for the dual root system, we have $(\Phi^\vee)^+ = (\Phi^+)^\vee$.

We use $C := (\langle \alpha_i, \alpha_j^\vee \rangle) \in \mathbb{Z}^{n \times n}$ to denote the Cartan matrix of $\Phi$. Clearly one can recover the root system $\Phi$ from the Cartan matrix $C$, which is encoded by its Dynkin diagram. The Dynkin diagram of $\Phi$ is the graph with vertex set $[n] := \{1, 2, \ldots, n\}$ obtained as follows: first for all $1 \leq i < j \leq n$ we draw $\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$ edges between $i$ and $j$; then, if $\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle \not\in \{0, 1\}$ for some $i$ and $j$, we draw an arrow on top of the edges between them, from $i$ to $j$ if $|\alpha_i| > |\alpha_j|$. If there are no arrows in the Dynkin diagram of $\Phi$ then we say that $\Phi$ is simply laced.

There are two important lattices related to $\Phi$, the root lattice $Q := \text{Span}_\mathbb{Z}(\Phi)$ and the weight lattice $P := \{v \in V : \langle v, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. The elements of $P$ are called the weights of $\Phi$. By the assumption that $\Phi$ is crystallographic, we have $Q \subseteq P$. 

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We use $\Omega := \{\omega_1, \ldots, \omega_n\}$ to denote the dual basis to the basis of simple coroots $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ (in other words, the $\omega_i$ are defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$); the elements of $\Omega$ are called \textit{fundamental weights}. Observe that $Q = \text{Span}_\mathbb{Z}(\Delta)$ and that $P = \text{Span}_\mathbb{Z}(\Omega)$.

We use $P_{\geq 0}^\mathbb{R} := \text{Span}_{\mathbb{R}_{\geq 0}}(\Omega)$, $P_{\geq 0} := \text{Span}_\mathbb{N}(\Omega)$ and similarly $Q_{\geq 0}^\mathbb{R} := \text{Span}_{\mathbb{R}_{\geq 0}}(\Delta)$, $Q_{\geq 0} := \text{Span}_\mathbb{N}(\Delta)$. Note that $P_{\geq 0}^\mathbb{R}$ and $Q_{\geq 0}^\mathbb{R}$ are dual cones; moreover, because the simple roots are pairwise non-acute, we have $P_{\geq 0}^\mathbb{R} \subseteq Q_{\geq 0}^\mathbb{R}$. The elements of $P_{\geq 0}$ are called \textit{dominant weights}. For every $\lambda \in P$ there exists a unique element in $W(\lambda) \cap P_{\geq 0}$ and we use $\lambda_{\text{dom}}$ to denote this element. We say $\lambda \in P_{\geq 0}$ is \textit{strictly dominant} if $\langle \lambda, \alpha_i^\vee \rangle > 0$ for all $i \in [n]$. A strictly dominant weight of great importance is the \textit{Weyl vector} $\rho := \sum_{i=1}^n \omega_i$. It is well-known (and easy to check) that $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

The connected components of $\{v \in V : \langle v, \alpha^\vee \rangle \neq 0 \text{ for all } \alpha \in \Phi\}$ are called the \textit{chambers} of $\Phi$. We use $C_0 := \{v \in V : \langle v, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Phi^+\}$ to denote the \textit{fundamental chamber}. The Weyl group acts freely and transitively on the chambers and hence every chamber is equal to $wC_0$ for some unique $w \in W$. Observe that $P_{\geq 0}^\mathbb{R}$ is the closure of $C_0$.

If $U \subseteq V$ is any subspace spanned by roots, then $\Phi \cap U$ is a root system in $U$, which we call a \textit{sub-root system} of $\Phi$. The root lattice of $\Phi \cap U$ is of course $\text{Span}_\mathbb{Z}(\Phi \cap U)$ while the weight lattice is $\pi_U(P)$. Moreover, $\Phi^+ \cap U$ is a set of positive roots for $\Phi \cap U$, although $\Delta \cap U$ may not be a set of simple roots for $\Phi \cap U$. We will always consider the positive roots of $\Phi \cap U$ to be $\Phi^+ \cap U$ unless explicitly stated otherwise. The case of \textit{parabolic sub-root systems} (where in fact $\Delta \cap U$ is a set of simple roots for $\Phi \cap U$) is of special significance: for $I \subseteq [n]$ we set $\Phi_I := \Phi \cap \text{Span}_\mathbb{R}(\{\alpha_i : i \in I\})$.

If there exists an orthogonal decomposition $V = V_1 \oplus V_2$ with $\emptyset \subsetneq V_1, V_2 \subsetneq V$ such that $\Phi = \Phi_1 \cup \Phi_2$ with $\Phi_i \subseteq V_i$ for $i = 1, 2$, then we write $\Phi = \Phi_1 \times \Phi_2$ and say the root system $\Phi$ is \textit{reducible}. Otherwise we say that it is \textit{irreducible}. (By fiat let us also declare that the empty set, although it is a root system, is not an irreducible root system.) In other words, a root system is irreducible if and only if its Dynkin diagram is connected. The famous \textit{Cartan-Killing classification} classifies all irreducible root systems up to isomorphism, where an \textit{isomorphism} of root systems is a bijection between roots induced from an invertible orthogonal map, potentially
composed with a global rescaling of the inner product. Figure 3-2 shows the Dynkin diagrams of all the irreducible root systems: these are the classical infinite series $A_n$ for $n \geq 1$, $B_n$ for $n \geq 2$, $C_n$ for $n \geq 3$, $D_n$ for $n \geq 4$, together with the exceptional root systems $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$. Our numbering of the simple roots is consistent with Bourbaki [21]. In every case the subscript in the name of the root system denotes the number of vertices of the Dynkin diagram, which is also the number of simple roots, that is, the rank of $\Phi$. These labels $A_n$, $B_n$, $G_2$, etc. are called the type of the root system, and we may also talk about, e.g., “Type A” root systems.

All constructions that depend on the root system $\Phi$ decompose in a simple way as a direct product of irreducible factors. Hence without loss of generality we will from now on **assume that $\Phi$ is irreducible**.

In an irreducible root system, there are at most two values of lengths $|\alpha|$ among the roots $\alpha \in \Phi$. Those roots whose lengths achieve the maximum value are called long, and those which do not are called short. The Weyl group $W$ acts transitively on the long roots, and it also acts transitively on the short roots. If $\Phi$ is simply laced, then all roots are long.
There is a natural partial order on \( P \) called the root order which has \( \mu \leq \lambda \) for \( \mu, \lambda \in P \) if \( \lambda - \mu \in Q_{\geq 0} \). When restricted to \( \Phi^+ \), this partial order is graded by height; the height of \( \alpha = \sum_{i=1}^{n} c_i \alpha_i \in \Phi \) is \( \sum_{i=1}^{n} c_i \). Because we have assumed that \( \Phi \) is irreducible, there is a unique maximal element of \( \Phi^+ \) according to root order, denoted \( \theta \) and called the highest root. The highest root is always long. We use \( \hat{\theta} \) to denote the unique (positive) root such that \( \hat{\theta}^\vee \) is the highest root of the dual root system \( \Phi^\vee \) (with respect to the choice of \( \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \) as simple roots). If \( \Phi \) is simply laced then \( \theta = \hat{\theta} \) and \( \theta \) is the unique root which is a dominant weight; if \( \Phi \) is not simply laced then \( \theta \) and \( \hat{\theta} \) are the two roots which are dominant weights. In the non-simply laced case we call \( \hat{\theta} \) the highest short root: it is the maximal short root with respect to the root ordering.

The root lattice \( Q \) is a full rank sublattice of \( P \); hence the quotient \( P/Q \) is some finite abelian group. Note that \( P/Q \cong \text{coker}(C^t) \) where we view the transposed matrix as a map \( C^t : \mathbb{Z}^n \to \mathbb{Z}^n \). (This group \( P/Q \) can be thought of as the “sandpile group” in our setting.) The order of this group is called the index of connection of \( \Phi \) and is denoted \( f := |P/Q| \). There is a nice choice of coset representatives of \( P/Q \), which we now describe. A dominant, nonzero weight \( \lambda \in P_{\geq 0} \setminus \{0\} \) is called minuscule if \( \langle \lambda, \alpha^\vee \rangle \in \{-1, 0, 1\} \) for all \( \alpha \in \Phi \). Let us use \( \Omega_m \) to denote the set of minuscule weights. Note that, since \( \Phi \) is irreducible and hence posses a highest root \( \theta \), we have \( \Omega_m \subseteq \Omega \), i.e., a minuscule weight must be a fundamental weight. In Figure 3-2, the vertices corresponding to minuscule weights are filled in. In fact, there are \( f - 1 \) minuscule weights and the minuscule weights together with zero form a collection of coset representatives of \( P/Q \). We use \( \Omega^0_m := \Omega_m \cup \{0\} \) to denote the set of these representatives.

There is another characterization of minuscule weights that we will find useful. Namely, for a dominant weight \( \lambda \in P_{\geq 0} \) we have that \( \lambda \in \Omega^0_m \) if and only if \( \lambda \) is the minimal element according to root order in \( (Q + \lambda) \cap P_{\geq 0} \).

This last characterization of minuscule weight can also be described in terms of certain polytopes called \((W)\)-permutohedra. Permutohedra will play a key role for us in our understanding of root-firing processes, so let us review these now. For \( v \in V \),
we define the *permutohedron* associated to \( v \) to be \( \Pi(v) := \text{ConvexHull} \ W(v) \), a convex polytope in \( V \). And for a weight \( \lambda \in P \), we define \( \Pi^Q(\lambda) := \Pi(\lambda) \cap (Q + \lambda) \), which we call the *discrete permutohedron* associated to \( \lambda \).

The following simple proposition describes the containment of permutohedra:

**Proposition 3.1.2.** For \( u, v \in P_{\geq 0} \) we have \( \Pi(u) \subseteq \Pi(v) \) if and only if \( v - u \in Q_{\geq 0} \). Hence for \( \mu, \lambda \in P_{\geq 0} \) we have \( \Pi^Q(\mu) \subseteq \Pi^Q(\lambda) \) if and only if \( \mu \leq \lambda \) (in root order).

**Proof.** First suppose that \( u \) and \( v \) are strictly inside the fundamental chamber \( C_0 \), i.e., that we have \( \langle u, \alpha_i^\vee \rangle > 0 \) and \( \langle v, \alpha_i^\vee \rangle > 0 \) for all \( i \in [n] \). By the *inner cone* of polytope at a vertex, we mean the affine convex cone spanned by the edges of the polytope incident to that vertex in the direction “outward” from that vertex. Note that a point belongs to a polytope if and only if it belongs to the inner cone of that polytope at every vertex. Since the walls of the fundamental chamber are orthogonal to the simple roots, it is easy to see that if \( u \) and \( v \) are strictly inside the fundamental chamber then the inner cone of \( \Pi(u) \) at \( u \) is spanned by the negatives of the simple roots, and ditto for the inner cone of \( \Pi(v) \) and \( v \). So if we do not have \( v - u \in Q_{\geq 0} \), then clearly \( u \) does not belong to \( \Pi(v) \). Hence suppose that \( v - u \in Q_{\geq 0} \). Every vertex of \( \Pi(u) \) belongs to the inner cone of \( \Pi(u) \) at \( u \); i.e., \( u - u' \in Q_{\geq 0} \) for all \( u' \in W(u) \). Thus for all \( u' \in W(u) \) we have \( v - u' \in Q_{\geq 0} \); i.e., every point in \( \Pi(u) \) is in the inner cone of \( \Pi(v) \) at \( v \). But then by the \( W \)-invariance of permutohedra, we conclude that every point in \( \Pi(u) \) is in the inner cone of \( \Pi(v) \) at every vertex of \( \Pi(v) \), and hence that \( \Pi(u) \subseteq \Pi(v) \), as claimed.

For arbitrary \( u, v \in P_{\geq 0} \), note \( \Pi(u) = \bigcap_{\varepsilon > 0} \Pi(u + \varepsilon \rho) \) and \( \Pi(v) = \bigcap_{\varepsilon > 0} \Pi(v + \varepsilon \rho) \), and \( u + \varepsilon \rho \) and \( v + \varepsilon \rho \) will be strictly inside the fundamental chamber for all \( \varepsilon > 0 \). Thus the result for arbitrary \( u, v \in P_{\geq 0} \) follows from the preceding paragraph. \( \square \)

So in light of Proposition 3.1.2, we see that minuscule weights can also be characterized as follows: for a dominant weight \( \lambda \in P_{\geq 0} \) we have that \( \lambda \in \Omega_0^m \) if and only if \( \Pi^Q(\lambda) = W(\lambda) \). For references for all these various characterizations of and facts about minuscule weights, see [13, Proposition 3.10] (who in particular credit Stembridge [74] for some of these facts).
3.2 Definition of central-firing

We are now ready to define the central root-firing process for \( \Phi \): the *central root-firing process*, or just *central-firing process* for short, is the binary relation \( \rightarrow_{\Phi^+} \) defined on the weight lattice \( P \) by

\[
\lambda \rightarrow_{\Phi^+} \lambda + \alpha, \text{ for } \lambda \in P \text{ and } \alpha \in \Phi^+ \text{ with } \lambda \text{ orthogonal to } \alpha.
\]

Observe that we define central-firing as a relation only on the weight lattice \( P \) and not all of \( V \): this is because, generalizing the discrete nature of chip-firing, we want our system to be discrete.

First of all, let us show that \( \rightarrow_{\Phi^+} \) is always terminating.

**Proposition 3.2.1.** The relation \( \rightarrow_{\Phi^+} \) is terminating.

**Proof.** For \( \lambda \in P \) define \( \varphi(\lambda) := \langle \rho - \lambda, \rho - \lambda \rangle \); in other words, \( \varphi(\lambda) \) is the length of the vector \( \rho - \lambda \). Suppose \( \lambda \rightarrow_{\Phi^+} \lambda + \alpha \) for \( \alpha \in \Phi^+ \). Then,

\[
\varphi(\lambda) - \varphi(\lambda + \alpha) = \langle \rho - \lambda, \rho - \lambda \rangle - \langle \rho - (\lambda + \alpha), \rho - (\lambda + \alpha) \rangle
\]

\[
= 2\langle \rho, \alpha \rangle - 2\langle \lambda, \alpha \rangle = 2\langle \rho, \alpha \rangle = \frac{4}{\langle \alpha, \alpha \rangle},
\]

where we use the fact that \( \langle \lambda, \alpha \rangle = 0 \) since \( \lambda \rightarrow_{\Phi^+} \lambda + \alpha \) means that \( \lambda \) is orthogonal to \( \alpha \). So each firing move causes the quantity \( \varphi(\lambda) \) to decrease by at least some fixed nonzero amount. But we have \( \varphi(\lambda) \geq 0 \) because it is measuring the length of a vector.

Thus indeed \( \rightarrow_{\Phi^+} \) is terminating. \( \square \)

Now let us explain how we can view central-firing as a variant of labeled chip-firing on the infinite path for all the classical types. For this we will want to use a particular realization of each of the root systems of Types A, B, C, and D. Here are the realizations we use\(^1\)

\(^1\)We use slightly nonstandard choices of positive roots which has \( e_i - e_j \) for \( i < j \) being positive. This is just to match the earlier description of labeled chip-firing, which sends lesser-labeled chips to the left. The standard realization would have \( e_i - e_j \) positive; this can be obtained from our choice by relabeling the coordinates in the obvious way.
• For $\Phi = A_n$, we take $V := \mathbb{R}^{n+1}/(1, 1, \ldots, 1)$, the quotient of $\mathbb{R}^{n+1}$ by the “all ones” vector, which can be identified with the hyperplane in $\mathbb{R}^{n+1}$ of vectors with coordinate sum zero. The elements of $\Phi$ are $e_j - e_i$ for all $i \neq j \in [n+1]$. The weight lattice is $\mathbb{Z}^{n+1}/(1, 1, \ldots, 1)$. The positive roots of $\Phi$ are $e_j - e_i$ for $i < j \in [n+1]$. The simple roots are $\alpha_i := e_{(n+2)\cdot i} - e_{(n+1)\cdot i}$ for $i \in [n]$.

• For $\Phi = B_n$, we take $V := \mathbb{R}^n$. The elements of $\Phi$ are $\pm e_j \pm e_i$ for all $i < j \in [n]$ and $\pm e_i$ for all $i \in [n]$. The weight lattice is $\mathbb{Z}^n \cup (\mathbb{Z}^n + (\frac{1}{2}, \ldots, \frac{1}{2}))$. The positive roots of $\Phi$ are $e_j \pm e_i$ for $i < j \in [n]$ and $e_i$ for $i \in [n]$. The simple roots are $\alpha_i := e_{(n+1)\cdot i} - e_{n\cdot i}$ for $i \in [n-1]$ and $\alpha_n := e_1$.

• For $\Phi = C_n$, we take $V := \mathbb{R}^n$. The elements of $\Phi$ are $\pm e_j \pm e_i$ for all $i < j \in [n]$ and $\pm 2e_i$ for all $i \in [n]$. The weight lattice is $\mathbb{Z}^n$. The positive roots of $\Phi$ are $e_j \pm e_i$ for $i < j \in [n]$ and $2e_i$ for $i \in [n]$. The simple roots are $\alpha_i := e_{(n+1)\cdot i} - e_{n\cdot i}$ for $i \in [n-1]$ and $\alpha_n := 2e_1$.

• For $\Phi = D_n$, we take $V := \mathbb{R}^n$. The elements of $\Phi$ are $\pm e_j \pm e_i$ for all $i < j \in [n]$. The weight lattice is $\mathbb{Z}^n \cup (\mathbb{Z}^n + (\frac{1}{2}, \ldots, \frac{1}{2}))$. The positive roots of $\Phi$ are $e_j \pm e_i$ for $i < j \in [n]$. The simple roots are $\alpha_i := e_{(n+1)\cdot i} - e_{n\cdot i}$ for $i \in [n-1]$ and $\alpha_n := e_1 + e_2$.

We want to view weights of the classical types as configurations of labeled chips on $\mathbb{Z}$. For a weight $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ we know how to do this: we view $v$ as the configuration $C_v$ of $n$ labeled chips which has chip $\bigcirc$ at vertex $v_i$ for all $i \in [n]$. We see from the above that sometimes we need to consider weights in $\mathbb{Z}^n + (\frac{1}{2}, \ldots, \frac{1}{2})$. A weight $v \in \mathbb{Z}^n + (\frac{1}{2}, \ldots, \frac{1}{2})$ corresponds to a configuration $C_v$ of $n$ labeled chips on the graph $G = \mathbb{Z} + \frac{1}{2}$ with vertices $V = \mathbb{Z} + \frac{1}{2}$ and edges $E = \{(i + \frac{1}{2}, i + \frac{3}{2}) : i \in \mathbb{Z}\}$. Finally, we need to be a little careful for $\Phi = A_n$ because its weight lattice is $\mathbb{Z}^{n+1}/(1, 1, \ldots, 1)$. But in fact we can view a Type $A_n$ weight $v \in \mathbb{Z}^{n+1}/(1, 1, \ldots, 1)$ as configuration of labeled chips by first taking any preimage $\tilde{v} \in \mathbb{Z}^{n+1}$ of $v$ under the quotient map and then setting $C_v := C_{\tilde{v}}$, a configuration of $n+1$ labeled chips on $\mathbb{Z}$. Choosing a different preimage amounts to shifting all of the chips over by some constant amount.
clearly this does not affect the behavior of the usual (i.e., Type A) labeled chip-firing process.

Now we can define the chip-firing moves which describe central-firing in all the classical types.

**Definition 3.2.2.** For a configuration $\mathcal{C}$ of labeled chips on $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, we define the following four kinds of moves on $\mathcal{C}$:

(a) for $i < j$, if chips $\bigcirc_i$ and $\bigcirc_j$ occupy the same vertex, move $\bigcirc_i$ one vertex leftward and $\bigcirc_j$ one vertex rightward;

(b) if chip $\bigcirc_i$ is at vertex 0, move it one vertex rightward (i.e., to vertex 1);

(c) if chip $\bigcirc_i$ is at vertex 0, move it two vertices rightward (i.e., to vertex 2);

(d) for $i < j$ and $a \in \mathbb{Z} \cup \mathbb{Z} + \frac{1}{2}$, if chip $\bigcirc_i$ is at vertex $a$ and chip $\bigcirc_j$ is at vertex $-a$,
   move both chips one vertex rightward.

**Proposition 3.2.3.** For $\Phi$ of classical type with the above specific realizations, we have $\lambda \xrightarrow{\Phi} \mu$ if and only if the configuration $\mathcal{C}_{\mu}$ of $N$ labeled chips on $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ can be obtained from the configuration $\mathcal{C}_{\lambda}$ by

- a series of moves of kind (a) if $\Phi = A_{N-1}$;
- a series of moves of kinds (a), (b), and (d) if $\Phi = B_N$;
- a series of moves of kinds (a), (c), and (d) if $\Phi = C_N$;
- a series of moves of kinds (a), and (d) if $\Phi = D_N$.

Finally, let us describe some specific chip configurations corresponding to the weights in $\Omega \cup \{0\}$. For each each $\Phi$ of classical type, the zero weight corresponds to the configuration of $N$ chips at the origin. For $\Phi$ of Types $A_{N-1}$, $B_N$, or $C_N$, the fundamental weight $\omega_i$ for $i \in [n - 1]$ corresponds to the configuration of the first $i$ chips at the origin and the last $n - i$ chips at vertex 1. For $\Phi = B_N$, the weight $\omega_N$ corresponds to all chips being at vertex $\frac{1}{2}$. For $\Phi = C_N$, the weight $\omega_N$ corresponds
to all chips being at vertex 1. Finally, for $\Phi = D_N$, the fundamental weight $\omega_i$ for $i \in [n - 2]$ corresponds to the configuration of the first $i$ chips being at the origin and the remaining $n - i$ chips being at vertex 1, $\omega_N$ corresponds to all chips being at vertex $\frac{1}{2}$, and $\omega_{N-1}$ differs from $\omega_N$ only in the position of chip $\Box$ which is at $-\frac{1}{2}$. See Figure 3-3 for an illustration of these configurations. One can check, using the description of central-firing given by Proposition 3.2.3, that for each configuration depicted in Figure 3-3, central-firing is confluent starting from that configuration for all of the root systems listed under that configuration.

We believe that this chip interpretation will help prove some parts of our main conjecture concerning confluence of central-firing because it allows chip-firing arguments similar to those used for the usual (i.e., Type A) labeled chip-firing in Chapter 2 to be applied to the other types as well.

### 3.3 Confluence of central-firing mod the Weyl group

In this section we study what happens to central-firing when we mod out by the action of the Weyl group on the weight lattice.

Let $X$ be a set, $\to$ a binary relation, and $G$ a group acting on $X$. For $x \in X$, we write $G.x$ to denote the orbit of $x$ under $G$, and we write $X/G$ for the set of orbits
of $X$ under $G$. The relation $\rightarrow$ descends to a relation, also denoted $\rightarrow$, on $X/G$ as follows: we have $G.x \rightarrow G.y$ if and only if there exists $x' \in G.x$ and $y' \in G.y$ such that $x' \rightarrow y'$. Note that the notation $G.x^* \rightarrow G.y$ is inherently ambiguous because it is not clear if it means that we mod out by the group action before or after taking the reflexive transitive closure. In what follows will take $G.x^* \rightarrow G.y$ to mean that there exists $t \geq 0$ and $x_0, x_1, \ldots, x_t \in X$ such that

$$G.x = G.x_0 \rightarrow G.x_1 \rightarrow \cdots \rightarrow G.x_t = G.y.$$  

However, in the case that we care about, central-firing modulo the Weyl group, this ambiguity is actually irrelevant and the two possible interpretations coincide as the next proposition shows. Of course, when $\Phi = A_{N-1}$, the relation $\rightarrow_{\Phi^+}$ on $P/W$ corresponds exactly to unlabeled chip-firing of $N$ chips on the infinite path. We refer to the quotient relation $\rightarrow_{\Phi^+}$ on $P/W$ as unlabeled central-firing.

**Proposition 3.3.1.** For $\lambda, \mu \in P$, we have that $W.\lambda \rightarrow_{\Phi^+} W.\mu$ if and only if there is some $\mu' \in W.\mu$ with $\lambda \rightarrow \mu'$.

**Proof.** Let $w\lambda \in W.\lambda$ be such that $w\lambda \rightarrow w\lambda + \alpha \in W.\mu$ for some $\alpha \in \Phi^+$ which satisfies $\langle w\lambda, \alpha^\vee \rangle = 0$. Since $w$ is an orthogonal transformation, $\langle \lambda, w^{-1}(\alpha)^\vee \rangle = 0$ as well. If $w^{-1}(\alpha) \in \Phi^+$, then we are done since we found a firing move $\lambda \rightarrow_{\Phi^+} \lambda + w^{-1}(\alpha)$ with $\lambda + w^{-1}(\alpha) \in W.\mu$. If $w^{-1}(\alpha) \in \Phi^-$, let $\mu' := s_{w^{-1}(\alpha)}(\lambda + w^{-1}(\alpha)) \in W.\mu$. Since we have $\langle \lambda, w^{-1}(\alpha)^\vee \rangle = 0$, it follows that $\mu' = \lambda - w^{-1}(\alpha)$ and now $-w^{-1}(\alpha)$ is a positive root, so we are done. \hfill $\square$

**Corollary 3.3.2.** For $\lambda, \mu \in P$, we have $W.\lambda^* \rightarrow_{\Phi^+} W.\mu$ if and only if there is $\mu' \in W.\mu$ with $\lambda^* \rightarrow \mu'$.

**Proposition 3.3.3.** The relation $\rightarrow_{\Phi^+}$ on $P/W$ is terminating.

**Proof.** Suppose that there exists an infinite path $W.\lambda_1 \rightarrow_{\Phi^+} W.\lambda_2 \rightarrow_{\Phi^+} \ldots$. Then by Proposition 3.3.1, there exists $\mu_2 \in W.\lambda_2$ such that $\lambda_1 \rightarrow_{\Phi^+} \mu_2$. By Proposition 3.3.1 again, there exists $\mu_3 \in W.\lambda_3$ such that $\mu_2 \rightarrow_{\Phi^+} \mu_3$, and so on. We obtain an infinite sequence $\lambda_1 \rightarrow_{\Phi^+} \mu_2 \rightarrow_{\Phi^+} \mu_3 \rightarrow_{\Phi^+} \ldots$ which contradicts Proposition 3.2.1. \hfill $\square$
Now we proceed to prove that unlabeled central-firing is confluent (from every initial orbit $W.\lambda$). In order to do so, we will use the diamond lemma (Lemma 1.1.1).

**Lemma 3.3.4.** The relation $\rightarrow_{\Phi^+}$ on $P/W$ is locally confluent.

**Proof.** Let $\lambda, \mu, \mu' \in P$ be such that $W.\lambda \rightarrow_{\Phi^+} W.\mu$ and $W.\lambda \rightarrow_{\Phi^+} W.\mu'$. By Proposition 3.3.1, we may choose $\mu$ and $\mu'$ so that $\lambda \rightarrow_{\Phi^+} \mu$ and $\lambda \rightarrow_{\Phi^+} \mu'$. Let $\alpha := \mu - \lambda$ and $\beta := \mu' - \lambda$. Thus $\alpha$ and $\beta$ are positive roots that are both orthogonal to $\lambda$. We may assume that $\alpha \neq \beta$. Consider now the affine 2-dimensional plane $H$ spanned by $\alpha$ and $\beta$ that passes through $\lambda$. If we can show that there exists some $\nu \in H$ such that $W.\nu \rightarrow_{\Phi^+} W.\nu$ and $W.\nu' \rightarrow_{\Phi^+} W.\nu$ then we are done with the proof. Therefore it is enough to show that for the sub-root system $\Phi'$ of $\Phi$ spanned by $\alpha$ and $\beta$, the relation $\rightarrow_{(\Phi')^+}$ on $P'/W'$ is confluent, where $P'$ and $W'$ denote the weight lattice and the Weyl group of $\Phi'$.

Thus we can now assume that $\Phi = \Phi'$ is a rank 2 root system. Note, in rank 2, that to establish confluence we only need to check confluence from $W.0$ (because there is at most one firing move from any other orbit). This is easily verified by hand in each of the four possible cases: $A_1 \times A_1, A_2, B_2, G_2$. We need to check that for any $\beta_1, \beta_2 \in \Phi^+$, there exists $\lambda \in P$ such that $W.\beta_1 \rightarrow_{\Phi^+} W.\lambda$ and $W.\beta_2 \rightarrow_{\Phi^+} W.\lambda$. For $A_1 \times A_1$ this is trivial, so we can assume $\Phi$ is irreducible. Then, if $\beta_1$ and $\beta_2$ have the same length we get $W.\beta_1 = W.\beta_2$ and so there is nothing to check. Thus we can assume that $\Phi$ is not simply laced and $\beta_1$ is short and $\beta_2$ is long. Since the answer only depends on $W.\beta_1$ and $W.\beta_2$, we are free to choose any short $\beta_1$ and long $\beta_2$. So for $\Phi = B_2$ we can take $\beta_1 = \alpha_2$ and $\beta_2 = \alpha_1 + 2\alpha_2$ and $\lambda = \beta_2$, since then $\langle \beta_1, \alpha_1 + \alpha_2 \rangle = 0$ and $\alpha_1 + \alpha_2 \in \Phi^+$. And for $\Phi = G_2$ we can take $\beta_1 = \alpha_1$ and $\beta_2 = 3\alpha_1 + 2\alpha_2$ and $\lambda = \beta_1 + \beta_2$, since then $\langle \beta_1, \beta_2 \rangle = 0$. (Here we use the numbering of the simple roots as in Figure 3-1 and 3-2.)

**Corollary 3.3.5.** The relation $\rightarrow_{\Phi^+}$ on $P/W$ is confluent and terminating.

**Remark 3.3.6.** Unlabeled central-firing is a generalization of classical chip-firing to other root systems $\Phi$. Another such generalization, studied in detail by Benkart,
Klivans, and Reiner [13], is $M$-matrix chip-firing with respect to the Cartan matrix $C$ of $\Phi$. Such Cartan matrix chip-firing is also confluent for all root systems, starting with any initial configuration. We note that these generalizations are somewhat “orthogonal” to each other: for example, in Type $A_{N-1}$, unlabeled central-firing corresponds to chip-firing of $N$ chips on the infinite path graph; whereas the Cartan matrix chip-firing corresponds to chip-firing of any number of chips on the cycle graph with $N$ vertices. However, in later chapters where we study “deformations” of central-firing we will see a direct connection to Cartan matrix chip-firing.

Corollary 3.3.5 says that to decide if central-firing is confluent from $\lambda$, we only need to verify that there is a unique Weyl chamber which every central-firing sequence from $\lambda$ terminates in. However, in practice this does not necessary help that much to resolve the question of confluence; e.g., the main difficulty in the analysis of labeled chip-firing in Chapter 2 was precisely to show that the labeled chip-firing process sorts the chips (from the appropriate initial configuration).

In many cases we can say exactly what the stabilization of $W.\lambda$ is.

**Proposition 3.3.7.** Suppose that $\lambda \in \Pi^Q(\rho + \omega)$ for some $\omega \in \Omega^0_{m}$. Then $W.(\rho + \omega)$ is the $\longrightarrow_{\Phi^+}$-stabilization of $W.\lambda$.

**Proof.** It is clear that $W.(\rho + \omega)$ is $\longrightarrow_{\Phi^+}$-stable since $\rho + \omega$ is strictly dominant. In fact, we claim that $\rho + \omega$ is the only strictly dominant weight in $\Pi^Q(\rho + \omega)$. Indeed, suppose that $\nu$ is strictly dominant and belongs to $\Pi^Q(\rho + \omega)$. Since it is strictly dominant, we have $\nu = \rho + \mu$ for some dominant weight $\mu$. Thus by Proposition 3.1.2, $(\rho + \omega) - \nu$ is an integer combination of simple roots with nonnegative coefficients. Therefore the same is true for $\omega - \mu$, and hence $\mu \in \Pi^Q(\omega)$ again by Proposition 3.1.2. By definition, this forces $\mu = \omega$ and thus $\nu = \rho + \omega$.

So the vertices of $\Pi(\rho + \omega)$ are the only weights in $\Pi^Q(\rho + \omega)$ that are $\longrightarrow_{\Phi^+}$-stable. Let us now mention a result that will follow from the “permutohedron non-escaping lemma” (Lemma 4.5.2), which we prove in a later chapter when studying deformations of central-firing:
Lemma 3.3.8. If \( \mu \in \Pi^Q(\rho + \mu'') \) for dominant weights \( \mu, \mu'' \in P_{\geq 0} \) then we have that \( \mu' \in \Pi^Q(\rho + \mu'') \) for any \( \mu' \in P \) such that \( \mu \stackrel{\Phi+}{\longrightarrow} \mu' \).

By Lemma 3.3.8 together with Proposition 3.2.1 we know that any central-firing sequence starting at a weight in \( \Pi^Q(\rho + \omega) \) must terminate at a weight in \( \Pi^Q(\rho + \omega) \). So such a firing sequence must terminate at a vertex of \( \Pi(\rho + \omega) \). Thus indeed we have \( W.\lambda \stackrel{\ast}{\longrightarrow} W.(\rho + \omega) \).

3.4 Unlabeled central-firing on Dynkin diagrams

For \( \Phi \) of classical type, the moves from Section 3.2 allow one to give a similar description of unlabeled central-firing in these types. For example, for Type A, forgetting the labels of the chips yields exactly the unlabeled central-firing process. In this section, we give a very different description of the same process. It turns out that when \( \Phi \) is simply laced, unlabeled central-firing can be reformulated as a certain numbers game with simple rules on the Dynkin diagram \( D \) of \( \Phi \). The goal of this section is to describe this interpretation of unlabeled central-firing for simply laced \( \Phi \).

For a dominant weight \( \lambda \in P_{\geq 0} \), we set \( I^0_\lambda := \{ i \in [n] : \langle \lambda, \alpha_i^\vee \rangle = 0 \} \).

Proposition 3.4.1. Suppose \( \Phi \) is simply laced. Let \( \lambda \in P_{\geq 0} \) be a dominant weight. Then if \( W.\lambda \stackrel{\Phi+}{\longrightarrow} W.\mu \), there is a dominant \( \mu' \in W.\mu \) such that \( \lambda \stackrel{\Phi+}{\longrightarrow} \mu' \).

Proof. Let \( \lambda \in P \) be dominant and suppose that \( W.\lambda \stackrel{\Phi+}{\longrightarrow} W.\mu \). By Proposition 3.3.1 we may assume that \( \lambda \stackrel{\Phi+}{\longrightarrow} \mu \) so let \( \beta \in \Phi^+ \) be such that \( \mu = \lambda + \beta \). Since \( \langle \lambda, \beta^\vee \rangle = 0 \), we have \( \beta \in \Phi^I_\lambda \). Let \( \Phi' \subseteq \Phi^I_\lambda \) be the irreducible sub-root system of \( \Phi^I_\lambda \) containing \( \beta \).

Let \( \theta' \) be the highest root of \( \Phi' \). We claim that \( \lambda + \theta' \) is a dominant weight that belongs to \( W.\mu \). First note that since \( \Phi \) is simply laced and \( \Phi' \) is irreducible, \( \theta' \) can be obtained from \( \beta \) by the action of the Weyl group \( W' \) of \( \Phi' \) (which stabilizes \( \lambda \)), and thus \( \lambda + \theta' \in W.\mu \). Second, let us show that \( \lambda + \theta' \) is dominant. For any simple root \( \alpha_i \), we have

\[
\langle \lambda + \theta', \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle + \langle \theta', \alpha_i^\vee \rangle.
\]
Figure 3-4: The affine Dynkin diagrams. The “affine vertex” 0 is filled in red.

Suppose the first term $\langle \lambda, \alpha_i^\vee \rangle$ in the right hand side is nonzero; then it must be positive. The second term $\langle \theta', \alpha_i^\vee \rangle$ is greater than or equal to $-1$ because $\Phi$ is simply laced. Therefore, their sum is nonnegative. Suppose now that $\langle \lambda, \alpha_i^\vee \rangle$ is zero. Then $\alpha_i$ is a simple root of $\Phi_{I^\lambda}$ and hence $\langle \theta', \alpha_i^\vee \rangle \geq 0$. This finishes the proof.

Remark 3.4.2. Proposition 3.4.1 does not hold in general when $\Phi$ is not simply laced. This is already apparent for $\Phi = B_2$ and $\Phi = G_2$ when starting from the fundamental weight corresponding to the long simple root.

Let us now explicitly describe the relation $\rightarrow$ on $P/W$ for simply laced root systems. To do so, we need to discuss affine Dynkin diagrams.

Proposition 3.4.3 (See [21] VI, §3). Associated to every connected, simply laced Dynkin diagram $D$ with vertex set $[n]$ is a (unique) affine Dynkin diagram, denoted $\tilde{D}$, with vertex set $[n] \cup \{0\}$ and which contains $D$ as a subgraph. These affine Dynkin diagrams are depicted in Figure 3-4.
We also need the following lemma relating affine Dynkin diagrams to highest roots.

**Lemma 3.4.4** (See [21, VI, §3]). If \( \Phi \) is simply laced and \( D \) is its Dynkin diagram, then we have \( \theta = \sum_{i=1}^{n} c_i \omega_i \), where \( c_i \) is the number of edges between \( i \) and 0 in \( \tilde{D} \).

**Definition 3.4.5.** Let \( D \) be a simply laced Dynkin diagram with vertex set \([n]\). Let \( \gamma : [n] \to \mathbb{N} \) be an assignment of nonnegative integers to the vertices of \( D \). An *unlabeled central-firing move* (a UCF move for short) is an application of the following sequence of steps to \( \gamma \):

1. choose a *zero connected component* \( X \) of \( \gamma \), that is, a connected component of the induced subgraph \( D[\{i \in [n] : \gamma(i) = 0\}] \) of \( D \) with vertex set \( \{i \in [n] : \gamma(i) = 0\} \);
2. complete \( X \) to an affine Dynkin diagram \( \tilde{X} \) with vertex set \( X \cup \{0\} \);
3. for every edge \( \{0, i\} \) of \( \tilde{X} \), increase \( \gamma(i) \) by 1;
4. for every vertex \( j \notin X \) that is adjacent to a vertex \( i \in X \), decrease \( \gamma(j) \) by 1.

We denote the resulting assignment of integers by \( \gamma' \) and write \( \gamma \overset{\text{UCF}}{\longrightarrow} \gamma' \). We say that \( \gamma' \) is obtained from \( \gamma \) via a UCF move along \( X \).

**Example 3.4.6.** Let us illustrate this definition by an example for \( \Phi \) of Type \( E_7 \). Consider an assignment \( \gamma \) shown in Figure 3-5 (top). It has two zero connected components: \( X_1 \) of Type \( D_5 \) and \( X_2 \) of Type \( A_1 \). Applying a UCF move to \( \gamma \) along \( X_1 \) (resp., along \( X_2 \)) produces assignments \( \gamma'_1 \) (resp., \( \gamma'_2 \)) shown in Figure 3-5 (middle-left), resp., (middle-right). Note that \( \gamma'_1 \) has a zero connected component of Type \( A_5 \) that contains \( X_2 \), and similarly, \( \gamma'_2 \) has a zero connected component of Type \( E_6 \) that contains \( X_1 \). Moreover, applying another UCF move to the corresponding zero connected component of \( \gamma'_1 \) (resp., of \( \gamma'_2 \)) actually produces the same result \( \gamma'' \) shown in Figure 3-5 (bottom).

**Example 3.4.7.** All states of the classical chip-firing process starting with four chips at the origin are shown on the left of Figure 3-6; meanwhile, all states of the unlabeled central-firing process starting from 0 in Type \( A_3 \) are shown on the right of Figure 3-6.
Figure 3-5: Applying UCF moves to the Dynkin diagram of $E_7$ (see Example 3.4.6). For each move, the component $X$ is shown in blue, the extra vertex 0 of $\tilde{X}$ is shown in red, changes from step (3) are shown in red, and changes from step (4) are shown in blue.
Figure 3-6: Applying classical chip-firing moves to four chips at the origin (left). Applying UCF moves to $0 \in P$ for $\Phi$ of Type $A_3$ (right).
It turns out that the UCF moves always “commute,” and define a binary relation that coincides with $\Phi^+$:

**Theorem 3.4.8.** Let $D$ be a simply laced Dynkin diagram corresponding to the root system $\Phi$. For each assignment $\gamma : [n] \to \mathbb{N}$ we define the corresponding dominant weight $\lambda(\gamma) := \sum_{i=1}^{n} \gamma(i) \omega_i$. Then:

(i) An assignment $\gamma'$ is obtained from $\gamma$ by a UCF move (i.e. $\gamma \rightarrow_{\text{UCF}} \gamma'$) if and only if we have $W.\lambda(\gamma) \rightarrow_{\Phi^+} W.\lambda(\gamma')$.

(ii) UCF moves always “commute.” More precisely, let $X_1$ and $X_2$ be two zero connected components of $\gamma$, and let $\gamma'_1$ (resp., $\gamma'_2$) be the assignment obtained from $\gamma$ by a UCF move along $X_1$ (resp., along $X_2$). Then $\gamma'_1$ has a zero connected component $X'_2 \supseteq X_2$, $\gamma'_2$ has a zero connected component $X'_1 \supseteq X_1$, and applying a UCF move to $\gamma'_1$ along $X'_2$ produces the same result as applying a UCF move to $\gamma'_2$ along $X'_1$.

**Proof.** We start with (i). Recall from the proof of Proposition 3.4.1 that if $\lambda$ is dominant then for any root $\beta \in \Phi$ such that $\langle \lambda, \beta^\vee \rangle = 0$, there exists a root which we denote $\theta'$ such that $\lambda + \theta'$ is dominant and $W.(\lambda + \beta) = W.(\lambda + \theta')$. Moreover, it is easy to see again from the proof of Proposition 3.4.1 that such a root $\theta'$ is unique: it is the highest root of the irreducible sub-root system $\Phi'$ of $\Phi_{\lambda}$ containing $\beta$. Now let $\gamma$ be such that $\lambda = \sum_{i=1}^{n} \gamma(i) \omega_i$. It is a basic fact that for every $\beta \in \Phi^+$ given by $\beta = \sum_{i=1}^{n} b_i \alpha_i$, the graph $D[\beta] := D[\{i \in [n] : b_i \neq 0\}]$ is connected. Thus we have that $\langle \lambda, \beta \rangle = 0$ if and only if $D[\beta]$ is contained in a zero connected component $X$ of $\gamma$. It is then easy to see that the set of simple roots of $\Phi'$, written in the coordinates of the fundamental weights, is exactly given by steps (3) and (4) of Definition 3.4.5. In other words, we have

$$\langle \theta', \alpha_i^\vee \rangle = \begin{cases} 
-1, & \text{if } i \notin x \text{ is connected to a vertex } j \in x; \\
d_{i,0}, & \text{if } i \in x \text{ and there are } d_{i,0} \text{ edges of } \tilde{X} \text{ between } i \text{ and } 0; \\
0, & \text{otherwise.} 
\end{cases}$$
That $\langle \theta', \alpha_i^\vee \rangle = d_{i,0}$ if $i \in x$ follows from Lemma 3.4.4 above. If $i \notin x$ is connected to a vertex $j \in x$, then, writing $\theta' = \sum_{i=1}^n c_i \alpha_i$, we will have $c_j > 0$ since $\theta'$ is the highest weight of $\Phi'$; meanwhile, clearly $c_i = 0$; hence, $\langle \theta', \alpha_i^\vee \rangle < 0$; but since $\Phi$ is simply laced this means that $\langle \theta', \alpha_i^\vee \rangle = -1$. That $\langle \theta', \alpha_i^\vee \rangle = 0$ if $i \notin x$ is not connected to any vertex in $X$ is clear. This finishes the proof of (i).

To show (ii), note that the moves $W.\lambda \rightarrow W.\mu_1$ and $W.\lambda \rightarrow W.\mu_2$ “commute” for any rank 2 simply laced root system: in $A_2$, there is only one class of roots modulo the Weyl group, and in $A_1 \times A_1$ there are two classes but the two possible moves do indeed “commute.” Thus part (ii) follows from part (i) as an immediate corollary.

Remark 3.4.9. To extend this Dynkin diagram numbers game for unlabeled central-firing beyond the simply laced setting, there are two obstacles that need to be overcome. The first is that in general we may have both a highest root $\theta'$ and highest short root $\tilde{\theta}'$ for the parabolic sub-root system corresponding to a zero connected component of our weight, and adding $\theta'$ and $\tilde{\theta}'$ will lead to different weights. This is not such a serious obstacle: we can just allow these two different kinds of moves. The second, more serious, obstacle is that, as mentioned in Remark 3.4.2, not every unlabeled central-firing move corresponds to a move that stays in the dominant chamber: thus, sometimes adding $\theta'$ or $\tilde{\theta}'$ will make some coordinates of our weight negative. To overcome this, we could reflect our weight back into the dominant chamber by playing what is called Mozes’s numbers game (see [59] or [31]) on our Dynkin diagram. But this second obstacle makes the description of the unlabeled chip-firing game much more convoluted than in the simply laced case.

3.5 Conjectures for confluence of central-firing

In this section we put forward a complete conjectural classification of those weights in $\Omega \cup \{0\}$ from which central-firing is confluent. It is based on extensive computations, which were carried out with the Sage mathematical software system [67] [66]. To first order, confluence has to do with whether the initial weight is equal to $\rho$ modulo the root lattice. Hence while questions about confluence for central-firing are subtle and
difficult, some genuinely root theoretic phenomena are discernible.

**Conjecture 3.5.1.** Let $\omega \in \Omega \cup \{0\}$ be a fundamental weight or zero. Then $\rightarrow_{\Phi^+}$ is confluent from $\omega$ if and only if $\omega \notin Q + \rho$, unless:

1. $\Phi = A_n$ in which case $\rightarrow_{\Phi^+}$ is confluent from $\omega$ if and only if
   \[
   \begin{cases}
   \omega = 0, \omega_1, \omega_n, & \text{if } n \text{ is odd;} \\
   \omega = \omega_{n/2}, \omega_{n/2+1}, & \text{if } n \text{ is even}.
   \end{cases}
   \]  

2. $\Phi = B_n$ in which case $\rightarrow_{\Phi^+}$ is confluent from $\omega_n$ despite the fact that $\omega_n \in Q + \rho$;

3. $\Phi = D_{4m+2}$ for $m \geq 1$ in which case $\rightarrow_{\Phi^+}$ is not confluent from $0$ even though we have $0 \notin Q + \rho$;

4. $\Phi = G_2$ in which case $\rightarrow_{\Phi^+}$ is confluent from both $\omega_1$ and $\omega_2$ even though $P = Q$.

Here the simple roots are numbered as in Figure 3-2.

More explicitly, the confluent weights (i.e., the weights from which $\rightarrow_{\Phi^+}$ is confluent) in $\Omega \cup \{0\}$ are listed in Table 3.1. In particular, observe that the weights corresponding to the exceptional cases (1)-(4), which are highlighted in red and green in the table, are quite rare, especially outside Type A.

**Remark 3.5.2.** According to Conjecture 3.5.1, for each pair $(\Phi, \omega)$ shown in Figure 3-3, central-firing is confluent from $\omega$. We encourage the reader to check that the result of applying the chip-firing moves to these configurations as described in Section 3.2 does not depend on the choices made along the way.

We will spend this section discussing which parts of Conjecture 3.5.1 are known and which remain to be proven. Let us start with “positive” results (i.e., results about confluence). The results from Chapter 2 imply:

**Theorem 3.5.3.** Conjecture 3.5.1 is true for $\omega = 0$ when $\Phi = A_n$ or $B_n$. 

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Table 3.1: Confluent and non-confluent initial weights in $\Omega \cup \{0\}$. A vertex corresponding to 0, resp., $\omega_i$ is labeled by 0, resp., $i$ (as in Figure 3-2). Weights from which central-firing is confluent correspond to filled vertices with boldface labels. If $\omega \in Q + \rho$ then the corresponding vertex is marked by $\rho$. If $\omega \not\in Q + \rho$ but is still not confluent then it is colored red. If $\omega \in Q + \rho$ but is still confluent then it is colored green.
Actually, when Φ is of Type $B_n$, it is easy to see that for any $\omega \in \Omega \setminus \{\omega_n\}$, we have that $0 \xrightarrow{\Phi^+} \omega$. So Theorem 3.5.3 implies almost all cases of Conjecture 3.5.1 for root systems of Type B:

**Corollary 3.5.4.** Conjecture 3.5.1 is true for $\omega \in \Omega \setminus \{\omega_n\}$ when $\Phi$ of Type $B_n$.

The only other positive results we know are for the exceptional root systems. For root systems of exceptional type, we have verified the conjecture using Sage [67] [66]. Namely, we have:

**Proposition 3.5.5.** Suppose that $\Phi = E_6, E_7, E_8, F_4,$ or $G_2$. Then Conjecture 3.5.1 is true for $\Phi$.

Now let us describe “negative” results (about non-confluence). Let us show that in simply laced cases, having $\omega \in Q + \rho$ implies that $\omega$ is not confluent.

**Proposition 3.5.6.** Suppose that $\Phi$ is simply laced. Let $\lambda \in P$ be a dominant weight that belongs to $\Pi^Q(\rho)$ but is not equal to $\rho$. Then $\xrightarrow{\Phi^+}$ is not confluent from $\lambda$.

**Proof.** We know by Propositions 3.3.7 and 3.4.1 that there exists a firing sequence

$$\lambda = \lambda_0 \xrightarrow{\Phi^+} \lambda_1 \xrightarrow{\Phi^+} \ldots \xrightarrow{\Phi^+} \lambda_t = \lambda_t + 1 \xrightarrow{\Phi^+} \rho$$

such that for each $0 \leq s \leq t + 1$, $\lambda_s$ is a dominant weight. Let $\alpha \in \Phi^+$ be such that $\lambda_t + \alpha = \rho$. In particular, we have $\langle \lambda_t, \alpha^\vee \rangle = 0$ and thus $\langle \rho, \alpha^\vee \rangle = 2$. Write $\alpha$ in the basis of simple roots:

$$\alpha = \sum_{i=1}^{n} a_i \alpha_i.$$  

Since $\rho$ is the sum of the fundamental weights, we get $\sum_{i=1}^{n} a_i = 2$ (note that this conclusion uses the fact that $\Phi$ is simply laced). Since $2\alpha_i \notin \Phi$, we get that $\alpha = \alpha_i + \alpha_j$ for some $i \neq j \in [n]$. Moreover, it must be the case that $i$ and $j$ are connected by an edge in the Dynkin diagram $D$ of $\Phi$ because otherwise $\alpha_i + \alpha_j$ would not be a root. Thus $\langle \alpha_i, \alpha_j^\vee \rangle = -1$. Let us now consider the weight $\mu = \lambda_t + \alpha_i$. We claim that $\mu$ is $\xrightarrow{\Phi^+}$-stable and that $\langle \lambda_t, \alpha_i^\vee \rangle = 0$, that is, $\lambda_t \xrightarrow{\Phi^+} \mu$. Indeed, we have

$$\langle \lambda_t, \alpha_i^\vee \rangle = \langle \rho - \alpha_i - \alpha_j, \alpha_i^\vee \rangle = 1 - 2 + 1 = 0.$$
Thus $\lambda_+ \rightarrow \mu$. On the other hand, $\mu = \rho - \alpha_j$ is a vertex of $\Pi^Q(\rho)$:

$$s_{\alpha_j}(\rho) = \rho - \langle \rho, \alpha_j \rangle \alpha_j = \mu.$$ 

In particular, it is $\Phi^+$-stable, which finishes the proof. \hfill \Box

This proposition immediately implies some parts of Conjecture 3.5.1.

**Corollary 3.5.7.** Suppose $\Phi$ is simply laced and $\omega \in \Omega \cup \{0\}$ satisfies $\omega \in Q + \rho$. Then $\Phi^+$ is not confluent from $\omega$.

Note that in Types $B_2$ and $G_2$, the result of this last corollary is false (and assuming Conjecture 3.5.1 it is false for $B_n$ for all $n \geq 2$), so the simply laced requirement is necessary. On the other hand, in the $C_n$ and $F_4$ cases the result of this corollary still appears to hold.

Let us also show that all the red vertices in the Type A part of Table 3.1 really are non-confluent:

**Proposition 3.5.8.** Suppose that $\Phi = A_{N-1}$ and consider a weight $\omega \in \Omega \cup \{0\}$. Then $\Phi^+$ is not confluent from $\omega$ unless $\omega$ is given by (3.1) (in which case it may or may not be confluent).

*Proof.* The case $\omega \equiv \rho$ modulo $Q$ follows from Corollary 3.5.7, thus we may assume that $\omega \neq 0$. So let $i \in [N - 1]$ be such that $\omega = \omega_i$.

Let us use the chip interpretation $C_{\omega_i}$ of $\omega_i$ from Section 3.2. In $C_{\omega_i}$ chip $\bigcirc$ is at position 1 while chip $\bigcirc$ is at the origin. Consider the *unlabeled* stabilization of the configuration obtained from $C_{\omega_i}$ by removing $\bigcirc$. It is easy to see that this unlabeled stabilization will necessarily have a *gap* at some position $j \in \mathbb{Z}$ (i.e., there will be no chip at position $j$ but there will be chips both at positions $j - 1$ and $j + 1$).

If $j > 0$, then starting from $C_{\omega_i}$ we can first stabilize all chips other than $\bigcirc$ and then stabilize $\bigcirc$ as well, and this will cause chip $\bigcirc$ to end up at position $j$. On the other hand, if $j \leq 0$, then the unlabeled stabilization of the configuration obtained from $C_{\omega_i}$ by removing $\bigcirc$ will have a gap at position $j - 1 \leq 0$. So in this case if we
start from $\mathcal{C}_{\omega_i}$ we can first stabilize all chips other than $1$ and then stabilize $1$ and this will cause chip $1$ to end up at position $j - 1$. In either case, we reached a final configuration where the chips were not sorted, which corresponds to a weight that is not dominant. However, we know that there is also a firing sequence that starts from the origin and always stays inside the dominant chamber. So we found two stabilizations of $\omega_i$ and are done with the proof.

We have now reviewed all known cases of Conjecture 3.5.1. But before concluding this chapter, let us explain how the method of “folding” could potentially be used in addressing this conjecture. Folding is a way of obtaining non-simply laced root systems from simply laced ones via Dynkin diagram automorphisms. It can often be used to reduce questions about root systems to the simply laced cases. Actually, we have already seen folding in action: it is essentially what we used at the end of Chapter 2 to deduce the confluence of the Type B version of labeled chip-firing from the Type A result.

We follow the description of folding given in [75]. Suppose we are given a simply laced root system $\Phi \subseteq V$ with Dynkin diagram $D$ and an automorphism $\sigma : [n] \to [n]$ of $D$ that does not send a vertex to its neighbor. From this data, one constructs another root system $\Phi'$ as follows. Let $J$ be the set of equivalence classes of $[n]$ modulo $\sigma$. For each $j \in J$, define the $j$-th simple root $\alpha_j$ of $\Phi'$ to be the sum of the corresponding simple roots of $\Phi$ (which are necessarily orthogonal to each other):

$$\alpha_j := \sum_{i \in j} \alpha_i.$$  

It turns out that $\{\alpha_j : j \in J\}$ is a set of simple roots of another root system $\Phi'$ whose Dynkin diagram is obtained from $D$ via folding along $\sigma$. Note that $\Phi'$ is naturally living inside $V^\sigma := \{v \in V : \sigma(v) = v\}$. Here we extended $\sigma$ to a map $V \to V$ by linearity from its action on simple roots. The fundamental weights $\omega_j$ for $\Phi'$ are again given by a similar expression:

$$\omega_j := \sum_{i \in j} \omega_i.$$  

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It is easy to check that indeed \( \langle \omega_j, (\alpha_j^\vee) \rangle = \delta_{j_1,j_2} \), where \( \delta \) is the Kronecker delta.

Thus the weight lattice for \( \Phi' \) is \( P^\sigma := \{ \lambda \in P: \sigma(\lambda) = \lambda \} \).

Let us now discuss the relationship between \( \Phi^+ \) and \( (\Phi')^+ \). By [75, Claim 4], each \( \sigma \)-orbit of \( \Phi \) consists of pairwise orthogonal roots. By [75, Claim 1], the roots of \( \Phi' \) are precisely of the form \( \beta = \sum_{\alpha \in B} \alpha \), where \( B \) is a single \( \sigma \)-orbit of \( \Phi \). Thus if \( \lambda \rightarrow \lambda + \beta \) for some \( \beta \in (\Phi')^+ \) then \( \lambda \rightarrow^* \lambda + \beta \) because we can just fire each root in \( B \) in an arbitrary order. We obtain the following result.

**Proposition 3.5.9.** Suppose \( \rightarrow_{\Phi^+} \) is confluent from some weight \( \lambda \in P^\sigma \). Then \( \rightarrow_{(\Phi')^+} \) is confluent from \( \lambda \) as well.

**Proof.** Let \( \mu \) be the unique \( \rightarrow_{\Phi^+} \)-stable point such that \( \lambda \rightarrow_{\Phi^+}^* \mu \). Then \( \sigma(\mu) \) would also be such a stable point, and thus we must have \( \sigma(\mu) = \mu \). Suppose that there is some \( \rightarrow_{(\Phi')^+} \)-stable point \( \mu' \in P^\sigma \) such that \( \lambda \rightarrow_{(\Phi')^+}^* \mu' \), and assume that \( \mu' \neq \mu \). Then by the above discussion we have that \( \lambda \rightarrow_{\Phi^+}^* \mu' \) and thus \( \mu' \) must not be a \( \rightarrow_{\Phi^+} \)-stable point. Thus there is a root \( \alpha \in \Phi^+ \) such that \( \langle \mu', \alpha^\vee \rangle = 0 \). Let \( B \) be the \( \sigma \)-orbit of \( \alpha \), then \( \beta := \sum_{\alpha' \in B} \alpha' \) is a positive root for \( \Phi' \) and since \( \mu' \) is \( \sigma \)-invariant, we still have that \( \langle \mu', \beta^\vee \rangle = 0 \). We have shown that if \( \mu' \in P^\sigma \) is a \( \rightarrow_{(\Phi')^+} \)-stable point in \( P^\sigma \) such that \( \lambda \rightarrow_{(\Phi')^+}^* \mu' \) then \( \mu' = \mu \). Since \( \rightarrow_{(\Phi')^+} \) is terminating, there has to be at least one such stable point, and thus it follows that \( \mu \) is the only \( \rightarrow_{(\Phi')^+} \)-stable point that satisfies \( \lambda \rightarrow_{(\Phi')^+}^* \mu \).

Proposition 3.5.9 can be directly applied to get some dependencies between various claims in Conjecture 3.5.1. Let us list a few interesting ones:

- If central-firing for \( \Phi = A_{2n-1} \) is confluent from the origin then central-firing for \( \Phi = B_n \) is confluent from the origin as well (this is the example of folding that we have already seen at the end of Chapter 2);

- If central-firing for \( \Phi = D_{n+1} \) is confluent from \( \omega_n + \omega_{n+1} \) then central-firing for \( \Phi = C_n \) is confluent from \( \omega_n \);

- If central-firing for \( \Phi = D_{n+1} \) is confluent from 0 (resp., from \( \omega_i \) for some \( i \in [n-1] \)) then central-firing for \( \Phi = C_n \) is confluent from 0 (resp., from \( \omega_i \)).
Remark 3.5.10. According to our computations, central-firing for $\Phi = D_6$ is not confluent from 0 even though for $\Phi = C_5$ it is. (This is generalized in Conjecture 3.5.1 to $D_{4m+2}$ and $C_{4m+1}$; but in fact we could not check computationally whether central-firing for $D_{10}$ is confluent from 0.) Similarly, one can easily check that central-firing for $\Phi = A_3$ is not confluent from $\omega_2$, but for $\Phi = B_2$ it is. Thus the converse to Proposition 3.5.9 fails to hold in many cases.

To conclude the chapter, let us list what remains to be done for Conjecture 3.5.1:

1. Show that central-firing for $\Phi = A_{2m+1}$ is confluent from $\omega_1$ and $\omega_{2m+1}$.
2. Show that central-firing for $\Phi = A_{2m}$ is confluent from $\omega_m$ and $\omega_{m+1}$.
3. Show that central-firing for $\Phi = B_n$ is confluent from $\omega_n$.
4. Show that central-firing for $\Phi = C_n$ is confluent from $\omega \in \Omega \cup \{0\}$ if and only if $\omega \not\in \rho + Q$.
5. Show that central-firing for $\Phi = D_{4m+2}$ is not confluent from 0.
6. Show that central-firing for $\Phi = D_n$ is confluent from $\omega \in \Omega \cup \{0\}$ if $\omega \not\in \rho + Q$, except in the case 5 above.

As mentioned in Section 3.2, one possible way to attack these problems is via the chip-firing interpretation of central-firing in the classical types.

In the following chapters of this thesis we will not pursue questions about confluence for central-firing further. Instead, we will introduce some “deformations” of central-firing which turn out to have remarkable geometric structure related to permutohedra, and which, unlike central-firing, are always confluent from all initial weights.
Chapter 4

Interval root-firing: confluence

In this chapter we introduce some “affine deformations” of the central-firing process studied in Chapter 3. The material in this chapter is joint work with Pavel Galashin, Thomas McConville, and Alexander Postnikov and appears in [36].

Continue to fix an irreducible, crystallographic root system $\Phi$ as in the previous chapter. The central-firing process allowed the firing of a positive root $\alpha \in \Phi^+$ from a weight $\lambda \in P$ whenever $\lambda$ is orthogonal to $\alpha$; this can of course equivalently be described in terms of the inner product on $V$ as

$$\lambda \rightarrow \lambda + \alpha, \text{ for } \lambda \in P \text{ and } \alpha \in \Phi^+ \text{ with } \langle \lambda, \alpha^\vee \rangle = 0.$$ 

The deformations of central-firing we consider involve changing the values of $\langle \lambda, \alpha^\vee \rangle$ at which we allow the firing move $\lambda \rightarrow \lambda + \alpha$ to be some wider interval. In fact, we study two very particular families of intervals. For $k \in \mathbb{N}$, the symmetric interval root-firing process is the binary relation $\rightarrow_{\text{sym},k}$ on $P$ defined by

$$\lambda \rightarrow_{\text{sym},k} \lambda + \alpha, \text{ for } \lambda \in P \text{ and } \alpha \in \Phi^+ \text{ with } \langle \lambda, \alpha^\vee \rangle + 1 \in \{-k, -k+1, \ldots, k\}$$

and the truncated interval root-firing process is the relation $\rightarrow_{\text{tr},k}$ on $P$ defined by

$$\lambda \rightarrow_{\text{tr},k} \lambda + \alpha, \text{ for } \lambda \in P \text{ and } \alpha \in \Phi^+ \text{ with } \langle \lambda, \alpha^\vee \rangle + 1 \in \{-k+1, -k+2, \ldots, k\}.$$
We refer to these as *interval-firing* processes for short. The symmetric interval-firing process is so-called because the symmetric poclosure $\xrightarrow{\text{sym}, k}$ of the relation $\xrightarrow{\text{sym}, k}$ is invariant under the Weyl group. The truncated process is so-called because the interval defining it is truncated by one element on the left compared to the symmetric interval. We call these deformations “affine” because they allow firing when our weight belongs to a certain affine hyperplane arrangement which deforms the Coxeter arrangement (as opposed to the central-firing process, which allows firing when our weight belongs to the central Coxeter arrangement). Note, however, that these processes are not truly “deformations” of central-firing in the sense that we cannot recover central-firing by specializing $k$.

Actually, one can see above that rather than record the intervals corresponding to the values of $\langle \lambda, \alpha^\vee \rangle$ at which we allow firing, we recorded the intervals corresponding to the values of $\langle \lambda, \alpha^\vee \rangle + 1 = \langle \lambda + \frac{\alpha}{2}, \alpha^\vee \rangle$ at which we allow firing. This turns out to be more natural in many respects. And with this convention, the intervals defining the symmetric and truncated interval-firing processes are exactly the same as the intervals defining the *(extended)* $\Phi^\vee$-Catalan and *(extended)* $\Phi^\vee$-Shi hyperplane arrangements [65, 7]. The Catalan and Shi arrangements are known to have many remarkable combinatorial and algebraic properties, such as freeness [26, 73, 78, 80]. Although we have no precise statement to this effect, empirically it seems that many of the remarkable properties of these families of hyperplane arrangements are reflected in the interval-firing processes.

What makes the symmetric and truncated interval-firing processes special? Well, the main result in this chapter is that these two processes are always confluent (and terminating) from all initial weights. This is very much in contrast to central-firing, for which the pattern of confluence is, as we have seen in Chapter 3, extremely complicated. Indeed, the symmetric and truncated intervals are essentially the only intervals we could use to get a process which is confluent from all initial weights. Observe that some of the confluent interval-firing processes are close to central-firing: we have $\lambda \xrightarrow{\text{sym}, 0} \lambda + \alpha$ when $\langle \lambda, \alpha^\vee \rangle = -1$, and $\lambda \xrightarrow{\text{tr}, 1} \lambda + \alpha$ when $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0\}$. This suggests central-firing is on the “cusp” of being confluent from all initial weights.
Figure 4-1: The $k = 1$ symmetric interval-firing process for $\Phi = A_2$.

In order to establish the confluence of these two interval-firing processes, rather than use any “chip-firing” interpretation we instead adopt a geometric perspective. These interval-firing processes are intimately related to permutohedra. The connection to permutohedra can already be seen in Figure 4-1 which depicts the $k = 1$ symmetric interval-firing process for $\Phi = A_2$. A key technical result in our proof of confluence is an explicit formula for traverse lengths in permutohedra. This formula for traverse lengths then implies a “permutohedron non-escaping lemma,” which says that our interval-firing processes must get “trapped” inside of certain permutohedra. Confluence follows from the permutohedron non-escaping lemma by considering at which stable points the process could possibly terminate.

### 4.1 Definition of interval-firing

In this section we formally define the interval-firing processes. We will work at a slightly larger higher of generality than discussed in the introduction to this chapter where we allow our deformation parameter to vary on each Weyl group orbit of weights separately. Hence, we use the notation $k \in \mathbb{Z}[\Phi]^W$ to mean that $k$ is an integer-valued function on the roots of $\Phi$ that is invariant under the action of the Weyl group. We write $a \leq b$ to mean that $a(\alpha) \leq b(\alpha)$ for all $\alpha \in \Phi$. We use the notation $k = k$ to mean that $k$ is constantly equal to $k$. We also use the obvious
notation $a\mathbf{a} + b\mathbf{b}$ for linear combinations of these functions. We use $\mathbb{N}[\Phi]^W$ to denote the set of $k \in \mathbb{Z}[\Phi]^W$ with $k \geq 0$. We write $\rho_k := \sum_{i=1}^n k(\alpha_i)\omega_i$. Since we have assumed that $\Phi$ is irreducible, there are at most two $W$-orbits of $\Phi$: the short roots and the long roots. If $\Phi$ is simply laced then there is in fact one single orbit and so we have $k = k$ for some constant $k \in \mathbb{Z}$; otherwise, we have two constants $k_s, k_l \in \mathbb{Z}$ so that $k(\alpha) = k_s$ if $\alpha$ is short and $k(\alpha) = k_l$ if $\alpha$ is long.

For $k \in \mathbb{N}[\Phi]^W$, the \textit{symmetric interval-firing process} is the binary relation $\lambda \xrightarrow{\text{sym, } k} \lambda + \alpha$, for $\lambda \in P$ and $\alpha \in \Phi^+$ with $\langle \lambda + \frac{\alpha}{2}, \alpha^\vee \rangle \in [-k(\alpha), k(\alpha)]$

and the \textit{truncated interval-firing process} is the binary relation $\lambda \xrightarrow{\text{tr, } k} \lambda + \alpha$, for $\lambda \in P$ and $\alpha \in \Phi^+$ with $\langle \lambda + \frac{\alpha}{2}, \alpha^\vee \rangle \in [-k(\alpha) + 1, k(\alpha)]$.

Let us now show some examples of these processes. Recall the digraph $\Gamma \xrightarrow{\cdot}$ associated to a relation $\rightarrow$, as defined in Chapter 1. From now own we will often think about $\rightarrow$ as $\Gamma \rightarrow$. So we use $\Gamma_{\text{sym, } k} := \Gamma \xrightarrow{\text{sym, } k}$ and $\Gamma_{\text{tr, } k} := \Gamma \xrightarrow{\text{tr, } k}$.

\textbf{Example 4.1.1}. The irreducible rank 2 root systems are $A_2$, $B_2$ and $G_2$. The positive roots and fundamental weights for these root systems are depicted in Figure 3-1. In Figures 4-2, 4-3, and 4-4 we depict the the truncated and symmetric interval-firing processes $\Gamma_{\text{tr, } k}$ and $\Gamma_{\text{sym, } k}$ for $k = 0, 1, 2$ for these three root systems. Of course these graphs are infinite, so we depict the “interesting part” of the graphs near the origin (which is circled in black). The colors in these drawings correspond to classes of weights modulo the root lattice (hence there are three colors in the $A_2$ graphs, two in the $B_2$ graphs, and one in the $G_2$ graphs). Note that as $k$ increases, the scale of the drawing is not maintained. Most, if not all, of the features of truncated and symmetric interval-firing that we care about are visible already in rank 2. Thus the reader is encouraged, while reading the rest of this thesis, to return to these figures and understand how each of the results apply to these two dimensional examples.
<table>
<thead>
<tr>
<th>k</th>
<th>$\Gamma_{tr,k}$</th>
<th>$\Gamma_{sym,k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td><img src="image" alt="Graph 0" /></td>
<td><img src="image" alt="Graph 0" /></td>
</tr>
<tr>
<td>1</td>
<td><img src="image" alt="Graph 1" /></td>
<td><img src="image" alt="Graph 1" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image" alt="Graph 2" /></td>
<td><img src="image" alt="Graph 2" /></td>
</tr>
</tbody>
</table>

Figure 4-2: The graphs $\Gamma_{tr,k}$ and $\Gamma_{sym,k}$ for $\Phi = A_2$ and $k = 0, 1, 2$. 
Figure 4-3: The graphs $\Gamma_{tr,k}$ and $\Gamma_{sym,k}$ for $\Phi = B_2$ and $k = 0, 1, 2$. 
<table>
<thead>
<tr>
<th>$k$</th>
<th>$\Gamma_{tr,k}$</th>
<th>$\Gamma_{sym,k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>![Graph for $k=0$]</td>
<td>![Graph for $k=0$]</td>
</tr>
<tr>
<td>1</td>
<td>![Graph for $k=1$]</td>
<td>![Graph for $k=1$]</td>
</tr>
<tr>
<td>2</td>
<td>![Graph for $k=2$]</td>
<td>![Graph for $k=2$]</td>
</tr>
</tbody>
</table>

Figure 4-4: The graphs $\Gamma_{tr,k}$ and $\Gamma_{sym,k}$ for $\Phi = G_2$ and $k = 0, 1, 2$. 
Remark 4.1.2. Via the same correspondence between weights of Type $A_{N-1}$ and configurations of $N$ labeled chips on $\mathbb{Z}$ described in Chapter 3, the Type A symmetric and truncated interval-firing processes can also be seen as “labeled chip-firing processes” that consist of the same labeled chip-firing moves, which send chip $i$ one vertex to the left and chip $j$ one vertex to the right for any $i < j$, but where we allow these moves to be applied under different conditions: namely, when the position of chip $j$ minus the position of chip $i$ is either in the interval $[-k-1, k-1]$ (in the symmetric case) or in the interval $[-k, k-1]$ (in the truncated case). For example, consider the smallest non-trivial case: symmetric interval-firing with $k = 0$. This corresponds to the “labeled chip-firing process” that allows the transposition of the chips $i$ and $j$ with $i < j$ when $i$ is one position to the right of $j$. It is immediately apparent that this process is confluent; for instance, the configuration

\[
\begin{array}{cccccccc}
7 & 4 & & & & & & \\
6 & 5 & 3 & 2 & & & & \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 & \\
\end{array}
\]

\[\xrightarrow{\text{sym,0}}\]

stabilizes to

\[
\begin{array}{cccccccc}
4 & 6 & & & & & & \\
7 & 5 & 3 & 2 & & & & \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 & \\
\end{array}
\]

In general the stabilization will weakly sort each collection of contiguous chips, while leaving the underlying unlabeled configuration of chips the same. The next smallest case is truncated interval-firing with $k = 1$, which corresponds to the “labeled chip-firing process” that allows both the transposition moves from the symmetric $k = 0$ case and the usual labeled chip-firing moves. The reader can verify that

\[
\begin{array}{cccccccc}
1 & 3 & & & & & & \\
2 & 4 & & & & & & \\
-2 & -1 & 0 & 1 & 2 & & & \\
\end{array}
\]

\[\xrightarrow{\text{tr,1}}\]

stabilizes to

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & & & & \\
-2 & -1 & 0 & 1 & 2 & & & \\
\end{array}
\]
Here it is less obvious that confluence holds (although it is not too hard to prove this fact directly via a diamond lemma argument). The reader is now encouraged to experiment with this labeled chip-firing interpretation of symmetric and truncated interval-firing for higher values of $k$. Note that increasing $k$ allows for the firing of chips $i$ and $j$ when they are further apart.

As mentioned earlier, in our further treatment of the interval-firing processes we will focus on the geometric picture (on display in Example 4.1.1) and not the chip-firing picture (discussed in Remark 4.1.2).

To close out this section, let us demonstrate that the interval-firing processes are always terminating. Because the collection $\Phi^+$ of vectors we are adding is acyclic, this is straightforward and works exactly as in Proposition 3.2.1.

**Proposition 4.1.3.** For $k \in \mathbb{N}[\Phi]^W$, the relations $\rightarrow_{\text{sym},k}$ and $\rightarrow_{\text{tr},k}$ are terminating.

**Proof.** It is enough to show this for $\rightarrow_{\text{sym},k}$ which has strictly more firing moves than $\rightarrow_{\text{tr},k}$. For $\lambda \in P$ define $\varphi(\lambda) := \langle \rho_{k+1} - \lambda, \rho_{k+1} - \lambda \rangle$; in other words, $\varphi(\lambda)$ is the length of the vector $\rho_{k+1} - \lambda$. Suppose $\lambda \rightarrow_{\text{sym},k} \lambda + \alpha$ for $\alpha \in \Phi^+$. Then,

$$\varphi(\lambda) - \varphi(\lambda + \alpha) = \langle \rho_{k+1} - \lambda, \rho_{k+1} - \lambda \rangle - \langle \rho_{k+1} - (\lambda + \alpha), \rho_{k+1} - (\lambda + \alpha) \rangle$$

$$= 2\langle \rho_{k+1}, \alpha \rangle - 2\langle \lambda, \alpha \rangle$$

$$\geq \frac{4}{\langle \alpha, \alpha \rangle} (k(\alpha) + 1 - k(\alpha)) = \frac{4}{\langle \alpha, \alpha \rangle},$$

where we use the fact that $\langle \lambda, \alpha \rangle \leq \frac{2}{\langle \alpha, \alpha \rangle} k(\alpha)$ since $\lambda \rightarrow_{\text{sym},k} \lambda + \alpha$. So each firing move causes the quantity $\varphi(\lambda)$ to decrease by at least some fixed nonzero amount. But we have $\varphi(\lambda) \geq 0$ because it is measuring the length of a vector. Thus indeed $\rightarrow_{\text{sym},k}$ is terminating. \qed

### 4.2 Symmetries of interval-firing processes

In this section we study the symmetries of the two interval-firing processes. Since the set of positive roots $\Phi^+$ is an “oriented” set of vectors, we do not expect the directed
graphs $\Gamma_{\text{sym},k}$ and $\Gamma_{\text{tr},k}$ to have many symmetries, and certainly none coming from the Weyl group. But if we consider instead the undirected graphs $\Gamma_{\text{sym},k}^{\text{un}}$ and $\Gamma_{\text{tr},k}^{\text{un}}$ (corresponding to the symmetric relations $\leftrightarrow_{\text{sym},k}$ and $\leftrightarrow_{\text{tr},k}$), we will see that both of these do in fact have symmetries coming from the Weyl group.

For the symmetric interval-firing process, the graph $\Gamma_{\text{sym},k}^{\text{un}}$ is invariant under the action of the whole Weyl group $W$. This explains the name “symmetric” for the process: it has the biggest possible group of symmetries. As for the truncated process, in order to understand its symmetries we need to introduce a certain subgroup of the Weyl group $C \subseteq W$. In fact this $C$ is an abelian group and satisfies $C \cong P/Q$. In our definition of $C$ we follow Lam and Postnikov \[49\].

The Coxeter number of $\Phi$, another fundamental invariant of the root system, is $h := \langle \rho, \hat{\theta}^\vee \rangle + 1$. (The Coxeter number is also equal to $h = 1 + \sum_{i=1}^{n} a_i$ where $\theta = \sum_{i=1}^{n} a_i \alpha_i$.) Lam and Postnikov \[49\] §5 defined the subgroup $C := \{ w \in W : \rho - w(\rho) \in hP \}$ of the Weyl group and explained (using the affine Weyl group, which we will not discuss here) that $C$ is naturally isomorphic to $P/Q$: the isomorphism is explicitly given by $w \mapsto \omega \in \Omega_m$ if and only if $\rho - w(\rho) = h\omega$. (Since $\rho - w(\rho) \in Q$ for any $w \in W$, a consequence of this description of the isomorphism is that $h \cdot (P/Q) = \{0\}$.) As they mention, this subgroup was also studied before by Verma \[79\], but in spite of its significance it does not seem to have any name other than $C$ in the root system literature. Lam and Postnikov gave another characterization \[49\ Proposition 6.4\] of $C$ that will be useful for us: $C = \{ w \in W : w(\{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee\}) = \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee\} \}$, where we use the suggestive notation $\alpha_0^\vee := -\hat{\theta}^\vee$.

**Theorem 4.2.1.** Let $k \in \mathbb{N}[\Phi]^W$. Set $\Gamma := \Gamma_{\text{sym},k}^{\text{un}}$ or $\Gamma := \Gamma_{\text{tr},k}^{\text{un}}$. Then,

- if $\Gamma = \Gamma_{\text{sym},k}^{\text{un}}$, the linear map $v \mapsto w(v)$ is an automorphism of $\Gamma$ for all $w \in W$;

- if $\Gamma = \Gamma_{\text{tr},k}^{\text{un}}$, the affine map $v \mapsto w(v - \frac{1}{h} \rho) + \frac{1}{h} \rho$ is an automorphism of $\Gamma$ for all $w \in C \subseteq W$.

\[1\] Lam and Postnikov worked in a completely dual setting to ours: that is, they described a copy of the coweight lattice modulo the coroot lattice inside of $W$; hence, they used $\theta$ instead of $\hat{\theta}$, etc.
**Proof.** If $\Gamma = \Gamma^\text{un}_{\text{sym},k}$ set $c := 0$, and if $\Gamma = \Gamma^\text{un}_{\text{tr},k}$ set $c := 1$. Consider the hyperplane arrangement $\mathcal{H} := \{H_{\alpha,v} : \alpha \in \Phi^+\}$ with hyperplanes $H_{\alpha,v} := \{v \in V : \langle v, \alpha^\vee \rangle = \frac{c}{2}\}$.

First we claim that if for $w \in W$ and $u \in V$ the affine map $\varphi : v \mapsto w(v - u) + u$ is an automorphism of $\mathcal{H}$ which maps $P$ to $P$, then it is an automorphism of $\Gamma$ (by an automorphism of the hyperplane arrangement, we mean an invertible affine map such that $\varphi$ permutes the hyperplanes in $\mathcal{H}$). Indeed, observe that there is an edge in $\Gamma$ between $\lambda$ and $\mu$ if and only if there is some $\alpha \in \Phi^+$ such that $\mu = \lambda + \alpha$ and $\max(\{|\langle \mu, \alpha^\vee \rangle - \frac{c}{2}|, |\langle \lambda, \alpha^\vee \rangle - \frac{c}{2}|\}) \leq k(\alpha) + 1 - \frac{c}{2}$. So suppose there is an edge between $\lambda$ and $\mu$ in the $\alpha$ direction. Then note that any $\varphi$ of this form will satisfy $\varphi(\mu) - \varphi(\lambda) = w(\alpha)$ and $\varphi(H_{\alpha,v}) = H_{\pm w(\alpha)v,\frac{\varepsilon}{2}}$ (where the sign $\pm$ is chosen so that $\pm w(\alpha) \in \Phi^+$). Moreover, since all Weyl group elements are orthogonal, and in particular, preserve distances, the distance from $\mu$ to $H_{\alpha,v}$ will be the same as the distance from $\varphi(\mu)$ to $H_{\pm w(\alpha)v,\frac{\varepsilon}{2}}$, and ditto for $\lambda$. But $|\langle \mu, \alpha^\vee \rangle - \frac{c}{2}|$ is precisely the distance from $\mu$ to $H_{\alpha,v}$, and ditto for $\lambda$. Hence indeed we will get that $\varphi(\mu) = \varphi(\lambda) + w(\alpha)$ and that

$$\max \left( \left\{ \left| \langle \varphi(\mu), (\pm w(\alpha))^\vee \rangle - \frac{c}{2} \right|, \left| \langle \varphi(\lambda), (\pm w(\alpha))^\vee \rangle - \frac{c}{2} \right| \right\} \right)$$

$$\leq k(\alpha) + 1 - \frac{c}{2} = k(\pm w(\alpha)) + 1 - \frac{c}{2},$$

which means there is an edge in $\Gamma$ between $\varphi(\lambda)$ and $\varphi(\mu)$ in the $\pm w(\alpha)$ direction. To see that conversely if there is an edge between $\varphi(\lambda)$ and $\varphi(\mu)$ in $\Gamma$, there is one between $\lambda$ and $\mu$, use that $\varphi$ is invertible and $\varphi^{-1}$ is of the same form.

In the case $c = 0$, the hyperplane arrangement $\mathcal{H}$ is just the Coxeter arrangement of $\Phi$ and it is easy to see that every $w \in W$ is an automorphism of $\mathcal{H}$.

Now consider the case $c = 1$, in which case $\mathcal{H}$ is (a scaled version of) the $\Phi^\vee$-Linial arrangement; see for instance [65] and [7]. We claim that $\varphi : v \mapsto w(v - \frac{c}{2}\rho) + \frac{c}{2}\rho$ is an automorphism of $\mathcal{H}$ for all $w \in C$. So suppose $x \in H_{\alpha,v,\frac{\varepsilon}{2}}$; we want to show that $\varphi(x) \in H_{\pm w(\alpha)v,\frac{\varepsilon}{2}}$ where the sign $\pm$ is chosen so that $\pm w(\alpha)$ is positive. (The reverse implication will then follow from consideration of $\varphi^{-1} = w^{-1}(v - \frac{c}{2}\rho) + \frac{c}{2}\rho$.) We have
\[ \langle \varphi(x), w(\alpha)^\vee \rangle = \frac{c}{2} - \left\langle \frac{c}{h} \rho, \alpha^\vee \right\rangle + \left\langle \frac{c}{h} \rho, w(\alpha)^\vee \right\rangle. \] (4.1)

Write \( \alpha^\vee = \sum_{i=1}^{n} a_i \alpha_i^\vee \), with the convention \( a_0 := 0 \). By a result of Lam-Postnikov mentioned above, there is a permutation \( \pi : \{0, 1, \ldots, n\} \rightarrow \{0, 1, \ldots, n\} \) such that \( w(\alpha^\vee) = \alpha_{\pi(i)}^\vee \) (with the aforementioned convention \( \alpha_0^\vee := -\hat{\theta}^\vee \) where \( \hat{\theta}^\vee \) is the highest root of \( \Phi^\vee \)). Thus, \( w(\alpha)^\vee = \sum_{i=1}^{n} a_i \alpha_{\pi(i)}^\vee \).

We will consider two cases. First suppose that \( a_{\pi^{-1}(0)} = 0 \). Then \( w(\alpha)^\vee \) is clearly a positive root, so \( \pm = + \); moreover, we have \( \langle \frac{c}{h} \rho, \alpha^\vee \rangle = \langle \frac{c}{h} \rho, w(\alpha)^\vee \rangle = \frac{c}{h} \cdot \sum_{i=1}^{n} a_i \). So from (4.1) we get that \( \langle \varphi(x), w(\alpha)^\vee \rangle = \frac{c}{2} \), that is, \( \varphi(x) \in H_{\pm w(\alpha)^\vee, \frac{c}{2}} \), as desired.

Now suppose that \( a_{\pi^{-1}(0)} \neq 0 \). We claim that this forces \( a_{\pi^{-1}(0)} = 1 \): indeed, otherwise the height of \( w(\alpha)^\vee \) would be strictly less than \( -(h-1) \), which is impossible because \( -\hat{\theta}^\vee \) has height \( -(h-1) \) and is the root in \( \Phi^\vee \) of smallest height. So indeed we have \( a_{\pi^{-1}(0)} = 1 \). Note also that in this case the height of \( w(\alpha)^\vee \) is a negative root and hence \( w(\alpha)^\vee \) is negative, so \( \pm = - \). Then we compute

\[ -\left\langle \frac{c}{h} \rho, \alpha^\vee \right\rangle + \left\langle \frac{c}{h} \rho, w(\alpha)^\vee \right\rangle = -\left\langle \frac{c}{h} \rho, \alpha_{\pi^{-1}(0)}^\vee \right\rangle + \left\langle \frac{c}{h} \rho, \alpha_0^\vee \right\rangle = -\frac{c}{h} - \left( \frac{c}{h} (h-1) \right) = -c. \]

Thus from (4.1) we get that \( \langle \varphi(x), -w(\alpha)^\vee \rangle = -\frac{c}{2} + c = \frac{c}{2} \), that is, \( \varphi(x) \in H_{\pm w(\alpha)^\vee, \frac{c}{2}} \), as desired.

Finally, the description of \( C \) given above says that \( \varphi(0) = w(0 - \frac{c}{h} \rho) + \frac{c}{h} \rho = c\omega \) for some \( \omega \in \Omega^0_m \). Hence indeed \( \varphi \) maps \( P \) to \( P \). \( \square \)

### 4.3 Sinks of symmetric interval-firing

Recall that our overall strategy for proving confluence of interval-firing processes is to show that they get “trapped” inside certain permutohedra, and then to analyze where these processes must terminate. In order to carry out this strategy, we need to understand what are the possible final points we terminate at, i.e., what are the stable points of these processes.

In this section we describe the \( \Gamma_{\text{sym,k}} \)-stable points, i.e., the sinks of \( \Gamma_{\text{sym,k}} \). We
will show in particular that there is a way to consistently label the sinks of $\Gamma_{\text{sym}, k}$ across all values of $k$. This labeling comes from a certain injective map $\eta_k : P \rightarrow P$ that will be of crucial importance for us for much of the rest of the thesis.

In order to define $\eta_k$ we need to review some basic facts about parabolic subgroups and parabolic cosets. Recall that the Weyl group $W$ is generated by the simple reflections $s_i := s_{\alpha_i}$ for $i = 1, \ldots, n$. For any $w \in W$ we use $\ell(w)$ to denote the length of $w$, which is the length of the shortest representation of $w$ as a product of simple reflections. An inversion of $w$ is a positive root $\alpha \in \Phi^+$ for which $w(\alpha)$ is negative. The length $\ell(w)$ is equal to the number of inversions of $w$. The identity is the only Weyl group element of length zero. The simple reflections are the only Weyl group elements of length one: $s_i$ sends $\alpha_i$ to $-\alpha_i$ and permutes $\Phi^+ \setminus \{\alpha_i\}$. A (right) descent of $w \in W$ is a simple reflection $s_i$ such that $\ell(ws_i) < \ell(w)$. The reflection $s_i$ is a descent of $w$ if and only if $\alpha_i$ is an inversion of $w$.

Recall that for $I \subseteq [n]$ we use $W_I$ to denote the corresponding parabolic subgroup of $W$, that is, the subgroup of $GL(V)$ generated by simple reflections $s_i$ for $i \in I$. Note that $W_I$ is (isomorphic to) the Weyl group of $\Phi_I$. For $\lambda \in P$ we define the parabolic permutohedron $\Pi_I(\lambda) := \text{ConvexHull} W_I(\lambda)$ and $\Pi^Q_I(\lambda) := \Pi_I(\lambda) \cap (Q + \lambda)$. An important property of parabolic subgroups is the existence of distinguished coset representatives: each (left) coset $wW_I$ in $W$ contains a unique element of minimal length. We use $W^I$ for the set of minimal length coset representatives of $W_I$. There is even an explicit description: $W^I := \{w \in W : s_i \text{ is not a descent of } w \text{ for all } i \in I\}$ (see for instance [15, §2.4]).

Recall that for any $\lambda \in P$ we use $\lambda_{\text{dom}}$ to denote the dominant element of $W(\lambda)$. And for a dominant weight $\lambda = \sum_{i=1}^n c_i \omega_i \in P_{\geq 0}$, we define $I_\lambda^0 := \{i \in [n] : c_i = 0\}$.

**Proposition 4.3.1.** For $\lambda \in P_{\geq 0}$, the stabilizer of $\lambda$ in $W$ is $W_{I_\lambda^0}^\lambda$.

**Proof.** This (straightforward proposition) is [16, Lemma 10.2B].

**Corollary 4.3.2.** For any $\lambda \in P$, $\{w \in W : w^{-1}(\lambda) \in P_{\geq 0}\}$ is a coset of $W_{I_{\lambda_{\text{dom}}}^0}^{\lambda_{\text{dom}}}$.

**Proof.** First let us show that if $w^{-1}(\lambda)$ is dominant then $(ww')^{-1}(\lambda)$ is dominant for any $w' \in W_{I_{\lambda_{\text{dom}}}^0}$. This is clear: $(ww')^{-1}(\lambda) = (w')^{-1}(w^{-1}(\lambda)) = (w')^{-1}(\lambda_{\text{dom}}) = \lambda_{\text{dom}}$. 

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Figure 4-5: A graphical depiction of the piecewise-linear map $\eta_k$.

since $w'$ is in the stabilizer of $\lambda_{\text{dom}}$ by Proposition 4.3.1. Next let us show that if $w^{-1}(\lambda)$ is dominant and $(w')^{-1}(\lambda)$ is dominant then $w = w''w'$ for some $w'' \in W_{\lambda_{\text{dom}}}$. This is also clear: $w^{-1}(w'(\lambda_{\text{dom}})) = w^{-1}(\lambda) = \lambda_{\text{dom}}$, so $w^{-1}w'$ is in the stabilizer of $\lambda_{\text{dom}}$, that is, $w^{-1}w' = w''$ for some $w'' \in W_{\lambda_{\text{dom}}}$ thanks to Proposition 4.3.1 as claimed.

In light of Corollary 4.3.2 for $\lambda \in P$ we define $w_\lambda$ to be the minimal length element of $\{w \in W : w^{-1}(\lambda) \in P_{>0}\}$. Hence, for $\lambda \in P_{>0}$ we have (by the Orbit-Stabilizer Theorem) that $W_{\lambda_{\text{dom}}} = \{w_\mu : \mu \in W(\lambda)\}$ and $w_\mu \neq w_{\mu'}$ for $\mu \neq \mu' \in W(\lambda)$. Another way to think about $w_\lambda$: $\lambda$ may belong to the closure of many chambers, but there will be a unique chamber $wC_0$ with $w$ of minimal length such that $\lambda$ belongs to the closure of $wC_0$ and this is when $w = w_\lambda$. Then for $k \in \mathbb{N}[\Phi]^W$, we define the map $\eta_k : P \to P$ by setting $\eta_k(\lambda) := \lambda + w_\lambda(\rho_k)$ for all $\lambda \in P$ (where, as above, we have $\rho_k := \sum_{i=1}^n k(\alpha_i)\omega_i$).

This map $\eta_k$ will be of crucial importance for us in our investigation of both the symmetric and truncated interval-firing processes and the relationship between these two processes. Figure 4-5 gives a graphical depiction of $\eta_k$: as we can see, this map “dilates” space by translating the chambers radially outwards; a point not inside any chamber travels in the same direction as the chamber closest to the fundamental chamber among those chambers whose closure the point lies in. The following
proposition lists some very basic properties of \( \eta_k \).

**Proposition 4.3.3.**

- For any \( k, m \in \mathbb{N}[\Phi]^W \), we have \( \eta_{k+m} = \eta_m(\eta_k) \).
- For any \( k \in \mathbb{N}[\Phi]^W \), the map \( \eta_k : P \to P \) is injective.

**Proof.** For the first bullet point: let \( \lambda \in P \). Set \( \lambda' := \eta_k(\lambda) = \lambda + w_\lambda(\rho_k) \). Observe that \( \lambda'_\text{dom} = w_\lambda^{-1}(\eta_k(\lambda)) = \lambda_{\text{dom}} + \rho_k \). Hence, \( I^0_{\lambda'_\text{dom}} \subseteq I^0_{\lambda_{\text{dom}}} \). This means the cosets of \( W_{I^0_{\lambda'_{\text{dom}}}} \) are unions of cosets of \( W_{I^0_{\lambda_{\text{dom}}}} \). But we just saw that \( w_\lambda \in w_\lambda W_{I^0_{\lambda_{\text{dom}}}} \), because \( w_\lambda^{-1}(\lambda') \) is dominant. So \( w_\lambda \) must be the minimal length element of \( w_\lambda W_{I^0_{\lambda_{\text{dom}}}} \) (since it is the minimal length element of a superset of \( w_\lambda W_{I^0_{\lambda_{\text{dom}}}} \)). Hence \( w_{\lambda'} = w_\lambda \).

This means that \( \eta_m(\eta_k(\lambda)) = \lambda + w_\lambda(\rho_k) + w_\lambda(\rho_m) = \lambda + w_\lambda(\rho_{k+m}) = \eta_{k+m}(\lambda) \) and thus the claim is proved.

For the second bullet point: suppose \( \lambda, \mu \in P \) with \( \eta_k(\lambda) = \eta_k(\mu) \). First of all, since \( \eta_k(\lambda)_{\text{dom}} = \lambda_{\text{dom}} + \rho_k \) and similarly for \( \mu \), we have \( \lambda_{\text{dom}} = \mu_{\text{dom}} \). Let \( m \gg 0 \in \mathbb{Z} \) be some very large constant. From the first bullet point we know \( \eta_{k+m}(\lambda) = \eta_{k+m}(\mu) \) and hence \( \lambda + w_\lambda(\rho_{k+m}) = \mu + w_\mu(\rho_{k+m}) \). But \( \rho_{k+m} \) is inside the fundamental chamber \( C_0 \), and hence \( w(\rho_{k+m}) = w'(\rho_{k+m}) \) if and only if \( w = w' \). Moreover, by taking \( m \) large enough we can guarantee that \( w(\rho_{k+m}) \) and \( w'(\rho_{k+m}) \) are very far away from one another for \( w \neq w' \). Hence \( \lambda + w_\lambda(\rho_{k+m}) = \mu + w_\mu(\rho_{k+m}) \) in fact forces \( w_\lambda = w_\mu \). But \( w_\lambda = w_\mu \) together with \( \lambda_{\text{dom}} = \mu_{\text{dom}} \) means \( \lambda = \mu \) and thus the claim is proved.

Now we proceed to explain how \( \eta_k \) labels the sinks of \( \Gamma_{\text{sym},k} \).

For a dominant weight \( \lambda = \sum_{i=1}^n c_i \omega_i \in P_{\geq 0} \), define \( I^0_{\lambda} := \{ i \in [n] : c_i \in \{0, 1\} \} \).

**Proposition 4.3.4.** Let \( \lambda \in P \) with \( \langle \lambda, \alpha_i^\vee \rangle \neq -1 \) for all \( \alpha \in \Phi^+ \). Then \( w_\lambda(\Phi^+_{I^0_{\lambda_{\text{dom}}}}) \) is a subset of positive roots.

**Proof.** It suffices to show that \( w_\lambda(\alpha_i) \) is positive for all \( i \in I^0_{\lambda_{\text{dom}}} \). Suppose that \( w_\lambda(\alpha_i) \) is negative for some \( i \in I^0_{\lambda_{\text{dom}}} \), i.e., that \( s_i \) is a descent of \( w_\lambda \). Note \( \langle \lambda_{\text{dom}}, \alpha_i^\vee \rangle \in \{0, 1\} \).

If \( \langle \lambda_{\text{dom}}, \alpha_i^\vee \rangle = 1 \), then \( \langle \lambda_{\text{dom}}, -\alpha_i^\vee \rangle = -1 \) so \( \langle \lambda, -w_\lambda(\alpha_i^\vee) \rangle = -1 \), which contradicts
that $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$. But since $w_\lambda$ is the minimal length representative of $w_\lambda W^0_{\lambda_{dom}}$, it cannot have any descents $s_j$ with $j \in I^0_{\lambda_{dom}}$. Hence we cannot have that $\langle \lambda_{dom}, \alpha_i^\vee \rangle = 0$ either. Thus it must be that $w_\lambda(\alpha_i)$ is positive for all $i \in I^0_{\lambda_{dom}}$. \qed

**Proposition 4.3.5.** For a dominant weight $\mu \in P_{\geq 0}$, we have that

$$W^{\mu}_{\lambda_{dom}} = \{w_\lambda: \lambda \in P, \lambda_{dom} = \mu, \langle \lambda, \alpha^\vee \rangle \neq -1 \text{ for all } \alpha \in \Phi^+\}.$$

**Proof.** Let $\lambda \in P$ with $\lambda_{dom} = \mu$ and first suppose that $\langle \lambda, \alpha^\vee \rangle = -1$ for some $\alpha \in \Phi^+$. Then we have $\langle w_\lambda^{-1}(\lambda), w_\lambda^{-1}(\alpha)^\vee \rangle = -1$. But since $w_\lambda^{-1}(\lambda) = \lambda_{dom}$ is dominant, this means $w_\lambda^{-1}(\alpha)$ is a negative root; moreover, the only way $\langle \lambda_{dom}, w_\lambda^{-1}(\alpha)^\vee \rangle = -1$ is possible is if all the simple coroots $\alpha_i^\vee$ appearing in the expansion of $-w_\lambda^{-1}(\alpha)^\vee$ have $i \in I^0_{\lambda_{dom}}$. This implies that $w_\lambda(\alpha_i)$ is negative for some $i \in I^0_{\lambda_{dom}}$. But then $s_i$ would be a descent of $w_\lambda$, and hence $w_\lambda$ cannot be the minimal length element of $w_\lambda W^{0,1}_{\lambda_{dom}}$.

If $\lambda \in P$ with $\lambda_{dom} = \mu$ satisfies $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$, then we have seen in Proposition 4.3.4 that $w_\lambda$ has no descents $s_i$ with $i \in I^0_{\mu}$ and hence indeed $w_\lambda \in W^{\mu}_{\lambda_{dom}}$. On the other hand, since $W^\mu_{\mu} \subseteq W^{0,1}_{\mu}$, the cosets of $W^{0,1}_{\mu}$ are unions of cosets of $W^\mu_{\mu}$ and hence the minimal length element of any coset of $W^{0,1}_{\mu}$ must be of the form $w_\lambda$ for some $\lambda \in P$ with $\lambda_{dom} = \mu$. \qed

**Lemma 4.3.6.** For any $k \in \mathbb{N}[\Phi]^W$, the sinks of $\Gamma_{sym,k}$ are

$$\{\eta_k(\lambda): \lambda \in P, \langle \lambda, \alpha^\vee \rangle \neq -1 \text{ for all } \alpha \in \Phi^+\}$$

**Proof.** First suppose that $\lambda \in P$ satisfies $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$. Let $\alpha \in \Phi^+$. If $\alpha \in w_\lambda(\Phi^0_{\lambda_{dom}})$, then $\langle \eta_k(\lambda), \alpha^\vee \rangle = \langle \lambda_{dom} + \rho_\kappa, w_\lambda^{-1}(\alpha)^\vee \rangle \geq k(\alpha)$ since $w_\lambda^{-1}(\alpha) \in \Phi^+$ by Proposition 4.3.4. So now consider $\alpha \notin w_\lambda(\Phi^0_{\lambda_{dom}})$. Then $w_\lambda^{-1}(\alpha)$ may be positive or negative, but $|\langle \lambda_{dom}, w_\lambda(\alpha)^\vee \rangle| \geq 2$ (because $\lambda_{dom}$ has an $\omega_i$ coefficient of at least 2 for some $i \notin I^0_{\lambda_{dom}}$ such that $\alpha_i^\vee$ appears in the expansion of $\pm w_\lambda(\alpha)^\vee$). Hence

$$|\langle \eta_k(\lambda), \alpha^\vee \rangle| = |\langle \lambda_{dom} + \rho_\kappa, w_\lambda^{-1}(\alpha)^\vee \rangle| \geq k(\alpha) + 2,$$

which means that $\langle \eta_k(\lambda), \alpha^\vee \rangle \notin [-k(\alpha) - 1, k(\alpha) - 1]$. Thus $\eta_k(\lambda)$ is a sink of $\Gamma_{sym,k}$.
Now suppose \( \mu \) is a sink of \( \Gamma_{\text{sym},k} \). Since \( \langle \mu, \alpha^\vee \rangle \notin [-k(\alpha) - 1, k(\alpha) - 1] \) for \( \alpha \in \Phi^+ \), in particular \( |\langle \mu, \alpha^\vee \rangle| \geq k(\alpha) \) for all \( \alpha \in \Phi^+ \). This means \( \langle \mu_{\text{dom}}, \alpha^\vee \rangle \geq k(\alpha) \) for all \( \alpha \in \Phi^+ \). Hence \( \mu_{\text{dom}} = \mu' + \rho_k \) for some dominant \( \mu' \in P_{\geq0} \). Suppose to the contrary that \( w_\mu \) is not the minimal length element of \( w_\mu W_{\mu'}^{0,1} \). Then there exists a descent \( s_i \) of \( w_\mu \) with \( i \in I_{\mu'}^{0,1} \). But then

\[
\langle \mu, -w_\mu(\alpha_i)^\vee \rangle = \langle \mu_{\text{dom}}, -\alpha_i^\vee \rangle = -\langle \mu', \alpha_i^\vee \rangle - \langle \rho_k, \alpha_i^\vee \rangle \geq -k(\alpha_i) - 1,
\]

and also \( \langle \mu, -w_\mu(\alpha_i)^\vee \rangle = -\langle \mu_{\text{dom}}, \alpha_i^\vee \rangle \leq 0 \). This would imply that \( \mu \) is not a sink of \( \Gamma_{\text{sym},k} \), since \( -w_\mu(\alpha_i) \in \Phi^+ \). So \( w_\mu \) must be the minimal length element of \( w_\mu W_{\mu'}^{0,1} \).

Thanks to Proposition 4.3.5, this means \( w_\mu = w_\lambda \) for some \( \lambda \in P \) with \( \lambda_{\text{dom}} = \mu' \) and \( \langle \lambda, \alpha^\vee \rangle \neq -1 \) for all \( \alpha \in \Phi^+ \). Moreover, \( \mu = w_\mu(\mu_{\text{dom}}) = \lambda + w_\lambda(\rho_k) = \eta_k(\lambda) \), as claimed.

\[\blacksquare\]

### 4.4 Traverse lengths of permutohedra

Our goal will now be to describe the connected components of \( \Gamma_{\text{sym},k} \), with the eventual aim of establishing confluence of \( \rightarrow_{\text{sym},k} \). (By connected component of a digraph \( \Gamma \), we mean a connected component of its underlying undirected graph \( \Gamma_{\text{un}} \).) We will show over the course of the next several sections that the connected components are contained in certain permutohedra; from this confluence will follow easily. First we need to discuss traverse lengths.

**Definition 4.4.1.** For a root \( \alpha \in \Phi \), an \( \alpha \)-string of length \( \ell \) is a subset of \( P \) of the form \( \{\mu, \mu - \alpha, \mu - 2\alpha, \ldots, \mu - \ell\alpha\} \) for some weight \( \mu \in P \). For a dominant weight \( \lambda \in P_{\geq0} \), an \( \alpha \)-traverse in the discrete permutohedron \( \Pi^Q(\lambda) \) is a maximal (as a set) \( \alpha \)-string that belongs to \( \Pi^Q(\lambda) \). Concretely, it is \( \{\mu, \mu - \alpha, \mu - 2\alpha, \ldots, \mu - \ell\alpha\} \subseteq \Pi^Q(\lambda) \) such that \( \mu + \alpha, \mu - (\ell + 1)\alpha \notin \Pi^Q(\lambda) \). Finally, for a dominant weight \( \lambda \in P_{\geq0} \), the traverse length \( I_\lambda \in \mathbb{Z}[\Phi^W] \) is given by

\[
I_\lambda(\alpha) := \text{the minimal length } \ell \text{ of an } \alpha \text{-traverse in } \Pi^Q(\lambda).
\]
Clearly, by the $W$-symmetry of permutohedra, the traverse length is $W$-invariant and hence really does belong to $\mathbb{Z}[\Phi]^W$.

**Lemma 4.4.2.** For $\lambda \in P$ and $\alpha \in \Phi$, any $\alpha$-traverse $\{\mu, \mu - \alpha, \ldots, \mu - \ell\alpha\} \subseteq \Pi^Q(\lambda)$ is symmetric with respect to the reflection $s_\alpha$, i.e., $s_\alpha(\mu - i\alpha) = \mu - (\ell - i)\alpha$ for all $i = 0, \ldots, \ell$. Its length is $\ell = \langle \mu, \alpha^\vee \rangle$. In particular, $\langle \mu, \alpha^\vee \rangle \geq 0$.

**Proof.** By the $W$-symmetry of discrete permutohedra, we have $s_\alpha(\Pi^Q(\lambda)) = \Pi^Q(\lambda)$, which implies the first sentence. The second sentence then follows from

$$\mu - \ell\alpha = s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha.$$ 

The last sentence is clear because the length $\ell$ must be nonnegative. \hfill $\square$

Lemma 4.4.2 implies the following reformulation of the definition of $l_\lambda$.

**Corollary 4.4.3.** For $\lambda \in P$, the traverse length $l_\lambda$ is given by

$$l_\lambda(\alpha) = \min\{\langle \mu, \alpha^\vee \rangle : \mu \in \Pi^Q(\lambda), \mu + \alpha \notin \Pi^Q(\lambda)\}.$$ 

Corollary 4.4.3 explains the connection of traverse length to interval-firing: we are going to prove that interval-firing processes get “trapped” inside of permutohedra because the traverse lengths of these permutohedra are large (and hence if $\mu$ is inside such a permutohedron but $\mu + \alpha$ is not, $\langle \mu, \alpha^\vee \rangle$ must be so large that it is outside the fierability interval of our process). To do this we need a formula for traverse length. In most cases, the traverse length of a permutohedron in a given direction $\alpha$ is realized on some edge of the permutohedron conjugate to $\alpha$ under the Weyl group. However, there are some strange exceptions to this general rule, for which we need the concept of “funny” weights.

**Definition 4.4.4.** If $\Phi$ is simply laced, then there are no funny weights. So suppose $\Phi$ is not simply laced. Then there is a unique long simple root $\alpha_l$ and short simple root $\alpha_s$ with $\langle \alpha_l, \alpha_s^\vee \rangle \neq 0$. We say the dominant weight $\lambda = \sum_{i=1}^n c_i \omega_i \in P_{\geq 0}$ is funny if $c_s = 0$ and $c_l \geq 1$ and $c_i \geq c_l$ for all $i$ such that $\alpha_i$ is long.
Example 4.4.5. For example, with the numbering of simple roots as in Figure 3-2, if \( \Phi = B_n \) then \( \lambda = \sum_{i=1}^n c_i \omega_i \in P_{\geq 0} \) is funny if \( c_1, \ldots, c_{n-2} \geq c_{n-1} \geq 1 \) and \( c_n = 0 \). If \( \Phi = C_n \), then \( \lambda \) is funny if \( c_{n-1} = 0 \) and \( c_n \geq 1 \).

For a dominant weight \( \lambda = \sum_{i=1}^n c_i \omega_i \in P_{\geq 0} \), define \( m_\lambda \in \mathbb{Z}[\Phi]^W \) by setting

\[
m_\lambda(\alpha) := \min(\{c_i : \alpha \in W(\alpha_i)\}).
\]

Theorem 4.4.6. For a dominant weight \( \lambda \in P_{\geq 0} \), we have

\[
l_\lambda(\alpha) = \begin{cases} 
m_\lambda(\alpha) - 1 & \text{if } \alpha \text{ is long and } \lambda \text{ is funny}, \\
m_\lambda(\alpha) & \text{otherwise}. \end{cases}
\]

Proof. Let \( \lambda = \sum_{i=1}^n c_i \omega_i \in P_{\geq 0} \). The \( \alpha_i \)-traverse \( \{\lambda, \lambda - \alpha, \ldots, \lambda - \ell \alpha = s_i(\lambda)\} \), which is contained in the edge \([\lambda, s_i(\lambda)]\) of the permutohedron \( \Pi(\lambda) \), has length equal to \( \ell = \langle \lambda, \alpha_i^\vee \rangle = c_i \). By the \( W \)-symmetry of the traverse length (and because any root is \( W \)-conjugate to some simple root), it follows that \( l_\lambda \leq m_\lambda \).

We will show that in most of the cases (except the case with long roots and funny weights) we actually have \( l_\lambda = m_\lambda \). We need to show that the length of any \( \alpha \)-traverse in \( \Pi^Q(\lambda) \) is greater than or equal to \( m_\lambda(\alpha) \), i.e., for \( \mu \in \Pi^Q(\lambda) \) such that \( \mu + \alpha \notin \Pi^Q(\lambda) \), we have \( \langle \mu, \alpha^\vee \rangle \geq m_\lambda(\alpha) \).

If \( m_\lambda(\alpha) = 0 \), then we automatically get \( l_\lambda(\alpha) = m_\lambda(\alpha) = 0 \), because \( l_\lambda(\alpha) \). So let us assume that \( m_\lambda(\alpha) \geq 1 \).

Let \( \mu \in \Pi^Q(\lambda) \) be such that \( \mu + \alpha \notin \Pi^Q(\lambda) \). Since \( \mu + \alpha \in Q + \lambda \), we deduce that \( \mu + \alpha \notin \Pi(\lambda) \). This means that the line segment \([\mu, \mu + \alpha]\) must “exit” the permutohedron \( \Pi(\lambda) \) at some point \( v \in V \), i.e., there exists a unique point \( v = \mu + t\alpha \), where \( t \in \mathbb{R} \), with \( v \in \Pi(\lambda) \) but \( \mu + q\alpha \notin \Pi(\lambda) \) for any \( q > t \). We have \( 0 \leq t < 1 \).

Let \( F \) be the minimal (by inclusion) face of \( \Pi(\lambda) \) that contains the point \( v \). For any vertex \( \nu \) of the face \( F \), we have \( \nu \in \Pi(\lambda) \) and \( \nu + \alpha \notin \Pi(\lambda) \).

The minimal value of the linear form \( \langle \cdot, \alpha^\vee \rangle \) on the face \( F \) should be reached at a vertex \( \nu \) of \( F \). By the \( W \)-symmetry of \( \Pi(\lambda) \), we assume without loss of generality
that this minimum is achieved at $\nu = \lambda$. So we have $\langle \lambda, \alpha^\vee \rangle \leq \langle v, \alpha^\vee \rangle$.

Let $\alpha = \sum_{i=1}^n a_i \alpha_i$, where the $a_i$ are either all nonnegative or all nonpositive. Then we have $\alpha^\vee = \sum_{i=1}^n \tilde{a}_i \alpha_i$ where $\tilde{a}_i = \frac{\langle \alpha_i, \alpha \rangle}{\langle \alpha, \alpha \rangle} a_i$.

Any root $\alpha$ is $W$-conjugate to at least one simple root that appears with nonzero coefficient in its expansion in terms of the simple roots. So there exists an index $j$ such that $\alpha_j \in W(\alpha)$ and $a_j = \tilde{a}_j \neq 0$. We have $c_j \geq m_\lambda(\alpha) \geq 1$.

If $\lambda$ is strictly in the fundamental chamber, then all edges of $\Pi(\lambda)$ coming out of $\lambda$ must be in the direction of a negative simple root. This is not true for general $\lambda \in P_{\geq 0}$, but the edges of $\Pi(\lambda)$ coming out of $\lambda$ that are not in the direction of a negative simple root must immediately leave the dominant chamber. Hence if we let $x \in V$ be some generic point in the interior of the face $F$ very close to $\lambda$, by acting by $W_{I\lambda}$ we can transport $x$ to the dominant chamber while fixing $\lambda$. Thus, we may assume that the affine span of $F$ is spanned by simple roots. So let $I \subseteq [n]$ be the minimal set of indices such that the face $F$ belongs to the affine subspace $\lambda + \text{Span}_\mathbb{R}\{\alpha_i : i \in I\}$.

We have $\mu = v - t\alpha = (\lambda - \sum_{i \in I} b_i \alpha_i) - t\alpha$ for real numbers $0 \leq t < 1$ and $b_i \geq 0$, $i \in I$. Thus $\langle \mu, \alpha^\vee \rangle = \langle v, \alpha^\vee \rangle - t\langle \alpha, \alpha^\vee \rangle = \langle v, \alpha^\vee \rangle - 2t \geq \langle \lambda, \alpha^\vee \rangle - 2t > \langle \lambda, \alpha^\vee \rangle - 2$. Moreover, since both $\langle \mu, \alpha^\vee \rangle$ and $\langle \lambda, \alpha^\vee \rangle - 2$ are integers, and the first is strictly greater than the second, we get

$$\langle \mu, \alpha^\vee \rangle \geq \langle \lambda, \alpha^\vee \rangle - 1 = \left( \sum_{i=1}^n \tilde{a}_i c_i \right) - 1.$$  

We already noted that the last expression involves at least one nonzero term $\tilde{a}_j c_j$ such that $\alpha_j \in W(\alpha)$. So $\tilde{a}_j c_j \geq c_j \geq m_\lambda(\alpha)$ and thus $\langle \mu, \alpha^\vee \rangle \geq m_\lambda(\alpha) - 1$.

We need to prove just a slightly stronger inequality $\langle \mu, \alpha^\vee \rangle \geq m_\lambda(\alpha)$.
If $\sum_{\alpha_i \in W(\alpha)} \tilde{a}_i \geq 2$, we get

$$\langle \mu, \alpha^\vee \rangle \geq \sum_{i=1}^{n} \tilde{a}_i c_i - 1 \geq \sum_{\alpha_i \in W(\alpha)} \tilde{a}_i c_i \geq 2m_\lambda(\alpha) - 1 \geq m_\lambda(\alpha),$$

as needed. So we now assume that $\sum_{\alpha_i \in W(\alpha)} \tilde{a}_i = 1$.

Since $\alpha$ does not belong to the subspace spanned by the $\alpha_i$ for $i \in I$, there is $r \notin I$ such that $a_r \geq 1$.

If $a_r = 1$, then, from the fact that $\lambda - \mu = (\sum_{i \in I} b_i \alpha_i) + t\alpha$ belongs to the root lattice $Q$ and thus is an integer linear combination of the simple roots, we deduce that in fact $t \in \mathbb{Z}$ and thus $t = 0$. In this case get $\langle \mu, \alpha^\vee \rangle \geq \langle \lambda, \alpha^\vee \rangle \geq a_j c_j \geq m_\lambda(\alpha)$, as needed. So we now assume that $c_r \geq 2$.

Then note that $\alpha_r \notin W(\alpha)$, because we assumed $\sum_{\alpha_i \in W(\alpha)} \tilde{a}_i = \sum_{\alpha_i \in W(\alpha)} a_i = 1$.

If there is an index $q$ such that $a_q \notin W(\alpha)$, $\tilde{a}_q \geq 1$ and $c_q \geq 1$, we have

$$\langle \mu, \alpha^\vee \rangle = \left( \sum_{i=1}^{n} \tilde{a}_i c_i \right) - 1 \geq \tilde{a}_j c_j + \tilde{a}_q c_q - 1 \geq \tilde{a}_j c_j \geq m_\lambda(\alpha),$$

as needed.

The only possibility which is not covered by the above discussion is when:

1. There is exactly one nonzero term $a_j \alpha_j$ in the expansion $\alpha = \sum_{i=1}^{n} a_i \alpha_i$ such that $\alpha_j \in W(\alpha)$. For this term, $a_j = 1$ and $c_j = m_\lambda(\alpha) \geq 1$.

2. There is at least one more more nonzero term $a_i \alpha_i$ in that expansion. For all such terms, $\alpha_i \notin W(\alpha)$, $a_i \geq 2$, and $c_i = 0$.

We claim that these conditions imply that $\alpha$ is a long root. This is easy to check by hand for $\Phi = B_n$, $C_n$, or $G_2$. One does not need to check Type $F_4$ separately, because in this case there are two long simple roots and two short simple roots, but the expansion of $\alpha$ involves either only one short simple root or only one long simple root.

We leave it as an exercise for the reader to find a uniform root theoretic argument of the fact that conditions (1) and (2) above imply that $\alpha$ is long.
Also, we claim that conditions (1) and (2) above imply that \( \lambda \) is a funny weight. Indeed, it is a well-known and simple fact that for any root \( \alpha = \sum_{i=1}^{n} a_i \alpha_i \), the set of \( i \in [n] \) for which \( a_i \neq 0 \) must be a connected subset of the Dynkin diagram. Hence indeed the \( \alpha_j \) in condition (1) must be the long simple root \( \alpha_l \), and one of the \( \alpha_i \) in condition (2) must be the short simple root \( \alpha_s \) (with notation as in Definition 4.4.4).

Note also that \( m_\lambda(\alpha) = c_l \) forces \( c_i \geq c_l \) for all \( i \) such that \( \alpha_i \) is long.

In this “long and funny” case we can only get the (slightly) weaker inequality:

\[
\langle \mu, \alpha^\vee \rangle \geq m_\lambda(\alpha) - 1.
\]

It remains to show that this last inequality is tight in this “long and funny” case. Let us concentrate on the 2-dimensional face of the permutohedron \( \Pi(\lambda) \) contained in the affine subspace \( \lambda + \text{Span}_\mathbb{R}(\{\alpha_l, \alpha_s\}) \) (with notation as in Definition 4.4.4).

This face is equivalent to the 2-dimensional \( W' \)-permutohedron \( \Pi_{W'}(\lambda') \) corresponding to the sub-root system \( \Phi' \) of rank 2 with simple roots \( \alpha_l \) and \( \alpha_s \), and fundamental weights \( \omega'_1 \) (corresponding to \( \alpha_l \)) and \( \omega'_2 \) (corresponding to \( \alpha_s \)), where \( W' \) is the Weyl group of \( \Phi' \), and \( \lambda' = m_\lambda(\alpha)\omega'_1 + 0\omega'_2 \).

The 2-dimensional root system \( \Phi' \) which must be equal to either \( B_2 \) or \( G_2 \). In this situation there in fact is a \( \mu \in \Pi_Q^{O}(\lambda') \) with \( \mu + \alpha \notin \Pi_Q^{O}(\lambda') \) for some long \( \alpha \in \Phi' \) such that \( \langle \mu, \alpha^\vee \rangle = m_\lambda(\alpha) - 1 \): indeed, we can take \( \alpha := \alpha_l \) and \( \mu := (m_\lambda(\alpha) - 1)\omega'_1 \) for \( B_2 \) or \( \alpha := \alpha_l \) and \( \mu := (m_\lambda(\alpha) - 1)\omega'_1 + \omega'_2 \) for \( G_2 \).

This finishes the proof of the theorem.

\[ \square \]

### 4.5 The permutohedron non-escaping lemma

We need to place some restrictions on our parameter \( k \) so that funny weights do not occur in our analysis of the relevant permutohedra traverse lengths. For this we have the notion of “goodness.”

**Definition 4.5.1.** If \( \Phi \) is simply laced, then every \( k \in \mathbb{N}[\Phi]^W \) is good. So suppose \( \Phi \) is not simply laced and let \( k \in \mathbb{N}[\Phi]^W \). Then there exist \( k_s, k_l \in \mathbb{Z} \) with \( k(\alpha) = k_s \)
Figure 4-6: The graph $\Gamma_{\text{sym}, k}$ in Example \[4.5.3\]. The permutohedron $\Pi(\rho_k)$ is shown in red.

if $\alpha$ is short and $k(\alpha) = k_l$ if $\alpha$ is long. We say $k$ is good if $k_s = 0 \Rightarrow k_l = 0$. Note in particular that if $k = k \geq 0$ is constant, then it is good.

Now we can prove the following permutohedron non-escaping lemma, which says that certain discrete permutohedra “trap” the symmetric interval-firing process inside of them.

**Lemma 4.5.2.** Let $k \in \mathbb{N}[\Phi]^W$ be good and let $\Gamma := \Gamma_{\text{sym}, k}^{\text{un}}$. Let $\lambda \in P_{\geq 0}$. Then there is no directed edge $(\mu, \mu')$ in $\Gamma$ with $\mu \in \Pi^Q(\eta_k(\lambda))$ and $\mu' \notin \Pi^Q(\eta_k(\lambda))$.

**Proof.** First suppose $\Phi$ is not simply laced and $k_l = 0$. Then also $k_1 = 0$, i.e., $k = 0$, since $k$ is good. Hence $\rho_k = 0$, so $\eta_k(\lambda) = \lambda$. If $\mu \in \Pi^Q(\lambda)$ but $\mu + \alpha \notin \Pi^Q(\lambda)$, then by Corollary \[4.4.3\] we have $\langle \mu, \alpha^\vee \rangle \geq l_{\lambda}(\alpha)$. Note that by definition $l_{\lambda}(\alpha) \geq 0$. But this means $\langle \mu, \alpha^\vee \rangle \geq k(\alpha)$, so indeed $(\mu, \mu + \alpha)$ cannot be a directed edge of $\Gamma$.

Now suppose either $\Phi$ is simply laced or $\Phi$ is not simply laced but $k_s \geq 1$. Then note that $\rho_k$ is not funny. Hence by Theorem \[4.4.6\] we conclude that $l_{\eta_k(\lambda)}(\alpha) \geq k(\alpha)$. If $\mu \in \Pi^Q(\eta_k(\lambda))$ but $\mu + \alpha \notin \Pi^Q(\eta_k(\lambda))$, then $\langle \mu, \alpha^\vee \rangle \geq l_{\eta_k(\lambda)}(\alpha)$ by Corollary \[4.4.3\]. This means $\langle \mu, \alpha^\vee \rangle \geq k(\alpha)$, so indeed $(\mu, \mu + \alpha)$ cannot be a directed edge of $\Gamma$. $\square$

**Example 4.5.3.** Lemma \[4.5.2\] is false in general without the goodness assumption. Indeed, suppose $\Phi = B_2$ and $k \in \mathbb{N}[\Phi]^W$ is given by $k_s := 0$ and $k_l := 1$. Then Figure 4-6 depicts (a portion of) the graph $\Gamma_{\text{sym}, k}$. In this picture we only show elements of the root lattice $Q$. The permutohedron $\Pi(\rho_k) = \Pi(\omega_1)$ is shown in red. Observe that although $0 \in \Pi^Q(\rho_k)$ and $\alpha_1 \notin \Pi^Q(\rho_k)$, we have an edge $(0, \alpha_1)$ in $\Gamma_{\text{sym}, k}$.
We also need a “lower-dimensional” version of the permutohedron non-escaping lemma that says that these interval-firing processes get trapped inside of permutohedra of parabolic subgroups of $W$. This is established in the following lemma and corollary.

**Lemma 4.5.4.** Let $k \in \mathbb{N}[\Phi]^W$ and $\Gamma := \Gamma_{\mathrm{sym},k}^\mathrm{un}$. Let $\lambda \in P_{\geq 0}$. Then if $(\mu, \mu + \alpha)$ is a directed edge in $\Gamma$ with $\mu \in \Pi^Q_{\lambda,1}(\eta_k(\lambda))$, we have $\alpha \in \Phi_{\lambda,1}^\vee$.

**Proof.** Write $\eta_k(\lambda) = \sum_{i=1}^n c_i \omega_i$. Assume to the contrary that there exists an edge $(\mu, \mu + \alpha)$ in $\Gamma$ such that $\mu \in \Pi^Q_{\lambda,1}(\eta_k(\lambda))$ but $\alpha$ does not belong to $\text{Span}_\mathbb{R}\{\alpha_i : i \in I\}$.

Note that $\alpha$ is a root (positive or negative) with $-k(\alpha) - 1 \leq \langle \mu, \alpha^\vee \rangle \leq k(\alpha) + 1$.

Let $\beta = \pm \alpha \in \Phi^+$ be the positive root. Then $\langle \mu, \beta^\vee \rangle \leq k(\alpha) + 1 \leq k(\beta) + 1$.

Since the point $\mu$ belongs to $\Pi^Q_{\lambda,1}(\eta_k(\lambda))$, we deduce that the same inequality $\langle \nu, \beta^\vee \rangle \leq k(\beta) + 1$ holds for some vertex $\nu$ of $\Pi^Q_{\lambda,1}(\eta_k(\lambda))$. We have $\nu = w(\eta_k(\lambda))$ where $w \in W_{\lambda,1}$. Hence we have that $\langle w(\lambda), \beta^\vee \rangle = \langle \lambda, w^{-1}(\beta)^\vee \rangle \leq k(\beta) + 1$ for some $w \in W_{\lambda,1}$.

The action of the parabolic subgroup $W_{\lambda,1}$ on $\beta^\vee$ does not change the coefficients $b_j$ of the expansion $\beta^\vee = \sum_{i=1}^n b_i \alpha_i^\vee$ for all $j \notin I$, and at least one of these coefficients $b_j$ should be strictly positive (because $\beta^\vee$ is a positive coroot that does not belong to $\text{Span}_\mathbb{R}\{\alpha_i : i \in I\}$). So the expansion $w^{-1}(\beta)^\vee = \sum_{i=1}^n b_i' \alpha_i^\vee$ contains some strictly positive coefficient, which means that $w^{-1}(\beta)^\vee$ is a positive coroot and thus we have $b_i' \geq 0$ for all $i$.

Moreover, any coroot is $W$-conjugate to some simple coroot that appears in its expansion with nonzero coefficient. These observations mean that we can find $j \notin I$ such that $b_j' = b_j \geq 1$, and also (possibly the same) $i$ such that $b_i' \geq 1$ and $\alpha_i \in W(\alpha)$. Note that for this $i$ we have $k(\alpha_i) = k(w^{-1}(\beta)) = k(\beta) = k(\alpha)$.

If $i = j$, we get $\langle \lambda, w^{-1}(\beta)^\vee \rangle \geq \langle \lambda, b_j' \alpha_j^\vee \rangle \geq \langle \lambda, \alpha_j^\vee \rangle = c_j \geq k(\alpha_j) + 2 = k(\alpha) + 2$ (because for $j \notin I$, $c_j \geq k(\alpha_j) + 2$). But this contradicts $\langle \lambda, w^{-1}(\beta)^\vee \rangle \leq k(\alpha) + 1$.

On the other hand, if $i \neq j$, we get

$$\langle \lambda, w^{-1}(\beta)^\vee \rangle \geq \langle \lambda, b_i' \alpha_i^\vee + b_j' \alpha_j^\vee \rangle \geq \langle \lambda, \alpha_i^\vee \rangle + \langle \lambda, \alpha_j^\vee \rangle = c_i + c_j$$
\[\geq k(\alpha_i) + (k(\alpha_j) + 2) \geq k(\alpha_i) + 2 = k(\alpha) + 2.\]

Again, we get a contradiction. \(\square\)

**Corollary 4.5.5.** Let \(k \in \mathbb{N}[\Phi]^W\) be good and \(\Gamma := \Gamma_{\text{sym}, k}^\text{un}\). Let \(\lambda \in P_{\geq 0}\). Then there is no directed edge \((\mu, \mu')\) in \(\Gamma\) with \(\mu \in \Pi^Q_{\lambda_1} (\eta_k(\lambda))\) and \(\mu' \notin \Pi^Q_{\lambda_1} (\eta_k(\lambda))\).

**Proof.** This follows by combining Lemmas 4.5.2 and 4.5.4: if we have a directed edge \((\mu, \mu + \alpha)\) with \(\mu \in \Pi^Q_{\lambda_1} (\eta_k(\lambda))\), then \(\alpha \in \Phi^+\) by Lemma 4.5.4; hence this firing move is equivalent (via projection) to the same move for the sub-root system \(\Phi^+\); so by Lemma 4.5.2 applied to that sub-root system, we have \(\mu + \alpha \in \Pi^Q_{\lambda_1} (\eta_k(\lambda))\). \(\square\)

### 4.6 Confluence of symmetric interval-firing

Now, as promised, we are ready to show that connected components of \(\Gamma_{\text{sym}, k}\) are contained inside permutohedra.

**Theorem 4.6.1.** Let \(k \in \mathbb{N}[\Phi]^W\) be good. Let \(\lambda \in P\) with \(\langle \lambda, \alpha^\vee \rangle \neq -1\) for all \(\alpha \in \Phi^+\). Let \(Y_\lambda := \{\mu \in P : \mu \xrightarrow{\mathcal{S}} \eta_k(\lambda)\}\) be the connected component of \(\Gamma_{\text{sym}, k}\) containing the sink \(\eta_k(\lambda)\). Then \(Y_\lambda\) is contained in \(w_\lambda \Pi^Q_{\lambda_1} (\eta_k(\lambda_{\text{dom}}))\).

**Proof.** First suppose that \(\lambda\) is dominant. By Corollary 4.5.5 there is no edge \((\mu, \mu')\) in \(\Gamma_{\text{sym}, k}\) where one of \(\mu, \mu'\) is in \(\Pi^Q_{\lambda_1} (\eta_k(\lambda))\) and the other is not, which implies that \(Y_\lambda\) is contained in \(\Pi^Q_{\lambda_1} (\eta_k(\lambda))\). Now suppose \(\lambda\) is not dominant. By the preceding argument, the result is true for \(\lambda_{\text{dom}}\). But then we have \(Y_\lambda = w_\lambda Y_{\lambda_{\text{dom}}}\) by the \(W\)-symmetry of \(\Gamma_{\text{sym}, k}^\text{un}\), i.e., by Theorem 4.2.1. \(\square\)

**Corollary 4.6.2.** Let \(k \in \mathbb{N}[\Phi]^W\) be good. Then \(\xrightarrow{\text{sym}, k}\) is confluent (and terminating).

**Proof.** We already saw in Proposition 4.1.3 that \(\xrightarrow{\text{sym}, k}\) is terminating. Thus, every connected component of \(\Gamma_{\text{sym}, k}\) contains at least one sink, and \(\xrightarrow{\text{sym}, k}\) is confluent as long as every connected component contains a unique sink.

So suppose that two sinks belong to the same connected component of \(\Gamma_{\text{sym}, k}\). By Lemma 4.3.6, we know that these sinks must be of the form \(\eta_k(\lambda)\) and \(\eta_k(\lambda')\) for \(\lambda, \lambda' \in P\) with \(\langle \lambda, \alpha^\vee \rangle \neq -1\) and \(\langle \lambda, \alpha^\vee \rangle \neq -1\) for all \(\alpha \in \Phi^+\).
By Theorem 4.6.1, \( \eta_k(\lambda) \in w_\lambda \Pi^Q_{\lambda_{dom}}(\eta_k(\lambda_{dom}')) \) and vice-versa. In particular we have that \( \eta_k(\lambda_{dom}) \in \Pi^Q(\eta_k(\lambda_{dom}')) \) and \( \eta_k(\lambda'_{dom}) \in \Pi^Q(\eta_k(\lambda_{dom})) \). Proposition 3.1.2 then says that \( \eta_k(\lambda_{dom}) - \eta_k(\lambda'_{dom}) \) and \( \eta_k(\lambda'_{dom}) - \eta_k(\lambda_{dom}) \) are both in \( Q_{\geq 0} \), which is possible only if \( \eta_k(\lambda_{dom}) = \eta_k(\lambda'_{dom}) \). That is, thanks to the injectivity of \( \eta_k \) established in Proposition 4.3.3 we must have \( \lambda_{dom} = \lambda'_{dom} \).

But then the fact that \( \eta_k(\lambda) \in w_\lambda \Pi^Q_{\lambda_{dom}}(\eta_k(\lambda_{dom})) \) means that \( \eta_k(\lambda) \) is a vertex of \( w_\lambda \Pi^Q_{\lambda_{dom}}(\eta_k(\lambda_{dom})) \), i.e., \( \eta_k(\lambda) = w_\lambda w(\eta_k(\lambda_{dom})) \) for some \( w \in W_{\lambda_{dom}} \). Note that \( (w_\lambda w)^{-1}(\eta_k(\lambda)) \) is dominant. We have seen in the the proof of Proposition 4.3.3 that this means \( (w_\lambda w)^{-1}(\lambda) \) is dominant as well, or in other words, that \( w_\lambda w = w_\lambda w' \) for some \( w' \in W_{\lambda_{dom}} \). This shows that \( w_\lambda \in W_{\lambda_{dom}} \). By Proposition 4.3.5, \( w_\lambda \) and \( w_\lambda' \) must both be the minimal length elements of the cosets of \( W_{\lambda_{dom}} \) they belong to. So \( w_\lambda = w_{\lambda'} \). That \( \lambda_{dom} = \lambda'_{dom} \) and \( w_\lambda = w_{\lambda'} \) implies that \( \lambda = \lambda' \), and consequently that \( \eta_k(\lambda) = \eta_k(\lambda') \), as required. \( \square \)

**Remark 4.6.3.** As far as we know, Theorem 4.6.1 and Corollary 4.6.2 may be true even in the case where \( k \) is not good. Indeed, it appears that \( \rightarrow_{\text{sym,k}} \) is confluent for all \( k \in \mathbb{N}[\Phi]^W \) and to prove this it would be sufficient, thanks to the diamond lemma (Lemma 1.1.1), to prove it for root systems of rank 2, of which there are only four: \( A_1 \times A_1, A_2, B_2, G_2 \). All \( k \) are good for simply laced root systems, so in fact one would need only check \( B_2 \) and \( G_2 \).

### 4.7 Full-dimensional components, saturated components, and Cartan matrix chip-firing as a limit

Let \( k \in \mathbb{N}[\Phi]^W \) be good. For \( \lambda \in P \), recall the notation \( Y_\lambda := \{ \mu \in P : \mu \rightarrow_{\text{sym,k}} \eta_k(\lambda) \} \) for the connected component of \( \Gamma_{\text{sym,k}} \) containing the sink \( \eta_k(\lambda) \) from the last section. By the results of the last section, all these components are distinct. In this section, we take a moment to highlight certain special components \( Y_\lambda \), namely:

- those which are *full-dimensional* in the sense that their affine hulls are the whole vector space: \( \text{AffineHull} Y_\lambda = V \);
those which are full-dimensional and saturated in the sense that they contain all lattice points in their convex hulls: \( Y_\lambda = (\text{ConvexHull} Y_\lambda) \cap (Q + \eta_k(\lambda)) \).

For the full-dimensional components: by a result we will prove later (Corollary \ref{5:2:2}), we have that \( Y_\lambda \) always contains \( W(\eta_k(\lambda_{\text{dom}})) \) for \( \lambda \in P_{\geq 0} \) with \( I^{0,1}_\lambda = [n] \). Hence by Theorem \ref{4:6:1} we see that the full-dimensional connected components of \( \Gamma_{\text{sym},k} \) are exactly \( Y_\lambda \) for \( \lambda \in P_{\geq 0} \) with \( I^{0,1}_\lambda = [n] \), i.e., those \( \lambda = \sum_{i=1}^n c_i \omega_i \in P \) with \( c_i \in \{0, 1\} \) for all \( i \in [n] \). Clearly there are \( 2^n \) such full-dimensional components.

For the full-dimensional and saturated components: by that same Corollary \ref{5:2:2} we see that \( Y_\lambda \) being full-dimensional and saturated is equivalent to having this component satisfy \( Y_\lambda = \Pi^Q(\eta_k(\lambda_{\text{dom}})) \). And then we have the following:

**Proposition 4.7.1.** Let \( k \in \mathbb{N}[\Phi]^W \) be good. Let \( \lambda \in P \) with \( \langle \lambda, \alpha^\vee \rangle \neq -1 \) for all \( \alpha \in \Phi^+ \). Let \( Y_\lambda := \{ \mu \in P : \mu \rightarrow_{\text{sym},k} \eta_k(\lambda) \} \) be the connected component of \( \Gamma_{\text{sym},k} \) containing the sink \( \eta_k(\lambda) \). Then \( Y_\lambda \) is equal to \( \Pi^Q(\eta_k(\lambda)) \) if and only if \( \lambda \in \Omega_0^0 \).

**Proof.** First note that if \( \lambda \) is a sink of \( \Gamma_{\text{sym},k} \) then so is \( \lambda_{\text{dom}} \) and by the confluence of \( \rightarrow_{\text{sym},k} \) there cannot be two sinks in a single connected component of \( \Gamma_{\text{sym},k} \), so it suffices to prove this proposition for dominant \( \lambda \in P_{\geq 0} \) with \( I^{0,1}_\lambda = [n] \). (Observe that if \( \lambda \in \Omega_0^0 \) then certainly it is of this form.)

By the polytopal characterization of minuscule weights, there exists a dominant weight \( \mu \in P_{\geq 0} \) with \( \mu \in \Pi^Q(\lambda) \) but \( \mu \neq \lambda \) if and only if \( \lambda \notin \Omega_0^0 \). Hence by Proposition \ref{3:1:2} there exists \( \mu \in P_{\geq 0} \) with \( \eta_k(\mu) \in \Pi^Q(\eta_k(\lambda)) \) but \( \eta_k(\mu) \neq \eta_k(\lambda) \) if and only if \( \lambda \notin \Omega_0^0 \). By applying \( W \), we see that there is a sink \( \eta_k(\mu) \) of \( \Gamma_{\text{sym},k} \) with \( \eta_k(\mu) \in \Pi^Q(\eta_k(\lambda)) \) but \( \eta_k(\mu) \notin W(\eta_k(\lambda)) \) if and only if \( \lambda \notin \Omega_0^0 \). Finally, by the permutohedron non-escaping lemma, Lemma \ref{4:5:2}, this means precisely that \( \Pi^Q(\eta_k(\lambda)) \) is its own connected component if and only if \( \lambda \in \Omega_0^0 \). \( \square \)

**Remark 4.7.2.** Proposition \ref{4:7:1} fails when \( k \) is not good, as can be seen in Example \ref{4:5:3} above: in this example, \( 0 \in \Pi^Q(\rho_k) \) but \( 0 \) does not belong to the connected component of \( \Gamma_{\text{sym},k} \) containing \( \rho_k = \eta_k(0) = \omega_1 \).

So we see that the full-dimensional and saturated components of \( \Gamma_{\text{sym},k} \) are exactly the \( Y_\omega \) for \( \omega \in \Omega_0^0 \). There are \( f \) of these, where we recall that \( f := \#P/Q \) is the
index of connection of $\Phi$. We mentioned earlier that $P/Q$ is the “sandpile group” in our setting, and in fact we have that $P/Q \simeq \text{coker}(C^t)$, where $C$ is the Cartan matrix of $\Phi$. Hence, these full-dimensional and saturated components suggest that interval-firing may possibly be connected to Cartan matrix chip-firing.

Indeed, let us conclude this section by explaining how Cartan matrix chip-firing (which, as mentioned, has been investigated by Benkart-Klivans-Reiner [13]) can be realized as a certain “limit” of symmetric interval-firing. Note that a Cartan matrix is always an M-matrix (see [13, Proposition 4.1]). By associating to each integer vector $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ the weight $\lambda = \sum_{i=1}^{n} c_i \omega_i \in P$, we can view Cartan matrix chip-firing as the relation $\rightarrow C$ on $P$ defined by

$$\lambda \rightarrow C \lambda$$

for $\lambda \in P$ and simple root $\alpha_i$, $i \in [n]$ with $\langle \lambda, \alpha_i^\vee \rangle \geq 2$.

For $\lambda = \sum_{i=1}^{n} c_i \omega_i \in P$ and $k \in \mathbb{N}$ set $B_k(\lambda) := \{ \sum_{i=1}^{n} c'_i \omega_i \in P : \sum_{i=1}^{n} |c_i - c'_i| \leq k \}$.

In other words, $B_k(\lambda)$ consists of those $\mu$ which are within distance $k$ of $\lambda$ in the “taxicab distance” on $P$. Note that for all $\lambda \in B_k(\rho_k)$, we have that $\langle \lambda, \alpha^\vee \rangle \geq k$ if $\alpha \in \Phi^+$ is not a simple root. In other words, for $\lambda \in B_k(\rho_k)$, if $\lambda \rightarrow C \lambda + \alpha$, then $\alpha = \alpha_i$ is some simple root. Moreover, for $\lambda \in B_k(\rho_k)$ we have $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for any simple root $\alpha_i$. Hence, for $\lambda \in B_k(\rho_k)$ the symmetric interval-firing relation reduces to

$$\lambda \rightarrow \text{sym}_k \lambda + \alpha_i$$

for a simple root $\alpha_i$, $i \in [n]$ with $\langle \lambda, \alpha_i^\vee \rangle \leq k - 1$.

Define $\Psi_k : P \rightarrow P$ by $\Psi_k(\lambda) := -\lambda + \rho_{k+1}$ (thus $\Psi_k$ is just a “reflection plus translation”). Then for $\lambda \in \Psi_k^{-1}(B_k(\rho_k)) = B_k(\rho)$ we have

$$\Psi_k(\lambda) \rightarrow \text{sym}_k \Psi_k(\lambda - \alpha_i)$$

for a simple root $\alpha_i$, $i \in [n]$ with $\langle \Psi_k(\lambda), \alpha_i^\vee \rangle \geq 2$.

Therefore the restriction of $\Psi_k^{-1}(\Gamma_{\text{sym},k})$ to $B_k(\rho)$ is exactly the same as the restriction of $\Gamma_{C,\rightarrow}$ to $B_k(\rho)$. But every $\lambda \in P$ belongs to $B_k(\rho)$ as $k \rightarrow \infty$. In this way, we recover Cartan matrix chip-firing as a certain $k \rightarrow \infty$ limit of symmetric interval-firing.
Benkart-Klivans-Reiner [13, Theorem 1.1] showed that the recurrent configurations for Cartan matrix chip-firing are \( \rho - \omega \) for \( \omega \in \Omega^0_m \). Observe \( \Psi_k(\rho - \omega) = \eta_k(\omega) \), so these recurrent configurations correspond exactly to the sinks of our full-dimensional and saturated components. In the same way, the \( 2^n \) stable configurations in \( \mathbb{N}^n \) for Cartan matrix chip-firing correspond to the sinks of our full-dimensional components.

We should stress, however, that confluence is much easier to establish for Cartan matrix chip-firing than for our interval-firing processes: for Cartan matrix chip-firing, confluence holds locally, which ultimately has to do with the fact that simple roots are pairwise non-acute. On the other hand, when firing arbitrary positive roots, confluence need not hold locally because two positive roots may form an acute angle. Hence while Cartan matrix chip-firing describes the limiting behavior of our interval-firing process, it does not explain why the system is confluent from every initial point. Indeed, we could have also obtained Cartan matrix chip-firing by taking the same \( k \to \infty \) limit of the process which has \( \lambda \to \lambda + \alpha \) for \( \lambda \in P, \alpha \in \Phi^+ \) when \( \langle \lambda, \alpha^\vee \rangle + 1 \in [-k + 2, k] \), but that process is not confluent.

### 4.8 Confluence of truncated interval-firing

So far in this chapter we have mostly focused on symmetric interval-firing. We now finally turn to truncated interval-firing. In this section we prove the confluence of \( \Gamma_{\text{tr},k} \).

Let us start by describing the sinks of \( \Gamma_{\text{tr},k} \).

**Lemma 4.8.1.** For any \( k \in \mathbb{N}[\Phi]^W \), the sinks of \( \Gamma_{\text{tr},k} \) are \( \{ \eta_k(\lambda) : \lambda \in P \} \).

**Proof.** Let \( \lambda \in P \). Let \( \alpha \in \Phi^+ \). Note that since \( w_\lambda \in W_{I_{\lambda_{\text{dom}}}^0} \), \( w_\lambda \) does not have a descent \( s_i \) with \( I_{\lambda_{\text{dom}}}^0 \) and thus \( w_\lambda \) has no inversions in \( \Phi_{I_{\lambda_{\text{dom}}}^0} \). Thus if \( \alpha \in w_\lambda(\Phi_{I_{\lambda_{\text{dom}}}^0}) \), then \( \langle \eta_k(\lambda), \alpha^\vee \rangle = \langle \lambda_{\text{dom}} + \rho_k, w_\lambda^{-1}(\alpha)^\vee \rangle \geq k(\alpha) \), since \( w_\lambda^{-1}(\alpha) \in \Phi^+ \). So now consider \( \alpha \notin w_\lambda(\Phi_{I_{\lambda_{\text{dom}}}^0}) \). Then \( w_\lambda^{-1}(\alpha) \) may be positive or negative, but \( |\langle \lambda_{\text{dom}}, w_\lambda(\alpha)^\vee \rangle| \geq 1 \) (because \( \lambda_{\text{dom}} \) has an \( \omega_i \) coefficient of at least 1 for some \( i \notin I_{\lambda_{\text{dom}}}^0 \) such that \( \alpha_i^\vee \) appears in the expansion of \( \pm w_\lambda(\alpha)^\vee \)). Hence

\[
|\langle \eta_k(\lambda), \alpha^\vee \rangle| = |\langle \lambda_{\text{dom}} + \rho_k, w_\lambda^{-1}(\alpha)^\vee \rangle| \geq k(\alpha) + 1,
\]
which means that $\langle \eta_k(\lambda), \alpha^\vee \rangle \notin [-k(\alpha), k(\alpha) - 1]$. So indeed $\eta_k(\lambda)$ is a sink of $\Gamma_{tr,k}$.

Now suppose $\mu$ is a sink of $\Gamma_{tr,k}$. Since $\langle \mu, \alpha^\vee \rangle \notin [-k(\alpha), k(\alpha) - 1]$ for all $\alpha \in \Phi^+$, in particular $|\langle \mu, \alpha^\vee \rangle| \geq k(\alpha)$ for all $\alpha \in \Phi^+$. This means that $\langle \mu_{dom}, \alpha^\vee \rangle \geq k(\alpha)$ for all $\alpha \in \Phi^+$. Hence $\mu_{dom} = \mu' + \rho_k$ for some dominant $\mu' \in P_{\geq 0}$. Suppose to the contrary that $w_\mu$ is not the minimal length element of $w_\mu W_{I_0^\mu}$. Then there exists a descent $s_i$ of $w_\mu$ with $i \in I_0^\mu$. But then

$$\langle \mu, -w_\mu(\alpha_i)^\vee \rangle = \langle \mu_{dom}, -\alpha_i^\vee \rangle = -\langle \mu', \alpha_i^\vee \rangle - \langle \rho_k, \alpha_i^\vee \rangle \geq -k(\alpha_i),$$

and also $\langle \mu, -w_\mu(\alpha_i)^\vee \rangle = -\langle \mu_{dom}, \alpha_i^\vee \rangle \leq 0$. This would mean $\mu$ is not a sink of $\Gamma_{tr,k}$, since $-w_\mu(\alpha_i) \in \Phi^+$. So $w_\mu$ must be the minimal length element of $w_\mu W_{I_0^\mu}$. This means $w_\mu = w_\lambda$ for some $\lambda \in P$ with $\lambda_{dom} = \mu'$. And $\mu = w_\mu(\mu_{dom}) = \lambda + w_\lambda(\rho_k) = \eta_k(\lambda)$, as claimed.

We now proceed to prove the confluence of truncated interval-firing. In some sense our proof of confluence here is less satisfactory than the one for symmetric interval-firing because we heavily rely on the diamond lemma, and reduction to rank 2, which is a kind of “trick” that obscures the underlying polytopal geometry (and requires us at one point to use the classification of rank 2 root systems). But we also do crucially use the permutohedron non-escaping lemma in the following lemma, which says that “small” permutohedra close to the origin are connected components of truncated interval-firing.

**Lemma 4.8.2.** Let $k \in \mathbb{N}[\Phi]^W$ be good. Then for all $\omega \in \Omega_0^n$, the (translated) discrete permutohedron $\Pi^Q(\rho_k) + \omega$ is a connected component of $\Gamma_{tr,k}$ and the unique sink of this connected component is $\rho_k + \omega$.

**Proof.** First let us prove a preliminary result: for any $\lambda \in P$ and $\omega \in \Omega_0^n$, we have that $(\lambda - \omega)_{dom} = \lambda_{dom} - w(w^{-1}_\lambda(\omega))$ for some $w \in W_{I_{\lambda_{dom}}^0}$. Indeed, since $\omega$ is minuscule or zero, $\langle -w'(\omega), \alpha^\vee \rangle \in \{-1, 0, 1\}$ for any $\alpha \in \Phi$ and $w' \in W$. So, $w^{-1}_\lambda(\lambda - \omega) = \lambda_{dom} - w^{-1}_\lambda(\omega)$ may not be dominant, but the only $\alpha_i$ for which $\langle \lambda_{dom} - w^{-1}_\lambda(\omega), \alpha_i^\vee \rangle < 0$ must have $i \in I_{\lambda_{dom}}^0$. Hence, if we let $w \in W_{I_{\lambda_{dom}}^0}$ be such that $\langle w(w^{-1}_\lambda(\omega)), \alpha_i^\vee \rangle \geq 0$ for all $i \in I_{\lambda_{dom}}^0$, then we will have $(\lambda - \omega)_{dom} = \lambda_{dom} - w(w^{-1}_\lambda(\omega))$ as claimed.
Now let us show that for any \( \omega \in \Omega_m^{0} \), the only sink of \( \Gamma_{\text{tr},k} \) in \( \Pi_{\rho_k}^Q + \omega \). Suppose \( \eta_k(\lambda) \in \Pi_{\rho_k}^Q + \omega \) for some \( \lambda \in P \). This means \( \eta_k(\lambda) - \omega \in \Pi_{\rho_k}^Q \), which means that \( (\eta_k(\lambda) - \omega)_{\text{dom}} = \lambda_{\text{dom}} + \rho_k - w(\lambda(\omega)) \in \Pi_{\rho_k}^Q \) for some \( w \in W_{\rho_k} \) (we are using that \( w_{\eta_k(\lambda)} = w_\lambda \), which we have seen before, and that \( W_{\rho_k} \subseteq W_{\rho_k}^\lambda \)). Hence Proposition 3.1.2 tells us that

\[
\rho_k - (\lambda_{\text{dom}} + \rho_k - w(\lambda(\omega))) = -(\lambda_{\text{dom}} - \omega) + (w(\lambda(\omega)) - \omega) \in Q_{\geq 0}.
\]

Now, since \( \lambda_{\text{dom}} \in (Q + \omega) \cap P_{\geq 0} \), we know that \( \lambda_{\text{dom}} - \omega \in Q_{\geq 0} \) (by one characterization of minuscule weights mentioned in Chapter 3). Also, \( \omega - w(\lambda(\omega)) \in Q_{\geq 0} \) by Proposition 3.1.2. Hence we conclude that \( \lambda_{\text{dom}} = \omega \) and \( w(\lambda(\omega)) = \omega \). But since \( w \in W_{\rho_k} \), we have that \( w(\lambda(\omega)) = w_\lambda(\omega) \), and thus \( w_\lambda(\omega) = \omega \), which forces \( w_\lambda \) to be the identity, i.e., we have \( \lambda = \omega \). So indeed the only sink of \( \Gamma_{\text{tr},k} \) in \( \Pi_{\rho_k}^Q + \omega \) is \( \rho_k + \omega \).

Let us prove the lemma first for \( \omega = 0 \). Since \( \rightarrow_{\text{tr},k} \) is terminating by Proposition 4.1.3 any \( \rightarrow_{\text{tr},k} \)-firing sequence starting at some \( \mu \in \Pi_{\rho_k}^Q \) has to terminate somewhere. By the permutohedron non-escaping lemma, Lemma 4.5.2, such a sequence must terminate somewhere inside \( \Pi_{\rho_k}^Q \); and since \( \rho_k \) is the only sink in \( \Pi_{\rho_k}^Q \), it must terminate at \( \rho_k \). So indeed \( \Pi_{\rho_k}^Q \) is a connected component of \( \Gamma_{\text{tr},k} \).

Finally, let \( \omega \in \Omega_m \) be arbitrary, and let \( w \in C \) be the element corresponding to \( \omega \) under the isomorphism \( C \simeq P/Q \). Then by the description of this isomorphism given above, \( w(0 - \rho/h) + \rho/h = \omega \), and hence \( w(\Pi_{\rho_k}^Q - \rho/h) + \rho/h = \Pi_{\rho_k}^Q + \omega \). So from the symmetry of \( \Gamma_{\text{tr},k}^{\text{in}} \) described in Theorem 4.2.1 we get that \( \Pi_{\rho_k}^Q + \omega \) is also a connected component of \( \Gamma_{\text{tr},k} \). \( \square \)

Now we consider truncated interval-firing for rank 2 root systems.

**Proposition 4.8.3.** Suppose \( \Phi \) is of rank 2. Let \( k \in \mathbb{N}[\Phi]^W \). Let \( \lambda \in P \) be such that \( \langle \lambda, \alpha' \rangle \in [-k(\alpha), k(\alpha)] \) and \( \langle \lambda, \beta' \rangle \in [-k(\beta), k(\beta)] \) for two linearly independent roots \( \alpha, \beta \in \Phi \). Suppose that either \( \Phi \) is simply laced or one of \( \alpha \) and \( \beta \) is short and the other is long. Let \( \omega \in \Omega_m^{0} \) be such that \( \rho_k - \lambda \in Q + \omega \). Then \( \lambda \in \Pi_{\rho_k}^Q + \omega \).
Proof. First let us show \( \lambda_{\text{dom}} = c_1 \omega_1 + c_2 \omega_2 \) with \( c_1 \in [0, k(\alpha_1)] \) and \( c_2 \in [0, k(\alpha_2)] \). Observe that \( \langle \lambda_{\text{dom}}, w\lambda(\alpha) \rangle \in [-k(\alpha), k(\alpha)] \) and similarly for \( \beta \). By replacing \( \alpha \) with \(-\alpha\) and \( \beta \) with \(-\beta\) if necessary, we can assume \( \langle \lambda_{\text{dom}}, w\lambda(\alpha) \rangle \in [0, k(\alpha)] \) and similarly for \( \beta \), and since \( \lambda_{\text{dom}} \) is dominant, we are free to assume that \( w\lambda(\alpha) \) is positive and similarly for \( \beta \). Note that \( w\lambda(\alpha) \) and \( w\lambda(\beta) \) are both nonnegative integer combinations of the simple coroots \( \alpha_1^\vee \) and \( \alpha_2^\vee \). Then, since \( \alpha \) and \( \beta \) are linearly independent, and since either \( \Phi \) is simply laced, in which case \( k(\alpha) = k(\beta) = k \), or one of \( \alpha \), \( \beta \) is short (say e.g. \( k(\alpha) = k_s \)) and the other is long (say e.g. \( k(\beta) = k_l \)), we can conclude in fact that \( \lambda_{\text{dom}} \in [0, k(\alpha_1)] \) and \( \lambda_{\text{dom}} \in [0, k(\alpha_2)] \).

So indeed, \( \lambda_{\text{dom}} = c_1 \omega_1 + c_2 \omega_2 \) with \( c_1 \in [0, k(\alpha_1)] \) and \( c_2 \in [0, k(\alpha_2)] \). If \( c_1 = k(\alpha_1) \) and \( c_2 = k(\alpha_2) \), then \( \lambda = \rho_k \) and the proposition is obvious in this case (note that we will have \( \omega = 0 \)). So assume without loss of generality that \( c_2 \leq k(\alpha_2) - 1 \).

Let \( \lambda' := \lambda - \omega \). We want to show \( \lambda' \in \Pi^Q(\rho_k) \). As we have seen in the proof of Lemma 4.8.2, we have \( \lambda'_{\text{dom}} = \lambda_{\text{dom}} - w(\omega) \) for some \( w \in W \). So let \( w \in W \) be such that \( \lambda'_{\text{dom}} = \lambda_{\text{dom}} - w(\omega) \) and write \( \lambda'_{\text{dom}} = c'_1 \omega_1 + c'_2 \omega_2 \). Since \( \langle -w(\omega), \alpha \rangle \in \{-1, 0, 1\} \) for any \( \alpha \in \Phi \), we have \( c'_1 \leq k(\alpha_1) + 1 \) and \( c'_2 \leq k(\alpha_2) \). First suppose \( c'_1 \leq k(\alpha_1) \). Together with \( c'_2 \leq k(\alpha_2) \), this implies that \( \rho_k - \lambda'_{\text{dom}} \in P_{\geq 0} \), and hence \( \rho_k - \lambda'_{\text{dom}} \in Q_{\geq 0} \). Thus we conclude \( \lambda' \in \Pi^Q(\rho_k) \) by Proposition 3.1.2.

So suppose that \( c'_1 = k(\alpha_1) + 1 \). This means \( \langle -w(\omega), \alpha_1^\vee \rangle = 1 \). Since \( \omega \) is the only dominant element of \( W(\omega) \), we must actually have that \( \langle -w(\omega), \alpha_2^\vee \rangle \leq 0 \) and hence \( c'_2 \leq k(\alpha_2) - 1 \). Write \( \rho_k - \lambda'_{\text{dom}} = a_1 \alpha_1 + a_2 \alpha_2 \) for some integers \( a_1, a_2 \in \mathbb{Z} \). Then \( c'_1 = k(\alpha_1) + 1 \) and \( c'_2 \leq k(\alpha_2) - 1 \) translate to

\[
2a_1 + \langle a_2, \alpha_1^\vee \rangle a_2 = -1; \\
\langle a_1, \alpha_2^\vee \rangle a_1 + 2a_2 \geq 1.
\]

By the classification of rank 2 root systems we have \( \langle a_2, \alpha_1^\vee \rangle, \langle a_1, \alpha_2^\vee \rangle \in \{-1, -2, -3\} \) with at least one of them equal to \(-1\). It is then not hard to check that all integer solutions \( a_1, a_2 \in \mathbb{Z} \) to the above system of inequalities must have \( a_1, a_2 \geq 0 \). Hence we conclude \( \rho_k - \lambda'_{\text{dom}} \in Q_{\geq 0} \), and thus \( \lambda' \in \Pi^Q(\rho_k) \) by Proposition 3.1.2. \( \square \)
Corollary 4.8.4. Suppose $\Phi$ is of rank 2. Let $k \in \mathbb{N}[\Phi]^W$ be good. Then $\rightarrow_{\text{tr},k}$ is confluent (and terminating).

Proof. We know $\rightarrow_{\text{tr},k}$ is terminating thanks to Proposition 4.1.3. Hence by the diamond lemma, Lemma 1.1.1, it is enough to prove that $\rightarrow_{\text{tr},k}$ is locally confluent.

First let us prove this when $\Phi$ is simply laced. Suppose $\lambda \rightarrow_{\text{tr},k} \lambda + \alpha$ and $\lambda \rightarrow_{\text{tr},k} \lambda + \beta$ for $\alpha, \beta \in \Phi^+$. Then by Proposition 4.8.3 we have that $\lambda \in \Pi^Q(\rho_k) + \omega$ where $\omega \in \Omega^0_m$ is such that $\rho_k - \lambda \in Q + \omega$. But by Lemma 4.8.1, $\Pi^Q(\rho_k) + \omega$ is a connected component of $\Gamma_{\text{tr},k}$ with unique sink $\rho_k + \omega$; since $\rightarrow_{\text{tr},k}$ is terminating this means that any $\rightarrow_{\text{tr},k}$-firing sequence starting at $\lambda$ eventually terminates at $\rho_k + \omega$. Hence we can bring $\lambda + \alpha$ and $\lambda + \beta$ back together again via $\rightarrow_{\text{tr},k}$-firings.

Note that confluence for $\Phi = A_1 \times A_1$ (for any $k \in \mathbb{N}[\Phi]^W$) reduces to confluence for $\Phi = A_1$, which is trivial. Thus in fact we have proved confluence for all simply laced root systems of rank 2, including those which are not irreducible.

So assume $\Phi$ is not simply laced. Suppose $\lambda \rightarrow_{\text{tr},k} \lambda + \alpha$ and $\lambda \rightarrow_{\text{tr},k} \lambda + \beta$ for $\alpha, \beta \in \Phi^+$. If one of $\alpha$ and $\beta$ is short and the other is long, then we can apply Proposition 4.8.3 and Lemma 4.8.1 as above to conclude that we can bring $\lambda + \alpha$ and $\lambda + \beta$ back together again via $\rightarrow_{\text{tr},k}$-firings. So suppose $\alpha$ and $\beta$ have the same length. Then let $\tilde{\Phi}$ be the set of all roots in $\Phi$ with the same length as $\alpha$ and $\beta$. This $\tilde{\Phi}$ will again be a rank 2 root system, and by construction a simply laced one. Hence by the result for simply laced root systems, we know that truncated interval-firing is confluent for $\tilde{\Phi}$; so in particular we can bring $\lambda + \alpha$ and $\lambda + \beta$ back together again via $\rightarrow_{\text{tr},k}$-firings. □

The confluence of truncated interval-firing for all root systems follows easily from confluence for rank 2 root systems.

Corollary 4.8.5. Let $k \in \mathbb{N}[\Phi]^W$ be good. Then $\rightarrow_{\text{tr},k}$ is confluent (and terminating).

Proof. We know $\rightarrow_{\text{tr},k}$ is terminating thanks to Proposition 4.1.3. Hence by the diamond lemma, Lemma 1.1.1, it is enough to prove that $\rightarrow_{\text{tr},k}$ is locally confluent.

Suppose that $\lambda \rightarrow_{\text{tr},k} \lambda + \alpha$ and $\lambda \rightarrow_{\text{tr},k} \lambda + \beta$ for $\alpha, \beta \in \Phi^+$. Restricting $\Phi$ to the span of $\alpha$ and $\beta$ gives a rank 2 sub-root system, for which we have proved confluence in
Corollary 4.8.4 (as remarked in the proof of that corollary, we in fact proved confluence for all rank 2 root systems, including those which are not irreducible). Hence we can bring $\lambda + \alpha$ and $\lambda + \beta$ back together just with truncated interval-firing moves inside that rank 2 sub-root system.

Remark 4.8.6. Our method of proof of confluence for $\rightarrow_{tr,k}$ fails when $k$ is not good; for instance, Lemma 4.8.2 is not true for general $k$, as can be seen in Example 4.5.3: here $0 \in \Pi^Q(\rho_k)$ but $0$ does not belong to the connected component of $\Gamma_{tr,k}$ containing $\rho_k$. However, we can actually deduce that $\rightarrow_{tr,k}$ is confluent for all $k \in \mathbb{N}[\Phi]^W$ from Corollary 4.8.5. Indeed, if $k \in \mathbb{N}[\Phi]^W$ is not good, then $k_s = 0$. But if $k_s = 0$ then we will never be able to fire any short root. In other words, if $k_s = 0$ then truncated interval-firing reduces to truncated interval-firing with respect to the long roots only; and the long roots form a simply laced root system, for which $\rightarrow_{tr,k}$ is known to be confluent from Corollary 4.8.5.

Remark 4.8.7. It appears that when $\Phi = A_2$ there are no intervals $[a, b]$ for which the relation $\lambda \rightarrow \lambda + \alpha$ for $\lambda \in P$, $\alpha \in \Phi^+$ with $\langle \lambda, \alpha^\vee \rangle + 1 \in [a, b]$ is confluent besides the symmetric and truncated intervals (and this probably would not be too hard to prove). If so, then the same would be true for all simply laced root systems because any irreducible root system of rank 3 or greater contains an $A_2$ sub-root system. This observation also severely restricts possible intervals defining confluent processes for all root systems, including the non-simply laced ones (although note that central-firing is confluent for $\Phi = B_2$).

Remark 4.8.8. To any root-firing process $\rightarrow$ on $P$ let us associate the hyperplane arrangement which contains the hyperplane $H = \{v \in V : \langle v, \alpha^\vee \rangle = c\}$ whenever we have a firing move $\lambda \rightarrow \lambda + \alpha$ with $\langle \lambda + \alpha^\vee \rangle = c$; i.e., we include a hyperplane orthogonal to $\alpha$ at the midpoint between $\lambda$ and $\lambda + \alpha$. As mentioned in the introduction to this chapter, under this correspondence the symmetric and truncated interval-firing processes correspond to the (extended) Catalan and Shi hyperplane arrangements [65, 7]. The confluence of symmetric and truncated interval-firing seems like it might have something to do with the freeness of the Catalan and Shi arrange-
Freeness is a certain deep algebraic property of hyperplane arrangements introduced by Terao [77]. Freeness of the (extended) Catalan and Shi hyperplane arrangements of a root system was conjectured by Edelman and Reiner [26] and proven by Yoshinaga [80] building on work of Terao [78]. Vic Reiner suggested that we look at other free deformations of Coxeter arrangements as a possible source of other confluent root-firing processes. We found one such process which, experimentally, appears confluent: for $k \in \mathbb{N}[\Phi]$ consider the relation $\lambda \rightarrow \lambda + \alpha$ for $\lambda \in P$, $\alpha \in \Phi^+$ with $\langle \lambda, \alpha^\vee \rangle + 1 \in [-k(\alpha) + 1, k(\alpha)]$ if $\alpha$ is long and $\langle \lambda, \alpha^\vee \rangle + 1 \in [-k(\alpha), k(\alpha)]$ if $\alpha$ is short. In other words, we use either the truncated or symmetric intervals depending on which Weyl group orbit our root lies in. This process corresponds to a Shi-Catalan hyperplane arrangement, as studied by Abe and Terao [2]. Other free variants of Coxeter arrangements include the ideal subarrangements of Coxeter arrangements [1, 3], but we have not been able to obtain confluent root-firing processes from these. Note that the freeness of the corresponding hyperplane arrangement certainly does not imply confluence of the process: for instance, reversing the direction of all the arrows for the truncated interval-firing process yields a process which is not confluent but which corresponds to the same Shi hyperplane arrangement. Nevertheless, it would be very interesting to understand the connection between freeness and confluence further.

**Remark 4.8.9.** Under the correspondence between root-firing processes and hyperplane arrangements discussed in Remark 4.8.8, the central-firing process corresponds not to the central Coxeter arrangement, but rather to the affine Linial arrangement. The Linial arrangement has many interesting combinatorial properties (see e.g. [65] and [7]), but is not free.

In the remaining chapters of this thesis, we study the symmetric and truncated interval-firing processes further by examining in more detail the connected components for these processes, and how these components “grow” as a function of $k$. This growth is evident already in Example 4.1.1. In particular, we will try to count the number of weights in a given connected component and this will lead us to define certain polynomials analogous the Ehrhart polynomial of a convex polytope.
Chapter 5

Interval root-firing: existence of Ehrhart-like polynomials

In this chapter we study the interval-firing processes from the previous chapter in more detail by counting the number of weights with given stabilization. The material in this chapter is joint work with Pavel Galashin, Thomas McConville, and Alexander Postnikov and appears in [36].

Continue to fix a root system $\Phi$ as in Chapter 4 and retain all the notation from that chapter. In this chapter, we investigate the set of weights with given symmetric or truncated interval-firing stabilization. Thus, for good $k \in \mathbb{N}[\Phi]^W$, we define the stabilization maps $s_k^{\text{sym}}: P \to P$ and $s_k^{\text{tr}}: P \to P$ by

$$s_k^{\text{sym}}(\mu) = \lambda \iff \text{the $\eta_k(\lambda)$-stabilization of } \mu \text{ is } \eta_k(\lambda);$$

$$s_k^{\text{tr}}(\mu) = \lambda \iff \text{the $\eta_k(\lambda)$-stabilization of } \mu \text{ is } \eta_k(\lambda).$$

These functions are well-defined since the symmetric and truncated interval-firing processes are confluent and terminating (Corollaries 4.6.2 and 4.8.5), the stable points of these processes must have the form $\eta_k(\lambda)$ for some $\lambda \in P$ (Lemmas 4.3.6 and 4.8.1), and the map $\eta_k$ is injective (Proposition 4.3.3).

Looking at Example 4.1.1, one can see that the set $(s_k^{\text{sym}})^{-1}(\lambda)$ (or $(s_k^{\text{tr}})^{-1}(\lambda)$) of weights with interval-firing stabilization $\eta_k(\lambda)$ looks “the same” across all values of $k$. 

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except that it gets “dilated” as \( k \) is scaled. In analogy with the Ehrhart polynomial \([27]\) of a convex lattice polytope, which counts the number of lattice points in dilations of the polytope, let us define for all \( \lambda \in P \) and all good \( k \in \mathbb{N}[\Phi]^{W} \) the quantities:

\[
L^\text{sym}_\lambda(k) := #(s^\text{sym}_k)^{-1}(\lambda);
L^\text{tr}_\lambda(k) := #(s^\text{tr}_k)^{-1}(\lambda).
\]

Our aim is to show that \( L^\text{sym}_\lambda(k) \) and \( L^\text{tr}_\lambda(k) \) are polynomials in \( k \). By “polynomial in \( k \)” we mean that, if \( \Phi \) is simply laced, then these \( L^\text{sym}_\lambda(k) \) and \( L^\text{tr}_\lambda(k) \) are single-variable polynomials in \( k \), where \( k(\alpha) = k \) for all \( \alpha \in \Phi \); and if \( \Phi \) is non-simply laced, then they are two-variable polynomials in \( k_s \) and \( k_l \), where \( k(\alpha) = k_s \) if \( \alpha \) is short and \( k(\alpha) = k_l \) if \( \alpha \) is long.

We are able to show that the \( L^\text{sym}_\lambda(k) \) are polynomials for all root systems \( \Phi \), and we are able to show that the \( L^\text{tr}_\lambda(k) \) are polynomials assuming that \( \Phi \) is simply laced. In fact, we show that all these polynomials have integer coefficients. Moreover, we conjecture that for all \( \Phi \) that these \( L^\text{sym}_\lambda(k) \) and \( L^\text{tr}_\lambda(k) \) are polynomials with nonnegative integer coefficients. We focus on this positivity conjecture in the next chapter.

We refer to these \( L^\text{sym}_\lambda(k) \) and \( L^\text{tr}_\lambda(k) \) as the symmetric and truncated Ehrhart-like polynomials because they count the size of some discrete subset of lattice points as that set is somehow “dilated.” But it is important to note that the sets \((s^\text{sym}_k)^{-1}(\lambda)\) and \((s^\text{tr}_k)^{-1}(\lambda)\) are in general not the set of lattice points of any convex polytope, or indeed, any convex set. This can already be seen in rank 2 (see Example [4.1.1]). Nevertheless, for some special \( \lambda \) (namely, \( \lambda \in \Omega^0_m \)) the polynomials \( L^\text{sym}_\lambda(k) \) and \( L^\text{tr}_\lambda(k) \) are (essentially) genuine Ehrhart polynomials; and so we do use Ehrhart theory to prove the polynomiality of \( L^\text{sym}_\lambda(k) \) and \( L^\text{tr}_\lambda(k) \). Note that, because they apparently have nonnegative integer coefficients, these polynomials are (as we explain below) most similar to the Ehrhart polynomials of zonotopes.
5.1 Symmetric Ehrhart-like polynomials

The Ehrhart polynomial $L_\mathcal{P}(k)$ of a convex lattice polytope $\mathcal{P}$ is a single-variable polynomial in $k$ which satisfies

$$L_\mathcal{P}(k) = \text{the number of lattice points in } k\mathcal{P} \text{ (the } k\text{th dilate of } \mathcal{P})$$

for all $k \geq 1$. Such polynomials were first investigated by Ehrhart [27], who proved that they exist for all lattice polytopes. A famous result of Stanley [72, Example 3.1] says that the Ehrhart polynomial of a lattice zonotope (i.e., a Minkowski sum of line segments) has nonnegative integer coefficients. A standard way to prove this result is to inductively pave the zonotope (see [12, §9.2]); this decomposition of a zonotope goes back to Shephard [69]. In the following theorem we apply this same paving technique to a slightly more general setting: namely, we show that if $\mathcal{P}$ is any fixed convex lattice polytope, and $\mathcal{Z}$ is a lattice zonotope, then for $k \geq 1$ the number of lattice points in $\mathcal{P} + k\mathcal{Z}$ is a polynomial with nonnegative integer coefficients in $k$. Stanley’s result corresponds to taking $\mathcal{P}$ to be a point. Although the proof is, as mentioned, standard, we have not found this theorem in the Ehrhart theory literature; and it turns out that this result is just what we need to prove that the symmetric Ehrhart-like polynomials $L_\lambda^{\text{sym}}(k)$ exist.

**Theorem 5.1.1.** Let $\Lambda$ be a lattice in $V$. Let $\mathcal{P}$ be any convex lattice polytope in $V$ (i.e., the vertices of $\mathcal{P}$ belong to $\Lambda$). Let $v_1, \ldots, v_m \in \Lambda$ be lattice elements. Then for
any \( k = (k_1, \ldots, k_m) \in \mathbb{N}^m \) the quantity

\[
\#(P + k_1[0, v_1] + \cdots + k_m[0, v_m]) \cap \Lambda
\]

is given by a polynomial in the \( k_1, \ldots, k_m \) with nonnegative integer coefficients.

**Proof.** For \( X = \{u_1, \ldots, u_\ell\} \subseteq V \) linearly independent, a half-open parallelepiped with edge set \( X \) is a convex set \( \mathcal{Z}_{h.o.}^X \) of the form

\[
\mathcal{Z}_{h.o.}^X = \sum_{i=1}^\ell \begin{cases}
[0, u_i) & \text{if } \varepsilon = 1; \\
(0, u_i] & \text{if } \varepsilon = -1,
\end{cases}
\]

for some choice of sign vector \((\varepsilon_1, \ldots, \varepsilon_\ell) \in \{-1, 1\}^\ell\). For \( X \subseteq \{v_1, \ldots, v_m\} \) let us use \( kX := \{k_ivi: v_i \in X\} \).

The key idea for this theorem: \( P + k_1[0, v_1] + \cdots + k_m[0, v_m] \) can be inductively decomposed (or “paved”) into disjoint pieces that are (up to translation) of the form

\[
F + \mathcal{Z}_{h.o.}^{kX},
\]

where \( X \subseteq \{v_1, \ldots, v_m\} \) is linearly independent and \( F \) is an open face of the polytope \( P \) which is affinely independent from \( \text{Span}_R(X) \). Figure 5-1 shows how this is done. Here by “open face” of \( P \) we mean a face minus its relative boundary. Note that vertices have empty relative boundary and hence vertices are open faces. (But observe that Figure 5-1 is slightly misleading in that we should technically show the whole polytope \( P \) decomposed into its open faces as well; instead the figure shows these pieces grouped into a single bigger piece.) The proof, by induction on \( m \), that this is possible works in exactly the same way as for paving a zonotope (see [12, Lemma 9.1]), so we do not go into the details. Then note that

\[
\# \left( (F + \mathcal{Z}_{h.o.}^{kX}) \cap \Lambda \right) = \# \left( (F + \mathcal{Z}_{h.o.}^{kX}) \cap \Lambda \right) \cdot \prod_{v_i \in X} k_i
\]

precisely because \( F \) is affinely independent from \( \mathcal{Z}_{h.o.}^{kX} \). Hence the desired polyno-
mial in $k_1, \ldots, k_m$ indeed exists: it is a sum over the pieces of this decomposition of $\#((F + \mathbb{Z}_X^{k_0}) \cap \Lambda) \cdot \prod_{i \in X} k_i$. (We are implicitly using the fact that this decomposition can be realized in a uniform way across all values of $k$).

**Corollary 5.1.2.** For any $\lambda \in P_{\geq 0}$, for all $k \in \mathbb{N}[\Phi]^W$ the quantity $\#\Pi^Q(\lambda + \rho_k)$ is given by a polynomial with nonnegative integer coefficients in $k$.

**Proof.** We are free to translate $\Pi^Q(\lambda + \rho_k)$ so that it contains the origin; i.e., clearly $\#\Pi^Q(\lambda + \rho_k)$ is the number of $Q$-points in $\Pi(\lambda) - \lambda - \rho_k$. One easy consequence of Proposition 3.1.2 is that $\Pi(\lambda + \lambda) = \Pi(\lambda) + \Pi(\lambda)$ for dominant weights $\lambda, \mu \in P_{\geq 0}$. Hence, because $\lambda$ is dominant, we have

$$\Pi(\lambda + \rho_k) - \lambda - \rho_k = (\Pi(\lambda) - \lambda) + (\Pi(k) - \rho_k).$$

It is well known that the regular permutohedron $\Pi(\rho)$ is a zonotope. In Type A, a standard way to prove this fact is to compute the Newton polytope of the Vandermonde determinant in two ways (see [12, Theorem 9.4]). The same argument, but with Weyl’s denominator formula (see [46, §24.3]) in place of the Vandermonde determinant, establishes that $\Pi(\rho) = \Pi(\rho) - \rho_k - \rho_k$. It is then a simple exercise to show that $\Pi(\rho_k) = \sum_{\alpha \in \Phi^+} k(\alpha)[-\alpha/2, \alpha/2]$. Therefore,

$$\Pi(\lambda + \rho_k) - \lambda - \rho_k = (\Pi(\lambda) - \lambda) + \sum_{\alpha \in \Phi^+} k(\alpha)[0, -\alpha],$$

and so the desired polynomial indeed exists thanks to Theorem 5.1.1.

**Theorem 5.1.3.** For any $\lambda \in P$, for good $k \in \mathbb{N}[\Phi]^W$ the quantity $L^{\text{sym}}(k) = \#(s_k^{-1}(\lambda))$ is given by a polynomial with integer coefficients in $k$.

**Proof.** First of all, if $\lambda$ has $\langle \lambda, \alpha^\vee \rangle = -1$ for some $\alpha \in \Phi^+$ then clearly we can take $L^{\text{sym}}(k) := 0$ because, thanks to Lemma 4.3.6, $\eta_k(\lambda)$ cannot be a sink of $\Gamma_{\text{sym}, k}$ in this case. So now assume that $\lambda$ satisfies $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$. If $I^{0,1}_{\lambda, \text{dom}} \neq [n]$, then, by Theorem 4.6.1 the connected component of $\Gamma_{\text{sym}, k}$ containing the sink $\eta_k(\lambda)$ is contained in $w_{\lambda, k} \Pi_{I^{0,1}_{\lambda, \text{dom}}} Q_{\alpha, k}(\lambda_{\text{dom}})$, which is contained in an affine translate of the strict
subspace $\text{Span}_R(w_\lambda \Phi_{\rho_0,1}^{\lambda_{\text{dom}}})$. By induction on rank we know the theorem is true for the sub-root system $w_\lambda \Phi_{\rho_0,1}^{\lambda_{\text{dom}}}$. Hence, the desired polynomial $I^\text{sym}_\lambda(k)$ is just the corresponding polynomial for the orthogonal projection of $\lambda$ onto $\text{Span}_R(w_\lambda \Phi_{\rho_0,1}^{\lambda_{\text{dom}}})$.

(Here we use the fact that the map $\eta_k$ respects this projection: but this is clear because the projection of $\lambda$ and the projection of $w_\lambda(\rho_k)$ onto the weight lattice of $w_\lambda \Phi_{\rho_0,1}^{\lambda_{\text{dom}}}$ are both dominant with respect to the choice of $w_\lambda \Phi_{\rho_0,1}^{\lambda_{\text{dom}}}$ as positive roots, which is a subset of $\Phi^+$ by Proposition 4.3.4.)

So now assume $I_{\lambda_{\text{dom}}}^{0,1} = [n]$. This means that $\lambda$ is dominant. Let $k \in \mathbb{N}[\Phi]^W$ be good. Set $S := \{\mu \in P : \langle \mu, \alpha^\vee \rangle \neq -1 \text{ for all } \alpha \in \Phi^+, \eta_k(\mu) \in \Pi^Q(\eta_k(\lambda))\}$; i.e., $S$ is the set of all labels of sinks of $\Gamma_{\text{sym},k}$ that are inside of $\Pi^Q(\eta_k(\lambda))$.

We claim that in fact $S = \{\mu \in P : \langle \mu, \alpha^\vee \rangle \neq -1 \text{ for all } \alpha \in \Phi^+, \mu \in \Pi^Q(\lambda)\}$. Indeed, for $\mu \in P$ with $\langle \mu, \alpha^\vee \rangle \neq -1 \text{ for all } \alpha \in \Phi^+$, we have $\eta_k(\mu) \in \Pi^Q(\eta_k(\lambda))$ if and only if $\eta_k(\mu_{\text{dom}}) \in \Pi^Q(\eta_k(\lambda))$. By Proposition 3.1.2 we have that $\eta_k(\mu_{\text{dom}}) \in \Pi^Q(\eta_k(\lambda))$ if and only if $\lambda + \rho_k - (\mu_{\text{dom}} + \rho_k) = \lambda - \mu_{\text{dom}} \in Q_{\geq 0}$, which, again by Proposition 3.1.2, is if and only if $\mu_{\text{dom}} \in \Pi^Q(\lambda)$, that is, if and only if $\mu \in \Pi^Q(\lambda)$. Note that this second description of $S$ is independent of $k$. Also note that for all $\mu \neq \lambda \in S$, either $I_{\mu_{\text{dom}}}^{0,1} \neq [n]$ or $\mu = \mu_{\text{dom}}$, and in the latter case we have that $\mu$ is strictly less than $\lambda$ in root order. Now, the permutohedron non-escaping lemma, Lemma 4.5.2 says that

$$\Pi^Q(\eta_k(\lambda)) = \bigcup_{\mu \in S} (s^\text{sym}_k)^{-1}(\mu).$$

Hence, rewriting, and taking cardinalities, we get

$$\#(s^\text{sym}_k)^{-1}(\lambda) = \#\Pi^Q(\eta_k(\lambda)) - \sum_{\mu \neq \lambda \in S} \#(s^\text{sym}_k)^{-1}(\mu).$$

The quantity $\#\Pi^Q(\eta_k(\lambda))$ is a polynomial in $k$ with integer coefficients thanks to Theorem 5.1.2. The quantity $\sum_{\mu \neq \lambda \in S} \#(s^\text{sym}_k)^{-1}(\mu)$ is a polynomial in $k$ with integer coefficients by induction on rank and on root order. Since the above equality holds for all good $k \in \mathbb{N}[\Phi]^W$, we conclude that $I^\text{sym}_\lambda(k) = \#(s^\text{sym}_k)^{-1}(\lambda)$ is indeed a polynomial.
Table 5.1: The polynomials $L_{\lambda}^{{\text{sym}}}(k)$ for the irreducible rank 2 root systems.

<table>
<thead>
<tr>
<th>$\Phi$</th>
<th>$\lambda$</th>
<th>$L_{\lambda}^{{\text{sym}}}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2$</td>
<td>0</td>
<td>$3k^2 + 3k + 1$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\omega_1$</td>
<td>$3k^2 + 6k + 3$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\omega_2$</td>
<td>$3k^2 + 6k + 3$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\omega_1 + \omega_2$</td>
<td>$6k + 6$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>0</td>
<td>$2k_l^2 + 4k_lk_s + k_s^2 + 2k_l + 2k_s + 1$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$\omega_1$</td>
<td>$4k_l + 4k_s + 4$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$\omega_2$</td>
<td>$2k_l^2 + 4k_lk_s + k_s^2 + 6k_l + 4k_s + 4$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$\omega_1 + \omega_2$</td>
<td>$4k_l + 4k_s + 8$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>0</td>
<td>$9k_l^2 + 12k_lk_s + 3k_s^2 + 3k_l + 3k_s + 1$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\omega_1$</td>
<td>$12k_l + 6k_s + 6$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\omega_2$</td>
<td>$6k_l + 6k_s + 6$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\omega_1 + \omega_2$</td>
<td>$6k_l + 6k_s + 12$</td>
</tr>
</tbody>
</table>

Table 5.1 records the polynomials $L_{\lambda}^{{\text{sym}}}(k)$ for the irreducible rank 2 root systems, for all $\lambda \in P_{\geq 0}$ with $I_{\lambda}^{0,1} = [n]$. Compare these polynomials to the graphs of the corresponding symmetric interval-firing processes in Example 4.1.1.

Remark 5.1.4. The evaluation of the polynomial $L_{\lambda}^{{\text{sym}}}(k)$ for $k \in \mathbb{N}[\Phi]^W$ not good may not count the number of weights in the connected component of $\Gamma_{\text{sym},k}$ containing $\eta_k(\lambda)$. For example, take $\Phi = B_2$ and $k$ defined by $k_s := 0$ and $k_l := 1$, as in Example 4.5.3. Then, with $\lambda := 0$, looking at Table 5.1 we see

$$L_{\lambda}^{{\text{sym}}}(k) = 2k_l^2 + 4k_lk_s + k_s^2 + 2k_l + 2k_s + 1 = 5,$$

while there are only four weights in the connected component of $\Gamma_{\text{sym},k}$ containing the sink $\eta_k(\lambda)$. (Here the “missing” weight is of course the origin.)
Conjecture 5.1.5. The polynomials $L^\text{sym}_\lambda(k)$ have nonnegative integer coefficients.

When $\lambda \in \Omega^0_{\text{sym}}$, we know thanks to Proposition 4.7.1 that $(s^\text{sym}_k)^{-1}(\lambda) = \Pi^Q(\lambda+\rho_k)$, so Corollary 5.1.2 implies that Conjecture 5.1.5 is true in this case. In the final chapter in this thesis we will actually prove Conjecture 5.1.5 in general. The first step in the proof of positivity will be to give a more refined version of Theorem 5.1.1 that gives an explicit formula for the number of lattice points in a polytope plus dilating zonotope.

5.2 Cubical subcomplexes

In order to proceed further in our investigation of the stabilization maps $s^\text{sym}_k$ and $s^\text{tr}_k$, and the relation between them, we need to understand a bit more about the connected components of $\Gamma_{\text{sym},k}$. We know that the connected component of $\Gamma_{\text{sym},k}$ containing the sink $\eta_k(\lambda)$ is contained in the discrete permutohedron $w_\lambda \Pi^Q_{\lambda_{\text{dom}}^0,1}(\lambda_{\text{dom}})$ (Theorem 4.6.1); but it can sometimes contain all of this permutohedron (see Proposition 4.7.1) and can sometimes contain relatively little of it. In this section we will show that there is a small amount of $w_\lambda \Pi^Q_{\lambda_{\text{dom}}^0,1}(\lambda_{\text{dom}})$ that this connected component must always contain.

The permutohedron $\Pi_f(\lambda)$ has the structure of a polyhedral complex. The cubical subcomplex of $\Pi_f(\lambda)$ is the union of all faces of $\Pi_f(\lambda)$ that are cubes; here a cube means a product of pairwise orthogonal intervals. We denote the cubical subcomplex by $\sqcup \Pi_f(\lambda)$. Note that every edge is a cube, and hence $\sqcup \Pi_f(\lambda)$ contains at least the 1-skeleton of $\Pi_f(\lambda)$, but it may contain more. We use $\sqcup^Q \Pi_f(\lambda) := \sqcup \Pi_f(\lambda) \cap (Q+\lambda)$.

Proposition 5.2.1. Let $\lambda \in P$ with $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$ and let $k \in \mathbb{N}[\Phi]^W$. Let $Y_\lambda := \{ \mu \in P : \mu \xrightarrow{\text{sym},k} \eta_k(\lambda) \}$ be the connected component of $\Gamma_{\text{sym},k}$ containing the sink $\eta_k(\lambda)$. Then $Y_\lambda$ contains the discrete cubical subcomplex $w_\lambda \sqcup^Q_{\lambda_{\text{dom}}^0,1}(\eta_k(\lambda_{\text{dom}}))$.

Proof. By the usual projection argument that we have by now carried out many times, we can assume that $I_{\lambda_{\text{dom}}}^0 = [n]$ and consequently that $\lambda$ is dominant.

For any simple root $\alpha_i$ we have that $\langle \eta_k(\lambda), \alpha_i^\vee \rangle \in \{ k(\alpha), k(\alpha) + 1 \}$. This means that we can “unfire” $\alpha_i$ from $\eta_k(\lambda)$; that is, $\langle \eta_k(\lambda) - \alpha_i, \alpha_i^\vee \rangle \leq k(\alpha) - 1$, so that there
will be an edge $\eta_k(\lambda) - \alpha_i\xrightarrow{\text{sym,k}} \eta_k(\lambda)$ of $\Gamma_{\text{sym,k}}$. In fact, we can keep “unfiring” the simple root $\alpha_i$ until we reach $s_{\alpha_i}(\eta_k(\lambda))$; i.e., in $\Gamma_{\text{sym,k}}$ there are sequence of edges

$$s_{\alpha_i}(\eta_k(\lambda))\xrightarrow{\text{sym,k}} s_{\alpha_i}(\eta_k(\lambda)) + \alpha_i\xrightarrow{\text{sym,k}} \cdots \xrightarrow{\text{sym,k}} \eta_k(\lambda) - \alpha_i\xrightarrow{\text{sym,k}} \eta_k(\lambda).$$

(Note that is is possible that $s_{\alpha_i}(\eta_k(\lambda)) = \eta_k(\lambda)$, in which case we would not actually be able to unfire $\alpha_i$ at all). This means that all the $(Q + \eta_k(\lambda))$-points of the entire edge of $\Pi(\eta_k(\lambda))$ between $\eta_k(\lambda)$ and $s_{\alpha_i}(\eta_k(\lambda))$ are reachable via unfirings from $\eta_k(\lambda)$. Moreover, if $\alpha_i$ and $\alpha_j$ are orthogonal, then unfiring one of these does not affect our ability to unfire the other, and hence in this way we can reach any $(Q + \eta_k(\lambda))$-point on a face of $\Pi(\eta_k(\lambda))$ that is the orthogonal product of edges coming out of the vertex $\eta_k(\lambda)$ in the direction of a negative simple root. Since in particular $s_i(\eta_k(\lambda))$ is reachable via firings and unfirings from $\eta_k(\lambda)$, by applying the $W$-symmetry of $\Gamma_{\text{sym,k}}$ (Theorem 4.2.1) we see that all vertices of $\Pi(\eta_k(\lambda))$ are so reachable. But note that any face of $\Pi(\eta_k(\lambda))$ can be transported via $W$ to a face containing $\eta_k(\lambda)$, such that the edges of this face which contain $\eta_k(\lambda)$ are in the direction of a negative simple root (see the proof of Theorem 4.4.6). We thus conclude that we can reach any $(Q + \eta_k(\lambda))$-point on any cubical face of $\Pi(\eta_k(\lambda))$ via firings and unfirings from $\eta_k(\lambda).$

\[\Box\]

**Corollary 5.2.2.** Let $\lambda \in P$ with $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$ and let $k \in \mathbb{N}[\Phi]^W$.

Let $Y_\lambda := \{\mu \in P : \mu\xrightarrow{\text{sym,k}} \eta_k(\lambda)\}$ be the connected component of $\Gamma_{\text{sym,k}}$ containing the sink $\eta_k(\lambda)$. Then $Y_\lambda$ contains $w_{\lambda}W_{\lambda_{\text{dom}}}^{\alpha_1}(\eta_k(\lambda_{\text{dom}}))$. In the special case $k = 0$, $Y_\lambda$ is in fact equal to $w_{\lambda}W_{\lambda_{\text{dom}}}^{\alpha_1}(\lambda_{\text{dom}})$.

**Proof.** Note that $\Box \Pi_I(\mu)$ contains at least the 1-skeleton of $\Pi_I(\mu)$. Thus $Y_\lambda$ contains $w_{\lambda}W_{\lambda_{\text{dom}}}^{\alpha_1}(\eta_k(\lambda_{\text{dom}}))$ by Proposition 5.2.1. Now suppose $k = 0$. If $\mu\xrightarrow{\text{sym,0}} \mu'$ then $\mu' = \mu + \alpha$ for some $\alpha \in \Phi^+$ with $\langle \mu, \alpha^\vee \rangle = -1$, which means that $\mu' = s_\alpha(\mu)$. Hence any two elements in a connected component of $\Gamma_{\text{sym,0}}$ must be related by a Weyl group element. By Corollary 4.6.2, each connected component of $\Gamma_{\text{sym,0}}$ contains only a single sink, and thus the component $Y_\lambda$ must be exactly $w_{\lambda}W_{\lambda_{\text{dom}}}^{\alpha_1}(\lambda_{\text{dom}})$. \[\Box\]
5.3 How interval-firing components decompose

In this section, we study how symmetric and truncated interval-firing components “decompose” into smaller components. Let us explain what we mean by “decompose” more precisely. For any \( k \in \mathbb{N}[\Phi]^W \), \( \Gamma_{\text{tr},k} \) is a subgraph of \( \Gamma_{\text{sym},k} \), so the connected components of \( \Gamma_{\text{sym},k} \) are unions of connected components of \( \Gamma_{\text{tr},k} \). Similarly, \( \Gamma_{\text{sym},k} \) is a subgraph of \( \Gamma_{\text{tr},k+1} \) and so the connected components of \( \Gamma_{\text{tr},k+1} \) are unions of connected components of \( \Gamma_{\text{sym},k} \). What we want to show, in both cases, is that the way these components decompose into smaller components is consistent with the way we label the components by their sinks \( \eta_k(\lambda) \).

That the connected components of \( \Gamma_{\text{sym},0} \) break into connected components of \( \Gamma_{\text{tr},k} \) in a way consistent with the map \( \eta_k \) turns out to be a simple consequence of the fact that these connected components contain parabolic coset orbits (i.e., a consequence of Corollary 5.2.2 from the previous section). This is established in the next lemma and corollary.

**Lemma 5.3.1.** For \( \lambda, \mu \in P \), if \( \lambda \) and \( \mu \) belong to the same connected component of \( \Gamma_{\text{sym},0} \), then \( \eta_k(\lambda) \) and \( \eta_k(\mu) \) belong to the same connected component of \( \Gamma_{\text{sym},k} \) for all \( k \in \mathbb{N}[\Phi]^W \).

**Proof.** Let \( \lambda, \mu \in P \) belong to the same connected component of \( \Gamma_{\text{sym},0} \). From Corollary 5.2.2 we get that \( \mu_{\text{dom}} = \lambda_{\text{dom}} \) and also that there is some \( w \in w_\mu W_{\lambda_{\text{dom}}}^{0,1} \) such that \( w^{-1}(\lambda) \) is dominant. But by Corollary 4.3.2 this means \( w \in w_\lambda W_{\lambda_{\text{dom}}}^{0,1} \), and since the cosets of \( W_{\lambda_{\text{dom}}}^{0,1} \) are unions of cosets of \( W_{\lambda_{\text{dom}}}^{0} \), this means \( w_\mu W_{\lambda_{\text{dom}}}^{0,1} = w_\lambda W_{\lambda_{\text{dom}}}^{0} \). Thus, Corollary 5.2.2 tells us that indeed \( \eta_k(\lambda) \) and \( \eta_k(\mu) \) belong to the same connected component of \( \Gamma_{\text{sym},k} \) for all \( k \in \mathbb{N}[\Phi]^W \).

**Corollary 5.3.2.** For all \( \mu \in P \) and all good \( k \in \mathbb{N}[\Phi]^W \), we have

\[
s^\text{sym}_k(\mu) = s^\text{sym}_0(s^\text{tr}_k(\mu)).
\]

**Proof.** Since \( \Gamma_{\text{tr},k} \) is a subgraph of \( \Gamma_{\text{sym},k} \), the \( \longrightarrow \)stabilization of \( \mu \) is the same as the \( \longrightarrow \)stabilization of the \( \longrightarrow \)stabilization of \( \mu \). But the \( \longrightarrow \)stabilization of \( \mu \) is
by definition $\eta_k(\lambda)$ where $\lambda := s^\text{tr}_k(\mu)$. Let $\lambda'$ be the sink of the connected component of $\Gamma_{\text{sym},0}$ containing $\lambda$; hence, $\lambda' = s^\text{sym}_0(\lambda)$. Then Lemma 5.3.1 says that $\eta_k(\lambda')$ is the sink of the connected component of $\Gamma_{\text{sym},k}$ containing $\eta_k(\lambda)$. In other words, the $\xrightarrow{\text{sym},k}$-stabilization of $\lambda$ is $\eta_k(\lambda')$, i.e., $s^\text{sym}_k(\mu) = \lambda' = s^\text{sym}_0(s^\text{tr}_k(\mu))$. 

We want an analog of Lemma 5.3.1 and Corollary 5.3.2 for truncated interval-firing. But to show that the connected components of $\Gamma_{\text{tr},k+1}$ break into connected components of $\Gamma_{\text{sym},k}$ in a way consistent with the map $\eta_k$ turns out to be much more involved. In fact, for technical reasons, we are able to achieve this only assuming that $\Phi$ is simply laced. Nevertheless, the first few steps towards giving truncated analogs of Lemma 5.3.1 and Corollary 5.3.2 do not require the assumption that $\Phi$ be simply laced, so we state them for general $\Phi$.

**Proposition 5.3.3.** Let $\lambda \in P$ be such that $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$. Suppose that $\lambda \xrightarrow{\text{tr},1} \lambda + \beta$ for some $\beta \in \Phi^+$. Then $\lambda \xrightarrow{\text{tr},1} \lambda + w_\lambda(\alpha_i)$ for some simple root $\alpha_i$. Moreover, in this case we have $\eta_k(\lambda) \xrightarrow{\text{tr},k+1} \eta_k(\lambda) + w_\lambda(\alpha_i)$ for all $k \in \mathbb{N}[\Phi]^W$.

**Proof.** If $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$, but $\lambda \xrightarrow{\text{tr},1} \lambda + \beta$ for some $\beta \in \Phi^+$, this must mean that $\langle \lambda, \beta^\vee \rangle = 0$. Applying $w_\lambda^{-1}$, we get $\langle w_\lambda^{-1}(\lambda), w_\lambda^{-1}(\beta)^\vee \rangle = 0$. Since $w_\lambda^{-1}(\beta)^\vee$ is either a positive sum or a negative sum of simple coroots, and because $w_\lambda^{-1}(\lambda) = \lambda_{\text{dom}}$ is dominant, this means there is some simple root $\alpha_i$ such that $\langle w_\lambda^{-1}(\lambda), \alpha_i^\vee \rangle = 0$. But then $\langle \lambda, w_\lambda(\alpha_i)^\vee \rangle = 0$. And note by Proposition 4.3.4 that indeed $w_\lambda(\alpha)$ is positive.

To prove the last sentence of the proposition: note that

$$\langle \eta_k(\lambda), w_\lambda(\alpha_i)^\vee \rangle = \langle \lambda + w_\lambda(\rho_k), w_\lambda(\alpha_i)^\vee \rangle = \langle w_\lambda^{-1}(\lambda), \alpha_i^\vee \rangle + \langle \rho_k, \alpha_i^\vee \rangle = 0 + k(\alpha) = k(\alpha);$$

so indeed, $\eta_k(\lambda) \xrightarrow{\text{tr},k+1} \eta_k(\lambda) + w_\lambda(\alpha_i)$. 

**Proposition 5.3.4.** Let $\lambda \in P$ be a weight such that $\langle \lambda, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$. Let $k \in \mathbb{N}[\Phi]^W$ be good, with $k \geq 1$. Let $\mu \in w_\lambda \Pi_{I^0_{\lambda_{\text{dom}}}} \eta_k(\lambda_{\text{dom}})$. Then $\mu$ and $\eta_k(\lambda)$ belong to the same connected component of $\Gamma_{\text{tr},k+1}$.

**Proof.** First let us prove this proposition when $\lambda$ is dominant and $I^0_{\lambda} = [n]$. In this case, $\eta_k(\lambda) \in \Pi(\rho_{k+1})$. Let $\omega \in \Omega^0_m$ be such that $\lambda \in Q + \rho + \omega$. Note that,
We claim that \( W \) responding sink of \( \Gamma \) actually gets us "trapped" in the correct connected component of 

\[ \text{Suppose that \( \mu \) and \( \nu \) belong to the same connected component of } \Gamma_{\text{sym},k} \text{ that \( \eta_k(\lambda) \) belongs to contains the Weyl orbit } W(\eta_k(\lambda)). \text{ Hence also the the connected component of } \Gamma_{\text{sym},k+1} \text{ that \( \eta_k(\lambda) \) belongs to contains } W(\eta_k(\lambda)). \text{ But this connected component is, as mentioned, } \Pi^Q(\rho_{k+1}) + \omega; \text{ in particular, it is a convex set intersected with } Q + \eta_k(\lambda). \text{ Since } \mu \text{ belongs to the convex hull of } W(\eta_k(\lambda)) \text{ and belongs to the coset } Q + \eta_k(\lambda), \text{ this means that } \mu \in \Pi^Q(\rho_{k+1}) + \omega. \text{ So indeed } \mu \text{ and } \eta_k(\lambda) \text{ belong to the same connected component of } \Gamma_{\text{sym},k+1} \text{ in this case.} \]

Now let us address general \( \lambda \). Note that \( w_\lambda \Phi_{\rho_{0,1}}^+ \) is a choice of positive roots for the sub-root system \( w_\lambda \Phi_{\rho_{0,1}}^+ \). Moreover, by Proposition \[4.3.4\] \( w_\lambda \Phi_{\rho_{0,1}}^+ \) is a subset of positive roots. Hence any truncated interval-firing move (with parameter \( k + 1 \))
we can carry out in \( w_\lambda \Phi_{\rho_{0,1}}^+ \) with choice of positive roots \( w_\lambda \Phi_{\rho_{0,1}}^+ \), we can actually carry out in the original root system \( \Phi \). But then note that \( \langle \lambda, w_\lambda(\alpha_i) \rangle \in \{0, 1\} \) for all \( i \in \rho_{0,1} \); hence the result follows from the previous paragraph by orthogonally projecting \( \lambda \) and \( \mu \) onto \( \text{Span}_R(w_\lambda \Phi_{\rho_{0,1}}^+) \).

The strategy will be to use Proposition \[5.3.3\] to say that whenever we have a \( \rightarrow \)-move from a sink of \( \Gamma_{\text{sym},0} \), we have a corresponding \( \rightarrow \)-move from the corresponding sink of \( \Gamma_{\text{sym},k} \); then we will apply Proposition \[5.3.4\] to say that that move actually gets us "trapped" in the correct connected component of \( \Gamma_{\text{sym},k+1} \). But we have reached the point where to carry out this strategy we must assume that \( \Phi \) is simply laced.

**Proposition 5.3.5.** Suppose that \( \Phi \) is simply laced. Let \( \mu \in P_{\geq 0} \) be dominant. Suppose \( \mu \rightarrow \lambda \) where \( \lambda = \mu + \alpha_i \) for a simple root \( \alpha_i \). Then \( \lambda \in W_{\rho_{0,1}}(\lambda_{\text{dom}}) \).

**Proof.** If \( \mu \) is dominant but \( \mu \rightarrow \lambda \), this must mean that \( \langle \mu, \alpha_i \rangle = 0 \). Let \( \Phi' \) be the irreducible sub-root system of \( \Phi_{\rho_{0}}^+ \) that contains \( \alpha_i \). Let \( \theta' \) be the highest root of \( \Phi' \). We claim that \( \lambda_{\text{dom}} = \mu + \theta' \). First of all, because \( \Phi' \) is also simply laced, the Weyl group \( W' \) of \( \Phi' \) acts transitively on \( \Phi' \) so that there is some \( w \in W' \) with \( w(\theta') = \alpha_i \).
But \( W' \subseteq W_{\mu_0} \), the stabilizer of \( \mu \), so we indeed have \( w(\mu + \theta') = \mu + \alpha_i = \lambda \). Now, why is \( \mu + \theta' \) dominant? Let \( D \) be the Dynkin diagram of \( \Phi \) (which is just an undirected graph since \( \Phi \) is simply laced). For \( I \subseteq [n] \) use \( D[I] \) to denote the restriction of the Dynkin diagram to the vertices in \( I \). Note that \( \Phi' = \Phi_I \) where \( I \) is (the set of vertices of) the connected component of \( D[I_0^0] \) containing \( \alpha_i \). Hence \( \theta' = \sum_{j \in I} c_j \alpha_j \) for some coefficients \( c_j \). First of all, \( \theta' \) is dominant in \( \Phi' \), so if \( j \in I \) then \( \langle \theta', \alpha_j' \rangle \geq 0 \) and hence certainly \( \langle \mu + \theta', \alpha_j' \rangle \geq 0 \). Now suppose \( j \notin I \) and \( j \) is not adjacent in \( D \) to any vertex in \( I \); then clearly \( \langle \theta', \alpha_j' \rangle = 0 \) and so again \( \langle \mu + \theta', \alpha_j' \rangle \geq 0 \). Finally, suppose \( j \notin I \) but \( j \) is adjacent in \( D \) to some vertex in \( I \); then, since \( \Phi \) is simply laced and \( \theta' \) is a positive root of \( \Phi \), we certainly have \( \langle \theta', \alpha_j' \rangle \geq -1 \); but \( \langle \lambda, \alpha_j' \rangle \geq 1 \) since \( j \notin I_0^0 \), and thus \( \langle \lambda + \theta', \alpha_j' \rangle \geq 0 \). So indeed \( \mu + \theta' \) is dominant and so \( \lambda_{\text{dom}} = \mu + \theta' \), as claimed.

Now, suppose for a moment that \( \Phi' \neq A_1 \). Then, writing \( \theta' = \sum_{j=1}^n c_j \omega_j \), we will have that \( c_j \in \{0, 1\} \) for all \( j \in I \); this can be seen for instance by noting that these coefficients \( c_j \) are precisely the number of edges between \( j \) and the “affine vertex” in the affine Dynkin diagram extending \( D[I] \) (this is Lemma 3.4.4). This means that \( W' \subseteq W_{\lambda_{\text{dom}}} \), and so \( w(\lambda_{\text{dom}}) = \lambda \) for some \( w \in W_{\lambda_{\text{dom}}} \); or in other words, we have \( \lambda \in W_{\lambda_{\text{dom}}} \). On the other hand, if \( \Phi' = A_1 \), then actually \( \theta' = \alpha_i \) and so \( \lambda = \lambda_{\text{dom}} \) and the claim is clear.

**Remark 5.3.6.** Note that Proposition 5.3.5 is in general false when \( \Phi \) is not simply laced. For example, take \( \Phi = B_2 \). Then, with \( \mu := 0 \) and \( \lambda := \alpha_1 \) (the long simple root, with numbering as in Figure 3.2), we have \( \mu \rightarrow \lambda \) but \( \lambda \notin W_{t_{\lambda_{\text{dom}}}} \).

**Proposition 5.3.7.** Suppose that \( \Phi \) is simply laced. Let \( \mu \in P_{\geq 0} \) be dominant. Suppose \( \mu \rightarrow \lambda \) where \( \lambda = \mu + \alpha_i \) for a simple root \( \alpha_i \). Then \( \eta_k(\mu) + \alpha \in \Pi_{\lambda_{\text{dom}}}^Q (\eta_k(\lambda_{\text{dom}})) \) for all \( k \geq 0 \).

**Proof.** The statement in the case \( k = 0 \) follows immediately from Proposition 5.3.5 so assume \( k \geq 1 \). By Proposition 5.3.5 we have that \( \lambda \in W_{t_{\lambda_{\text{dom}}}} \), which means, by Proposition 3.1.2, that \( \lambda_{\text{dom}} - \lambda \) is a nonnegative sum of simple roots in \( I_{\lambda_{\text{dom}}}^0 \). Since \( \mu \) is dominant we have \( \eta_k(\mu) = \mu + k \rho \). Then note that \( \eta_k(\mu) + \alpha = \lambda + k \rho = \mu + k \rho + \alpha \).
is actually dominant as well, because $\mu$ is dominant, and $k\rho + \alpha$ is dominant since $\Phi$ is simply laced. Further, observe that $\eta_k(\lambda_{\text{dom}}) - (\eta_k(\mu) + \alpha) = \lambda_{\text{dom}} - \lambda$. But then the fact that $\eta_k(\lambda_{\text{dom}}) - (\eta_k(\mu) + \alpha)$ is a nonnegative sum of simple roots in $I_{\lambda_{\text{dom}}}^{\alpha}$, together with the fact that $\eta_k(\mu) + \alpha$ is dominant, implies, via Proposition 3.1.2, that we have $\eta_k(\mu) + \alpha \in \Pi_{I_{\lambda_{\text{dom}}}^{\alpha}}^Q (\eta_k(\lambda_{\text{dom}}))$.

**Proposition 5.3.8.** Suppose that $\Phi$ is simply laced. Let $\mu \in P$ satisfy $\langle \mu, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$. Suppose that $\mu \rightarrow \lambda$ where $\lambda = \mu + w_\mu(\alpha_i)$ for some simple root $\alpha_i$. Then for all $k \geq 0$, $\eta_k(\mu)$ and $\eta_k(\lambda)$ belong to the same connected component of $\Gamma_{\text{tr},k+1}$.

**Proof.** If $k = 0$ the claim is obvious. So assume $k \geq 1$.

Let $\lambda'$ be the sink of the connected component of $\Gamma_{\text{sym},0}$ containing $\lambda$; hence by Corollary 5.2.2, we have that $\lambda' \in w_\lambda W_{I_{\lambda_{\text{dom}}}^{\alpha}}^r (\lambda_{\text{dom}})$, so in particular $\lambda_{\text{dom}}' = \lambda_{\text{dom}}$. Now, if $\langle \mu, \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$ and $\mu \rightarrow \lambda$ this means that $\langle \mu, w_\mu(\alpha_i)^\vee \rangle = 0$. Hence we also have $\lambda'_{\text{dom}} \rightarrow \mu_{\text{dom}} + \alpha_i$. Then $\mu_{\text{dom}} + \alpha_i \in W_{I_{\lambda_{\text{dom}}}^{\alpha}}^r (\lambda_{\text{dom}})$ by Proposition 5.3.5 and so by applying $w_\mu$ we get $\lambda \in w_\mu W_{I_{\lambda_{\text{dom}}}^{\alpha}}^r (\lambda_{\text{dom}})$. This implies $\lambda' \in w_\mu W_{I_{\lambda_{\text{dom}}}^{\alpha}}^r (\lambda_{\text{dom}})$, so that $(w_\mu w)^{-1}(\lambda')$ is dominant for some $w \in W_{I_{\lambda_{\text{dom}}}^{\alpha}}^r$. But because of Corollary 4.3.2 that means that $w_\mu w = w_\lambda w'$ for some $w' \in W_{I_{\lambda_{\text{dom}}}^{\alpha}}^r$.

By Proposition 5.3.7 we get that $\eta_k(\mu_{\text{dom}}) + \alpha_i \in \Pi_{I_{\lambda_{\text{dom}}}^{\alpha}}^Q (\eta_k(\lambda_{\text{dom}}))$. By applying $w_\mu$ we get $\eta_k(\mu) + w_\mu(\alpha_i) \in w_\mu \Pi_{I_{\lambda_{\text{dom}}}^{\alpha}}^Q (\eta_k(\lambda_{\text{dom}}))$. Note that since $w \in W_{I_{\lambda_{\text{dom}}}^{\alpha}}^r$, we have that $w_\mu \Pi_{I_{\lambda_{\text{dom}}}^{\alpha}}^Q (\eta_k(\lambda_{\text{dom}})) = w_\lambda w' \Pi_{I_{\lambda_{\text{dom}}}^{\alpha}}^Q (\eta_k(\lambda_{\text{dom}}))$. Similarly, $w' \in W_{I_{\lambda_{\text{dom}}}^{\alpha}}^r \subseteq W_{I_{\lambda_{\text{dom}}}^{\alpha}}^{\rho,1}$ implies that $w' \lambda' w' \Pi_{I_{\lambda_{\text{dom}}}^{\alpha}}^Q (\eta_k(\lambda_{\text{dom}})) = w_\lambda \Pi_{I_{\lambda_{\text{dom}}}^{\alpha}}^Q (\eta_k(\lambda_{\text{dom}}))$. Hence, we can conclude that $\eta_k(\mu) + w_\mu(\alpha_i) \in w_\lambda \Pi_{I_{\lambda_{\text{dom}}}^{\alpha}}^Q (\eta_k(\lambda_{\text{dom}}))$. Since $\lambda'$ is a sink of $\Gamma_{\text{sym},0}$ (and thus, by Lemma 4.3.6 satisfies $\langle \lambda', \alpha^\vee \rangle \neq -1$ for all $\alpha \in \Phi^+$), we can apply Proposition 5.3.4 to conclude that $\eta_k(\lambda')$ and $\eta_k(\mu) + w_\mu(\alpha_i)$ belong to the same connected component of $\Gamma_{\text{tr},k+1}$.

But since $\lambda$ and $\lambda'$ belong to the same connected component of $\Gamma_{\text{sym},0}$, Lemma 5.3.1 tells us that $\eta_k(\lambda)$ and $\eta_k(\lambda')$ belong to the same connected component of $\Gamma_{\text{sym},k}$, and hence also belong to the same connected component of $\Gamma_{\text{tr},k+1}$. Then note by Proposition 5.3.3 that we have $\eta_k(\mu) \rightarrow \eta_k(\mu) + w_\mu(\alpha_i)$, so $\eta_k(\mu)$ and $\eta_k(\mu) + w_\mu(\alpha_i)$ belong to the same connected component of $\Gamma_{\text{tr},k+1}$. Putting it all together, $\eta_k(\mu)$
and $\eta_k(\lambda)$ belong to the same connected component of $\Gamma_{tr,k+1}$, as claimed. \hfill \Box

Finally, we are able to prove the desired analogs of Lemma 5.3.1 and Corollary 5.3.2 in the simply laced case.

**Lemma 5.3.9.** Suppose that $\Phi$ is simply laced. For $\lambda, \mu \in P$, if $\lambda$ and $\mu$ belong to the same connected component of $\Gamma_{tr,1}$, then $\eta_k(\lambda)$ and $\eta_k(\mu)$ belong to the same connected component of $\Gamma_{tr,k+1}$ for all $k \geq 0$.

**Proof.** Clearly it suffices to prove this when $\lambda$ is a sink of $\Gamma_{tr,1}$. So let us describe one way to compute the $\longrightarrow_{tr,k+1}$-stabilization of $\eta_k(\mu)$. If $\mu$ is not a sink of $\Gamma_{sym,0}$, then by Lemma 5.3.1 we know that $\eta_k(\mu)$ is in the same connected component of $\Gamma_{sym,k}$ as $\eta_k(\mu')$, where $\mu'$ is the sink of the component of $\Gamma_{sym,0}$ containing $\mu$; so then to compute the $\longrightarrow_{tr,k+1}$-stabilization of $\eta_k(\mu)$ we instead compute the $\longrightarrow_{tr,k+1}$-stabilization of $\eta_k(\mu')$. So now assume that $\mu$ is a sink of $\Gamma_{sym,0}$. Then, if $\mu$ is not a sink of $\Gamma_{tr,1}$, by Proposition 5.3.3 there is a simple root $\alpha_i$ with $\mu \longrightarrow_{tr,1} \mu'$ where $\mu' = \mu + w_\mu(\alpha_i)$. By Proposition 5.3.8 we get that $\eta_k(\mu)$ and $\eta_k(\mu')$ are in the same connected component of $\Gamma_{tr,k+1}$; so again to compute the $\longrightarrow_{tr,k+1}$-stabilization of $\eta_k(\mu)$ we instead compute the $\longrightarrow_{tr,k+1}$-stabilization of $\eta_k(\mu')$. Because $\longrightarrow$ is terminating, this procedure will eventually terminate; in fact, it must terminate at computing the $\longrightarrow_{tr,k+1}$-stabilization of $\eta_k(\mu)$ where $\mu$ is a sink of $\Gamma_{tr,1}$. But there is only one sink of the connected component of $\Gamma_{tr,1}$ containing $\mu$, namely, $\lambda$; so the lemma is proved. \hfill \Box

**Corollary 5.3.10.** Suppose that $\Phi$ is simply laced. Then for all $\mu \in P$ and all $k \geq 0$, we have

$$s_{k+1}^{tr}(\mu) = s_1^{tr}(s_k^{sym}(\mu)).$$

**Proof.** This follows from Lemma 5.3.9 in the same way that Corollary 5.3.2 follows from Lemma 5.3.1. Since $\Gamma_{sym,k}$ is a subgraph of $\Gamma_{sym,k+1}$, the $\longrightarrow_{tr,k+1}$-stabilization of $\mu$ is the same as the $\longrightarrow_{tr,k+1}$-stabilization of the $\longrightarrow_{sym,k}$-stabilization of $\mu$. But the $\longrightarrow_{sym,k}$-stabilization of $\mu$ is by definition $\eta_k(\lambda)$ where $\lambda := s_k^{sym}(\mu)$. Let $\eta_1(\lambda')$ be the sink of the connected component of $\Gamma_{tr,1}$ containing $\lambda$; hence, $\lambda' = s_1^{tr}(\lambda)$. Then Lemma 5.3.1 says that $\eta_k(\eta_1(\lambda')) = \eta_{k+1}(\lambda')$ (this equality follows from Proposition 4.3.3) is the
sink of the connected component of $\Gamma_{tr,k+1}$ containing $\eta_k(\lambda)$. In other words, the $\rightarrow_{tr,k+1}$-stabilization of $\lambda$ is $\eta_{k+1}(\lambda')$, i.e., $s_{k+1}^{tr}(\mu) = \lambda' = s_1^{tr}(s_k^{sym}(\mu))$. 

We expect that (with the appropriate care regarding the goodness of $k \in \mathbb{N}[\Phi]^W$) Lemma 5.3.1 and Corollary 5.3.2 should hold in the non-simply laced case as well, but, as we mentioned in Remark 5.3.6, our method of proof does not work there.

## 5.4 Truncated Ehrhart-like polynomials

The existence of the truncated Ehrhart-like polynomials, in the simply laced case, follows easily from the fact that truncated components decompose into symmetric ones in a consistent way (together with the existence of the symmetric Ehrhart-like polynomials).

**Theorem 5.4.1.** Suppose that $\Phi$ is simply laced. Then, for any $\lambda \in P$, for all $k \geq 1$ the quantity $L_{\lambda}^{tr}(k) = \#(s_k^{tr})^{-1}(\lambda)$ is given by a polynomial in $k$ with integer coefficients.

**Proof.** By Corollary 5.3.10 for any $k \geq 1$ and any $\lambda \in P$ we have

$$\#(s_k^{tr})^{-1}(\lambda) = \#(s_k^{sym})^{-1}(s_1^{tr})^{-1}(\lambda)) = \sum_{\mu \in (s_1^{tr})^{-1}(\lambda)} L_{\mu}^{sym}(k-1).$$

The right-hand side of this expression is an evaluation of a polynomial (with integer coefficients) because of Theorem 5.1.3. Since this identity holds for all $k \geq 1$, we conclude that the desired polynomial $L_{\lambda}^{tr}(k)$ does exist. \hfill \Box

**Conjecture 5.4.2.** For any $\Phi$ and $\lambda \in P$, for all good $k \in \mathbb{N}[\Phi]^W$ the quantity $L_{\lambda}^{tr}(k)$ is given by a polynomial with nonnegative integer coefficients in $k$.

Note that the fact we can take $k = 0$ in Conjecture 5.4.2 means that the constant term of the $L_{\lambda}^{tr}(k)$ polynomials should be 1 (which, compared to the symmetric polynomials, makes them even more like Ehrhart polynomials of zonotopes).
<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$L^\text{tr}_\lambda(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$3k^2 + 3k + 1$</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$3k^2 + 3k + 1$</td>
</tr>
<tr>
<td>$-\omega_1 + \omega_2$</td>
<td>$2k + 1$</td>
</tr>
<tr>
<td>$-\omega_2$</td>
<td>$k + 1$</td>
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<td>$2k + 1$</td>
</tr>
<tr>
<td>$-\omega_1$</td>
<td>$k + 1$</td>
</tr>
<tr>
<td>$\omega_1 + \omega_2$</td>
<td>$2k + 1$</td>
</tr>
<tr>
<td>$-\omega_1 + 2\omega_2$</td>
<td>$k + 1$</td>
</tr>
<tr>
<td>$2\omega_1 - \omega_2$</td>
<td>$k + 1$</td>
</tr>
<tr>
<td>$-2\omega_1 + \omega_2$</td>
<td>$k + 1$</td>
</tr>
<tr>
<td>$\omega_1 - 2\omega_2$</td>
<td>$k + 1$</td>
</tr>
<tr>
<td>$-\omega_1 - \omega_2$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Table 5.2: The polynomials $L^\text{tr}_\lambda(k)$ for $\Phi = A_2$. 
speaking, our Theorem 5.4.1 does not establish that these polynomials have constant term 1 even in the simply laced case.

**Remark 5.4.3.** Table 5.2 records the polynomials $L^\text{tr}_\lambda(k)$ for $\Phi = A_2$, for all $\lambda \in P$ with $I_{\lambda_{\text{dom}}}^{0,1} = [n]$. Compare these polynomials to the graphs of the $A_2$ truncated interval-firing processes in Example 4.1.1. In agreement with Conjecture 5.4.2, all these polynomials have constant coefficient 1. Note that, for $\lambda \in P$ with $L^\text{sym}_\lambda(k) \neq 0$, the constant term of $L^\text{sym}_\lambda(k)$ is by definition equal to the number of vertices in the connected component of $\Gamma_{\text{sym},0}$ containing $\lambda$, which by Lemma 5.3.1 is also equal to the number of connected components of $\Gamma_{\text{tr},k}$ contained in the connected component of $\Gamma_{\text{sym},k}$ with sink $\eta_k(\lambda)$ for all $k \geq 0$.

We know that Conjecture 5.4.2 holds for $\lambda \in \Omega_0^m$. That is because, for $\lambda \in \Omega_0^m$, Lemma 4.8.2 tells us that $(s^\text{tr}_k)^{-1}(\lambda) = \Pi(\rho_k) + \lambda$, and hence $\#(s^\text{tr}_k)^{-1}(\lambda)$ is literally the Ehrhart polynomial of a zonotope.

Polynomials with nonnegative integer coefficients occupy a special place in algebraic combinatorics. Of course it would be great, in the course of positively resolving Conjectures 5.1.5 and 5.4.2, to also give a combinatorial interpretation of the coefficients of the coefficients of these polynomials. (In fact, this is precisely what we do in the next chapter to positively resolve Conjecture 5.1.5.) It would also be extremely interesting to relate these polynomials to the representation theory or algebraic geometry attached to the root system $\Phi$, and establish positivity in that way. These polynomials arose for us in the course of a purely combinatorial investigation, but it is hard to imagine that they do not have some deeper significance if they indeed have nonnegative integer coefficients.

**Remark 5.4.4.** It is also worth considering how the stabilization maps $s^\text{sym}_k$ and $s^\text{tr}_k$ interact with the symmetries of $\Gamma_{\text{sym},k}$ and $\Gamma_{\text{tr},k}$ coming from Theorem 4.2.1. For the symmetric stabilization maps: if $\lambda \in P$ and $w \in W_{\lambda_{\text{dom}}}^{0,1}$, then it is not hard to deduce from Lemma 5.3.1 that

\[(s^\text{sym}_k)^{-1}(w(\lambda)) = w((s^\text{sym}_k)^{-1}(\lambda))\]
for all good \( k \in \mathbb{N}[\Phi]^W \). Of course this implies that

\[
L^\text{sym}(w(\lambda))(k) = L^\text{sym}(\lambda)(k),
\]

in this case. Meanwhile, it appears that if \( w \in C \subseteq W \) and \( \varphi : P \to P \) is the affine map \( \varphi : v \mapsto w(v - \rho/h) + \rho/h \), then

\[
(s_k^\text{tr})^{-1}(\varphi(\lambda)) = \varphi((s_k^\text{tr})^{-1}(\lambda))
\]

for all \( \lambda \in P \) and all good \( k \in \mathbb{N}[\Phi]^W \). But even in the simply laced case, where we have Lemma 5.3.9 at our disposal, in order to conclude that \( s_k^\text{tr} \) indeed respects the symmetry \( \varphi \) in this way, we would need to know that this is the case for \( k = 1 \); and, as we mention in the next section, we do not currently have a great understanding of \( \Gamma_{\text{tr},1} \). So to show that the truncated stabilization maps and polynomials have the expected symmetries coming from the subgroup \( C \) would require some more work.

### 5.5 Iterative descriptions of the stabilization

Finally, let us focus a little more on what our decomposition results tell us about the relationship between the polynomials \( L^\text{sym}_\lambda(k) \) and \( L^\text{tr}_\lambda(k) \), and between the stabilization map \( s_k^\text{sym} \) and \( s_k^\text{tr} \). So, let us assume that \( \Phi \) is simply laced for the remainder of this section. It is clear that Corollaries 5.3.2 and 5.3.10 imply the following identities relating these polynomials for all \( \lambda \in P \) and all \( k \geq 1 \):

\[
L^\text{sym}_\lambda(k) = \sum_{\mu \in (s_k^\text{sym})^{-1}(\lambda)} L^\text{tr}_\mu(k);
\]

\[
L^\text{tr}_\lambda(k) = \sum_{\mu \in (s_k^\text{tr})^{-1}(\lambda)} L^\text{sym}_\mu(k - 1).
\]

What is more, these corollaries also immediately imply some striking, iterative descriptions of the stabilization functions:
Figure 5-2: The map $s_{1}^{\text{sym}}: P \to P$ for $\Phi = A_{1}$. We write $L_{\lambda}^{\text{sym}}(k)$ above each weight $\lambda \in P$.

**Corollary 5.5.1.** Suppose that $\Phi$ is simply laced. Then for all $\mu \in P$ and all $k \geq 1$:

- $s_{1}^{\text{sym}}(\mu) = s_{0}^{\text{sym}}(s_{1}^{\text{tr}}(\mu))$;
- $s_{k}^{\text{sym}}(\mu) = (s_{1}^{\text{sym}})^{k}(\mu)$;
- $s_{k}^{\text{tr}}(\mu) = s_{1}^{\text{tr}}((s_{1}^{\text{sym}})^{k-1}(\mu))$.

Corollary 5.5.1 says that the information of all of the stabilization maps is contained just in $s_{0}^{\text{sym}}$ and $s_{1}^{\text{tr}}$. Now, $s_{0}^{\text{sym}}$ is pretty simple to understand: for example, its fibers are just parabolic Weyl coset orbits (see Corollary 5.2.2). So somehow all of the complexity of all truncated and symmetric interval-firing processes (or, at least all the complexity related to stabilization for these interval-firing processes) is contained just in $\Gamma_{\text{tr},1}$. Admittedly, we do not understand $\Gamma_{\text{tr},1}$ very well. It would be very interesting, for example, to try to find an explicit description of the connected components of $\Gamma_{\text{tr},1}$.

Finally, we end the chapter by discussing another surprising consequence of Corollary 5.5.1: for all $\lambda \in P$ and all $k \geq 1$,

$$\#((s_{1}^{\text{sym}})^{k})^{-1}(\lambda) = L_{\lambda}^{\text{sym}}(k).$$

In other words, we have a map $f: X \to X$ from some discrete set to itself, such that the sizes $\#(f^{k})^{-1}(x)$ of fibers of iterates of this map are given by polynomials (in $k$) for every point $x \in X$. In fact, we have many such maps, one for each simply laced root system. This is a very special property for a self-map of a discrete set to have.

In the next two examples we show what this looks like in the simplest cases.
Example 5.5.2. Although we have so far been eschewing one-dimensional examples, in fact $s_1^{\text{sym}}$ is interesting even for $A_1$. Figure 5-2 depicts $s_1^{\text{sym}}$ for $\Phi = A_1$. Of course in this picture we draw an arrow from $\mu$ to $\lambda$ to mean that $s_1^{\text{sym}}(\mu) = \lambda$. The colors of the vertices correspond to classes of weights modulo the root lattice. We write the polynomials $L_\lambda^{\text{sym}}(k)$ above the weights in this figure. One can verify by hand that in this case $\#((s_1^{\text{sym}})^k)^{-1}(\lambda) = L_\lambda^{\text{sym}}(k)$ for all $\lambda \in P$ and all $k \geq 0$.

Example 5.5.3. Note that when $\Phi = A_2$, we have $\rho \in Q$ and hence $s_1^{\text{sym}}$ preserves the root lattice and so descends to a map $s_1^{\text{sym}}: Q \to Q$. Figure 5-3 depicts $s_1^{\text{sym}}: Q \to Q$ for $\Phi = A_2$. (As with our previous drawings for rank 2 interval-firing processes, we of course only depict the “interesting,” finite portion of this function near the origin.) Compare this figure to the symmetric interval-firing graphs for $A_2$ in Example 4.1.1 and the polynomials $L_\lambda^{\text{sym}}(k)$ for $A_2$ recorded in Table 5.1. Observe that indeed $((s_1^{\text{sym}})^k)^{-1}(0) = \Pi^Q(k \rho)$ for all $k \geq 1$. Also observe that $((s_1^{\text{sym}})^k)^{-1}(\alpha_1 + \alpha_2)$ is the set of $Q$-lattice points on the boundary of $\Pi((k + 1)\rho)$.

In the next chapter we continue the study of these Ehrhart-like polynomials further. We will focus on giving explicit formulas for the Ehrhart-like polynomials, with the aim of establishing the positivity conjectures.
Figure 5-3: The map $s_1^\text{sym}: Q \to Q$ for $\Phi = A_2$. The origin is the central point (i.e., the one with a loop). The root $\alpha_1 + \alpha_2$ is the point immediately north-east of the origin.
Chapter 6

Interval root-firing: formulas for Ehrhart-like polynomials

In this final chapter we provide formulas for the Ehrhart-like polynomials introduced in Chapter 5. The material in this chapter is joint work with Alexander Postnikov and appears in [44].

Continue to fix $\Phi$ and retain all the set-up and notation from the previous two chapters. In this chapter we prove “half” of the positivity conjectures about Ehrhart-like polynomials from Chapter 5 by providing an explicit, positive formula for the symmetric Ehrhart-like polynomials. The basic idea is that the permutohedron non-escaping lemma (Lemma 4.5.2) implies that for $\lambda \in P_{\geq 0}$ with $I_{\lambda}^{0,1} = [n]$ we have

$$(s_{k}^{\text{sym}})^{-1}(\lambda) = \Pi_{Q}^{Q}(\lambda + \rho_{k}) \setminus \bigcup_{\mu \neq \lambda \in P_{\geq 0}, \mu \leq \lambda} \Pi_{Q}^{Q}(\mu + \rho_{k}),$$

where we remind the reader that $\mu \leq \lambda$ means $\mu$ is less than $\lambda$ in root order, i.e., $\mu \leq \lambda$ means that $\lambda - \mu \in Q_{\geq 0}$. So we can apply inclusion-exclusion, together with some fundamental facts about root order established by Stembridge [74], in order to give a formula for $I_{\lambda}^{\text{sym}}(k) = (s_{k}^{\text{sym}})^{-1}(\lambda)$, as long as we have a precise enough formula for $\#\Pi_{Q}^{Q}(\lambda + \rho_{k})$.

In order to establish the requisite formula for $\#\Pi_{Q}^{Q}(\lambda + \rho_{k})$, we need to show
that slices of permutohedra satisfy a subtle integrality property. The bulk of this chapter is devoted to establishing this integrality property of slices of permutohedra. Although this integrality property can be formulated in a totally uniform way, we resort to a case-by-case analysis in our proof. We leave it as an open problem to find a uniform proof of this integrality property, which seems like it could have some representation-theoretic significance.

At the end of the chapter we discuss future directions, including formulas for the truncated Ehrhart-like polynomials. Indeed, the formula we obtain for the symmetric Ehrhart-like polynomials very naturally suggests a conjectural formula for the truncated Ehrhart-like polynomials. But actually that conjectural formula for the truncated Ehrhart-like polynomials turns out to be false in general! Nevertheless, as we discuss, it may hold in some special cases such as Type A and B.

6.1 Lattice points in polytope plus dilating zonotope

In Chapter 5 we mentioned a result of Stanley [72] which says that the Ehrhart polynomial $L_Z(k)$ of a lattice zonotope $Z$ has nonnegative integer coefficients. In fact, he gave the following explicit formula:

$$L_Z(k) = \sum_{X \subseteq \{v_1, \ldots, v_m\}, \text{X is linearly independent}} r\text{Vol}_{\mathbb{Z}^n}(X) k^\#X, \quad (6.1)$$

where $Z := \sum_{i=1}^m [0, v_i]$ is the Minkowski sum of the lattice vectors $v_1, \ldots, v_m \in \mathbb{Z}^n$. (See [12] §9 for another presentation of this result.) Our Theorem 5.1.1 above gave a slight extension of Stanley’s result: we showed that for any convex lattice polytope $P$ and any lattice zonotope $Z$, the number of lattice points in $P + kZ$ is given by a polynomial with nonnegative integer coefficients in $k$. The case where $P$ is a point recaptures Stanley’s result. However, in Theorem 5.1.1 we did not give any explicit formula for the coefficients of the polynomial analogous to the formula (6.1) for zonotopes. The first thing we need to do in the present chapter is provide such a
formula, whose simple proof we also go over now. In fact, this result is stated most naturally in its “multi-parameter” formulation:

**Theorem 6.1.1.** Let $\mathcal{P}$ be a convex lattice polytope in $\mathbb{R}^n$, and $v_1, \ldots, v_m \in \mathbb{Z}^n$. Set $\mathcal{Z} := \sum_{i=1}^m [0, v_i]$, and for $k = (k_1, \ldots, k_m) \in \mathbb{N}^m$ define $k\mathcal{Z} := \sum_{i=1}^m k_i [0, v_i]$. Then for any $k \in \mathbb{N}^m$ we have

$$\#(\mathcal{P} + k\mathcal{Z}) \cap \mathbb{Z}^n = \sum_{X \subseteq \{v_1, \ldots, v_m\}, \text{X is linearly independent}} \#(\text{quot}_X(\mathcal{P}) \cap \text{quot}_X(\mathbb{Z}^n)) \cdot r\text{Vol}_{\mathbb{Z}^n}(X) k^X$$

where $k^X := \prod_{x \in X} k_i$ and $\text{quot}_X : \mathbb{R}^n \to \mathbb{R}^n / \text{Span}_\mathbb{R}(X)$ is the canonical quotient map.

**Proof.** The standard proof of Stanley’s formula for the Ehrhart polynomial of a lattice zonotope (and indeed the proof originally given by Stanley [72]) is via “paving” the zonotope, i.e., decomposing it into disjoint half-open parallelepipeds (see [12, §9]). This decomposition goes back to Shephard [69]. In the proof of Theorem 5.1.1 we explained how the technique of paving can be adapted to apply to $\mathcal{P} + k\mathcal{Z}$ as well. But we can actually establish the claimed formula for $\#(\mathcal{P} + k\mathcal{Z}) \cap \mathbb{Z}^n$ just from some general properties of “multi-parameter” Ehrhart polynomials. Actually we need only the following result:

**Lemma 6.1.2.** Let $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_m$ be convex lattice polytopes in $\mathbb{R}^n$. Then for non-negative integers $k_0, \ldots, k_m \in \mathbb{N}$, the number of lattice points in $k_0 \mathcal{Q}_0 + \cdots + k_m \mathcal{Q}_m$ is a polynomial (with real coefficients) in $k_0, \ldots, k_m$ of total degree at most the dimension of the smallest affine subspace containing all of $\mathcal{Q}_0, \ldots, \mathcal{Q}_m$.

Lemma 6.1.2 is due to McMullen [55, Theorem 6].

First of all, Lemma 6.1.2 immediately gives that $\#(\mathcal{P} + k\mathcal{Z}) \cap \mathbb{Z}^n$ is a polynomial in $k$: we can just take $\mathcal{Q}_0 := \mathcal{P}$, $\mathcal{Q}_i := [0, v_i]$ for $i \in [m]$, and set $k_0 := 1$. We use $f(k)$ to denote this polynomial.

Now we check that each coefficient of $f(k)$ agrees with the claimed formula. So fix some $a = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and set $k^a := \sum_{i=1}^m a_i$. We will check that the coefficient of $k^a$ is as claimed. By substituting $k_i := 0$ for any $i$ for which $a_i = 0$, we
can assume that \( a_i \neq 0 \) for all \( i \in [m] \). Thus, set \( X := \{v_1, \ldots, v_m\} \). We can count the number of lattice points in \( \mathcal{P} + k\mathbb{Z} \) by dividing them into “slices” which lie in affine translates of \( \text{Span}_\mathbb{R}(X) \). Accordingly, let \( u_1, u_2, \ldots, u_\ell \in \mathbb{Z}^n \) be such that:

- \( (u_i + \text{Span}_\mathbb{R}(X)) \cap \mathcal{P} \neq \emptyset \) for all \( i \in [\ell] \);
- if \( (u + \text{Span}_\mathbb{R}(X)) \cap \mathcal{P} \neq \emptyset \) for some \( u \in \mathbb{Z}^n \), then \( u + \text{Span}_\mathbb{R}(X) = u_i + \text{Span}_\mathbb{R}(X) \) for some \( i \in [\ell] \);
- \( u_i + \text{Span}_\mathbb{R}(X) \neq u_j + \text{Span}_\mathbb{R}(X) \) for \( i \neq j \in [\ell] \).

Set \( \mathcal{P}_i := \mathcal{P} \cap (u_i + \text{Span}_\mathbb{R}(X)) \) for \( i = 1, \ldots, \ell \) and observe that

\[
(\mathcal{P} + k\mathbb{Z}) \cap \mathbb{Z}^n = \bigcup_{i=1}^\ell (\mathcal{P}_i + k\mathbb{Z}) \cap \mathbb{Z}^n.
\]

Because \( \mathbb{Z} \) is full-dimensional inside of \( \text{Span}_\mathbb{R}(X) \), for each \( i \in [\ell] \) there is \( x_i \in \mathbb{N}^m \) such that \( \mathcal{P}_i \) is contained, up to lattice translation, in \( x_i\mathbb{Z} \). Hence we obtain the inequalities

\[
\sum_{i=1}^\ell \#(k\mathbb{Z} \cap \mathbb{Z}^n) \leq f(k) \leq \sum_{i=1}^\ell \#((k + x_i)\mathbb{Z} \cap \mathbb{Z}^n),
\]

for all \( k \in \mathbb{N}^m \). First consider the case where either \( k^a \neq k^X \), or \( X \) is not linearly independent. Then \( \sum_{i=1}^m a_i \) is strictly greater than the dimension of \( \text{Span}_\mathbb{R}(X) \). But by Lemma 6.1.2 the left- and right-hand sides of (6.2) are polynomials in \( k \) of degree at most the dimension of \( \text{Span}_\mathbb{R}(X) \). So in this case it must be that the coefficient of \( k^a \) in \( f(k) \) is zero. Now assume that \( k^a = k^X \) and \( X \) is linearly independent. In this case, \( \mathbb{Z} \) is a parallelepiped whose relative volume is in fact \( r\text{Vol}(X) \): see for example [12, Lemma 9.8]. Hence, when we make the substitution \( k_i := k \) for all \( i \in [m] \), the leading coefficient of both the left- and right-hand sides of (6.2) is \( \ell \cdot r\text{Vol}(X) \); furthermore, the degree of both of these polynomials is the dimension of \( \text{Span}_\mathbb{R}(X) \), which is the same as \( \#X \). Thus we conclude that \( \ell \cdot r\text{Vol}(X) \) is also the coefficient of \( k^a \) in \( f(k) \). But then note that \( \ell = \#(\text{quot}_X(\mathcal{P}) \cap \text{quot}_X(\mathbb{Z}^n)) \), finishing the proof of the theorem.

**Remark 6.1.3.** Although we will not need this, we believe that Theorem 6.1.1 holds
In general, the formula in Theorem 6.1.1 is not ideal from a combinatorial perspective because in order to compute the quantity \( \# (\text{quot}_X(\mathcal{P}) \cap \text{quot}_X(\mathbb{Z}^n)) \) we have to consider every rational point in \( \mathcal{P} \). But in particularly nice situations we may actually have that \( \text{quot}_X(\mathcal{P}) \cap \text{quot}_X(\mathbb{Z}^n) = \text{quot}_X(\mathcal{P} \cap \mathbb{Z}^n) \) for all \( X \subseteq \{v_1, \ldots, v_m\} \). In fact, this is exactly what happens in the case of Theorem 6.1.1 that is relevant to interval-firing: the Minkowski sum of a permutohedron and a dilating regular permutohedron.

Before we prove this integrality property of permutohedra, let us show how it can fail in the more general situation of arbitrary lattice polytopes plus lattice zonotopes.

**Example 6.1.4.** Let \( \mathcal{P} := \text{ConvexHull}\{(0, 3), (1, 4), (2, 0)\} \subseteq \mathbb{R}^2 \). Let \( v := (1, 1) \in \mathbb{Z}^2 \) and set \( \mathcal{Z} := [0, v] \), a zonotope (in fact, a line segment). Figure 6-1 depicts \( \mathcal{P} \) as the region shaded in blue, and \( \mathcal{P} + \mathcal{Z} \) as the region shaded in blue together with the region shaded in red. The dashed red lines are all the affine subspaces of the form \( u + \text{Span}_\mathbb{R}(v) \) for which \( u + \text{Span}_\mathbb{R}(v) \cap \mathcal{P} \neq \emptyset \). There are six such subspaces. However, only four of these subspaces satisfy \( u + \text{Span}_\mathbb{R}(v) \cap (\mathcal{P} \cap \mathbb{Z}^2) \neq \emptyset \). In other words, we have \( \#\text{quot}_X(\mathcal{P} \cap \mathbb{Z}^2) = 4 < 6 = \#\text{quot}_X(\mathcal{P}) \cap \text{quot}_X(\mathbb{Z}^2) \) when \( X := \{v\} \). We can
verify that \((\mathcal{P} + k\mathbb{Z}) \cap \mathbb{Z}^2 = 6k + 5\), in agreement with Theorem 6.1.1.

The reason Example 6.1.4 fails to satisfy \(\text{quot}_X(\mathcal{P}) \cap \text{quot}_X(\mathbb{Z}^n) = \text{quot}_X(\mathcal{P} \cap \mathbb{Z}^n)\) is that the polytope \(\mathcal{P}\) is too “thin” in the direction of \(X\). So in order to show that permutohedra do satisfy this integrality property, we need, roughly speaking, to show that they cannot be too “thin” in any direction spanned by roots. Intuitively, the \(W\)-invariance of permutohedra prevents them from being “thin” in any given root direction (because otherwise they would be “thin” in every root direction). But this is just a rough intuition for why permutohedral slices might satisfy the requisite integrality property. The actual argument, which we give in the next section, is rather involved and eventually requires us to invoke the classification of root systems.

### 6.2 Lattice points in permutohedron plus dilating regular permutohedron

In this section we prove the subtle integrality property of slices of permutohedra and hence deduce the requisite formula for the number of lattice points in a permutohedron plus dilating regular permutohedron. Before we do that, we need a few preparatory results. We will often need to consider projections of weights onto sub-root systems. Thus, for \(X \subseteq \Phi\) we use \(\pi_X : V \to \text{Span}_\mathbb{R}(X)\) to denote the orthogonal (with respect to \(\langle \cdot, \cdot \rangle\)) projection of \(V\) onto \(\text{Span}_\mathbb{R}(X)\). Note that for \(\lambda \in P\) it is always the case that \(\pi_X(\lambda)\) is a weight of \(\Phi \cap \text{Span}_\mathbb{R}(X)\), although it need not be a weight of \(\Phi\). Similarly, if \(\lambda \in P_{\geq 0}\) is dominant then \(\pi_X(\lambda)\) is a dominant weight of \(\Phi \cap \text{Span}_\mathbb{R}(X)\). For \(I \subseteq [n]\) we use the notation \(\pi_I := \pi_{\{\alpha_i : i \in I\}}\). Observe that \(\pi_I(\sum_{i=1}^n c_i \omega_i) = \sum_{i \in I} c_i \omega'_i\), where \(\{\omega'_i : i \in I\}\) is the set of fundamental weights of \(\Phi_I\).

We need a result giving a “standard form” for sub-root systems.

**Proposition 6.2.1.** Let \(X \subseteq \Phi^+\). Let \(v \in V\) be such that \(\pi_X(v) = 0\). Then there exists some \(w \in W\) and \(I \subseteq [n]\) such that \(w \text{Span}_\mathbb{R}(X) = \text{Span}_\mathbb{R}\{\alpha_i : i \in I\}\) and \(wv \in P_{\geq 0}^\mathbb{R}\).
Proof. This result is a slight extension of a result of Bourbaki [21, Chapter IV, §1.7, Proposition 24], which is equivalent to the present proposition but without the requirement $wv \in P_{\geq 0}$. If $v = 0$, then $wv \in P_{\geq 0}$ is automatically satisfied for any $w \in W$, so let us assume that $v \neq 0$.

Following Bourbaki, let us explain one way to choose a set of positive roots. Namely, suppose that $\preceq$ is a total order on $V$ compatible with the real vector space structure in the sense that if $u \preceq v$ then $u + u' \preceq v + u'$ and $\kappa u \preceq \kappa v$ for all $u, u', v \in V$ and $\kappa \in \mathbb{R}_{\geq 0}$. Then $\{ \alpha \in \Phi : 0 \preceq \alpha \}$ will be a valid choice of positive roots for $\Phi$.

We proceed to define an appropriate total order $\preceq$. Let $\beta_1, \ldots, \beta_\ell$ be a choice of simple roots for $\Phi \cap \text{Span}_\mathbb{R}(X)$. Then let $v_1, \ldots, v_n$ be an ordered basis of $V$ such that: $v_1 = v; \; v_{(n-\ell)+i} = \beta_i$ for all $i = 1, \ldots, \ell; \; v_1$ is orthogonal to all of $v_2, \ldots, v_n$. (Such a basis exists because $\pi_X(v) = 0$ implies $v$ is orthogonal to all of $\beta_1, \ldots, \beta_\ell$.) Then let $\preceq$ be the lexicographic order on $V$ with respect to the ordered basis $v_1, \ldots, v_n$; that is to say, $\sum_{i=1}^n a_i v_i \preceq \sum_{i=1}^n a'_i v_i$ means that either $\sum_{i=1}^n a_i v_i = \sum_{i=1}^n a'_i v_i$ or there is some $i \in [n]$ such that $a_j = a'_j$ for all $1 \leq j < i$ and $a_i < a'_i$.

It is clear that $\beta_1, \ldots, \beta_\ell$ are minimal (with respect to $\preceq$) in $\{ \alpha \in \Phi : 0 \preceq \alpha \}$, which implies that they are simple roots of $\Phi$ for the choice $\{ \alpha \in \Phi : 0 \preceq \alpha \}$ of positive roots.

Moreover, for any $u = \sum_{i=1}^n a_i v_i \in V$ we have that $\langle v, u \rangle = a_1$ because $v$ is orthogonal to $v_2, \ldots, v_n$. Hence for any $u \in V$ with $0 \preceq u$ we have $\langle v, u \rangle \geq 0$. This means in particular that $\langle v, \alpha^\vee \rangle \geq 0$ for any $\alpha \in \Phi$ with $0 \preceq \alpha$.

Since all choices of positive roots are equivalent up to the action of the Weyl group, there exists $w \in W$ such that $w\{ \alpha \in \Phi : 0 \preceq \alpha \} = \Phi^+$. This $w$ transports $\{ \beta_1, \ldots, \beta_\ell \}$ to a subset of simple roots, so we get $w \text{Span}_\mathbb{R}(X) = \text{Span}_\mathbb{R}\{ \alpha_i : i \in I \}$. That $wv \in P_{\geq 0}$ follows from the previous paragraph.

Next we need to review some basic facts about root order which appear in the seminal paper of Stembridge [74] (but may have been known in some form earlier). First of all, we have that dominant weights are always maximal in root order in their Weyl group orbits.

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Proposition 6.2.2 ([74, Lemma 1.7]). For any $\lambda \in P$ we have that $\lambda \leq \lambda_{\text{dom}}$.

Now let us consider root order restricted to the set of dominant weights. Root order on all of $P$ is trivially a disjoint union of lattices: it is isomorphic to $f$ copies of $\mathbb{Z}^n$ where $f$ is the index of $Q$ in $P$ (this index $f$ is called the \textit{index of connection} of the root system $\Phi$). Stembridge proved, what is much less trivial, that the root order on $P_{\geq 0}$ is also a disjoint union of $f$ lattices. Let us explain how he did this. For $\lambda = \sum_{i=1}^{n} a_i \alpha_i, \mu = \sum_{i=1}^{n} a'_i \alpha_i \in P$ with $\lambda - \mu \in Q$, we define their \textit{meet} to be

$$\lambda \land \mu := \sum_{i=1}^{n} \min(a_i, a'_i) \alpha_i.$$ 

This is obviously the meet of $\lambda$ and $\mu$ in $P$ with respect to the partial order $\leq$. Stembridge proved the following about this meet operation:

Proposition 6.2.3 ([74, Lemma 1.2]). Let $\lambda, \mu \in P$ with $\lambda - \mu \in Q$. Let $i \in [n]$ and suppose that $\langle \lambda, \alpha^\vee_i \rangle \geq 0$ and $\langle \mu, \alpha^\vee_i \rangle \geq 0$. Then $\langle \lambda \land \mu, \alpha^\vee_i \rangle \geq 0$. Hence, in particular, if $\lambda, \mu \in P_{\geq 0}$ then $\lambda \land \mu \in P_{\geq 0}$ as well.

Strictly speaking, Proposition 6.2.3 only implies that $(P_{\geq 0}, \geq)$ is a disjoint union of meet-semilattices; a little more is needed to show that it is a disjoint union of lattices. At any rate, Proposition 6.2.3 compels us to ask what the minimal elements of $(P_{\geq 0}, \geq)$ are; there will again be $f$ of these, one for every coset of $Q$ in $P$ (because $P_{\geq 0}^R \subseteq Q_{\geq 0}^R$, every element of $(P_{\geq 0}, \geq)$ has to be greater than or equal to a minimal element). Actually, we explained earlier in Chapter 3 that these minimal elements are zero or minuscule weights, but let us now reiterate this fact because it will be so crucial in all of our following analysis.

Recall that a dominant, nonzero weight $\lambda \in P_{\geq 0}$ is called \textit{minuscule} if we have that $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$.

Proposition 6.2.4 ([74, Lemma 1.12]). The minimal elements of $(P_{\geq 0}, \geq)$ are precisely the minuscule weights of $\Phi$ together with zero.

Proposition 6.2.4 together with Proposition 3.1.2 gives another characterization of minuscule weights (which we also mentioned earlier): we have that $\Pi(\lambda) = W(\lambda)$
for $\lambda \in P_{\geq 0}$ if and only if $\lambda$ is zero or minuscule. Another simple property of minuscule weights that we will use repeatedly is: if $\lambda \in P$ is zero or a minuscule weight of $\Phi$, then $\pi_{X}(\lambda)$ is a zero or a minuscule weight of $\Phi \cap \text{Span}_{\mathbb{R}}(X)$ for any $X \subseteq \Phi$.

We are now ready to give the proof of the integrality property of slices of permutohedra.

**Lemma 6.2.5.** Let $\lambda \in P_{\geq 0}$ be a dominant weight, let $\mu \in Q + \lambda$, and let $X \subseteq \Phi^{+}$. Suppose that $\Pi(\lambda) \cap (\mu + \text{Span}_{\mathbb{R}}(X)) \neq \emptyset$. Then $\Pi^{Q}(\lambda) \cap (\mu + \text{Span}_{\mathbb{R}}(X)) \neq \emptyset$.

**Proof.** Define $\mu_{X}^{0} \in V$ to be the unique vector in the affine subspace $\mu + \text{Span}_{\mathbb{R}}(X)$ for which $\pi_{X}(\mu_{X}^{0}) = 0$. We claim that $\mu_{X}^{0} \in \Pi(\lambda)$. Indeed, $\mu_{X}^{0}$ is the “inner-most” vector in $\mu + \text{Span}_{\mathbb{R}}(X)$, so if any vector of $\mu + \text{Span}_{\mathbb{R}}(X)$ lies in $\Pi(\lambda)$ then $\mu_{X}^{0}$ must as well. To explain this more formally, let $W'$ denote the Weyl group of the sub-root system $\Phi \cap \text{Span}_{\mathbb{R}}(X)$. There is of course the natural inclusion $W' \subseteq W$. Observe that for any $u \in \text{Span}_{\mathbb{R}}(X)$ we have $0 \in \text{Convex Hull} W'(u)$ (for instance, by Proposition 3.1.2). But since $\text{Convex Hull} W'(u) = \mu_{X}^{0} + \text{Convex Hull} W'(\pi_{X}(u))$ for any $u \in \mu + \text{Span}_{\mathbb{R}}(X)$, we conclude that we have $\mu_{X}^{0} \in \text{Convex Hull} W'(u)$ for any vector $u \in \mu + \text{Span}_{\mathbb{R}}(X)$. Hence we have $\mu_{X}^{0} \in \Pi(u)$ for any $u \in \mu + \text{Span}_{\mathbb{R}}(X)$. By supposition there exists some $u \in \Pi(\lambda) \cap (\mu + \text{Span}_{\mathbb{R}}(X))$, so $\mu_{X}^{0} \in \Pi(u) \subseteq \Pi(\lambda)$ as claimed.

Because of the $W$-invariance of $\Pi(\lambda)$, if the statement of this lemma is true for $X$ and $\mu$, then it is true for $wX$ and $w\mu$ as well. Hence, by Proposition 6.2.1 we may assume that $\text{Span}_{\mathbb{R}}(X) = \text{Span}_{\mathbb{R}}\{\alpha_{i} : i \in I\}$ for some $I \subseteq [n]$ and that $\mu_{X}^{0} \in P_{\geq 0}^{R}$. Note importantly that $\mu_{X}^{0}$ need not be a weight of $\Phi$: in general it is just a vector in $V$, and even if $\mu_{X}^{0}$ is a weight of $\Phi$ it need not belong to the coset $Q + \lambda$.

Having made some assumptions about $X$ and $\mu_{X}^{0}$, let us now show that we can also make some assumptions about $\mu$. First of all, note that $\sum_{\alpha \in \Phi_{I}} \alpha$ has positive inner product with every $\alpha_{i}^{\vee}$ for $i \in I$ (because it is equal to twice the Weyl vector of $\Phi_{I}$). Thus by repeatedly adding the vector $\sum_{\alpha \in \Phi_{I}} \alpha$ to $\mu$, we can assume that $\langle \mu, \alpha_{i}^{\vee} \rangle \geq 0$ for all $i \in I$. Furthermore, we claim $\lambda \wedge \mu \in \mu + \text{Span}_{\mathbb{R}}(X)$. Indeed, $\langle \mu, \omega_{i}^{\vee} \rangle = \langle \mu_{X}^{0}, \omega_{i}^{\vee} \rangle$ for any $i \notin I$. But since $\mu_{X}^{0} \in \Pi(\lambda)$, we have that $\langle \mu_{X}^{0}, \omega_{i}^{\vee} \rangle \leq \langle \lambda, \omega_{i}^{\vee} \rangle$ for any $i \notin I$ by
Proposition 3.1.2. So $\langle \mu, \omega^\vee_i \rangle \leq \langle \lambda, \omega^\vee_i \rangle$ for any $i \notin I$, which implies that $\mu - (\lambda \land \mu)$ belongs to $\text{Span}_R \{ \alpha_i : i \in I \}$. Because we have assumed that $\langle \mu, \alpha^\vee_i \rangle \geq 0$ for all $i \in I$, by Proposition 6.2.3 we conclude that $\langle \lambda \land \mu, \alpha^\vee_i \rangle \geq 0$ for all $i \in I$ as well. In other words, we know that $\pi_I(\lambda \land \mu)$ is dominant in $\Phi_I$. But by Proposition 6.2.4 the minimal, in root order, dominant weights are either zero or minuscule. Hence there exists some weight $\nu \in \mu + \text{Span}_R(X)$ for which $\pi_I(\nu)$ is either zero or a minuscule weight of $\Phi_I$, and that $\nu \leq \lambda \land \mu$. By replacing $\mu$ with $\nu$, we can thus assume that $\pi_I(\mu)$ is zero or a minuscule weight of $\Phi_I$, and that $\mu \leq \lambda$.

To summarize the above, without loss of generality we from now on in the proof of this lemma assume the following list of additional conditions:

(a) $\text{Span}_R(X) = \text{Span}_R \{ \alpha_i : i \in I \}$ for some $I \subseteq [n]$;

(b) the unique vector $\mu_0^X \in \mu + \text{Span}_R(X)$ with $\pi_X(\mu_0^X) = 0$ satisfies $\mu_0^X \in P_{\geq 0}^R$;

(c) $\pi_I(\mu)$ is zero or a minuscule weight of $\Phi_I$;

(d) $\mu \leq \lambda$.

Let us now give two rank 2 examples of what the setting of this lemma might look like after we have reduced to a case satisfying conditions (a)-(d) above. In these examples we follow the numbering of the simple roots from Figure 3-2.

**Example 6.2.6.** Suppose $\Phi = B_2$, $\lambda = \omega_1 + \omega_2$, $\mu = -\omega_1 + \omega_2$ and $X = \{ \alpha_1 \}$. This is depicted on the left of Figure 6-2. In this figure, the permutohedron $\Pi(\lambda)$ is the region shaded in blue. The dominant cone $P_{\geq 0}^R$ is the region shaded in green. The affine subspace $\mu + \text{Span}_R(X)$ is the dashed red line (in fact in this case it is a linear subspace). Points in the coset $Q + \lambda$ are represented by black circles; other points of interest are marked by yellow circles circles. It is easy to verify that conditions (a)-(d) hold in this case: for example, $\pi_I(\mu) = \mu = \frac{1}{2} \alpha_1$ is a minuscule weight of $\Phi_{\{1\}}$. Observe that $\mu_0^X = 0$ is a weight of $\Phi$, but that it does not belong to the coset $Q + \lambda$.

**Example 6.2.7.** Suppose $\Phi = G_2$, $\lambda = \omega_1 + \omega_2$, $\mu = \omega_1$ and $X = \{ \alpha_2 \}$. This is depicted on the right of Figure 6-2. In this figure, the permutohedron $\Pi(\lambda)$ is
Figure 6-2: Examples 6.2.6 and 6.2.7 of what the setting of Lemma 6.2.5 might look like for the rank 2 root systems $B_2$ and $G_2$.

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the lemma. We assert that if Claim I is true, then for \( \mu \) satisfying conditions (a)-(d) above we have \( \mu \in \Pi(\lambda) \). First observe that \( \mu \) and \( I \) satisfying (a)-(d) also satisfy the conditions of Claim I. Then note that by Proposition 6.2.3 we have \( \lambda \land \mu_{\text{dom}} \in P_{\geq 0} \). Moreover, because \( \mu \leq \mu_{\text{dom}} \) by Proposition 6.2.2 and because \( \mu \leq \lambda \) by (d), we have that \( \mu \leq \lambda \land \mu_{\text{dom}} \). Hence, Claim I says \( \mu_{\text{dom}} = \lambda \land \mu_{\text{dom}} \), which implies \( \mu_{\text{dom}} \leq \lambda \). So by Proposition 3.1.2 we get that \( \mu_{\text{dom}} \in \Pi(\lambda) \) and hence that \( \mu \in \Pi(\lambda) \), thus finishing the proof of the lemma.

We now proceed to prove Claim I. Unfortunately, we were unable to find a uniform proof of this claim; instead, our proof will be type-by-type. First let us deal with the classical types. In the classical types, something even stronger than Claim I is true; namely, we have the following:

**Claim II.** Suppose that \( \Phi \) is of classical type (i.e., of Type \( A_n, B_n, C_n, \) or \( D_n \)). Let \( I \subseteq [n] \). Let \( \mu \in P \) be such that \( \mu - \pi_I(\mu) \in P_{\geq 0} \) and \( \pi_I(\mu) \) is zero or a minuscule weight of \( \Phi_I \). Set \( J := \{ i \in [n] : \langle \mu, \omega_i \rangle \neq \langle \mu_{\text{dom}}, \omega_i \rangle \} \). Then \( \pi_J(\mu_{\text{dom}}) \) is zero or a minuscule weight of \( \Phi_J \).

Let us explain why this claim is indeed stronger, i.e., why Claim II implies Claim I in the case that \( \Phi \) is of classical type. So suppose \( \Phi \) is of classical type and that Claim II holds. Let \( I \) and \( \mu \) be as in Claim II and suppose \( \nu \in P_{\geq 0} \) with \( \mu \leq \nu \leq \mu_{\text{dom}} \). Let \( J \) be as in Claim II. By the definition of \( J \) we have \( \mu_{\text{dom}} - \mu \in \Phi_J \). This means that \( \nu - \mu \in \Phi_J \) as well. Thus, with respect to the root order of \( \Phi_J \), we have that \( \pi_J(\nu) \leq \pi_J(\mu_{\text{dom}}) \). But also, \( \pi_J(\nu) \) is a dominant weight of \( \Phi_J \). Since \( \pi_J(\mu_{\text{dom}}) \) is zero or a minuscule weight of \( \Phi_J \), and since by Proposition 6.2.4 it is thus minimal among dominant weights of \( \Phi_J \) in root order, we conclude that \( \pi_J(\nu) = \pi_J(\mu_{\text{dom}}) \). Hence,

\[
\nu - \mu = \pi_J(\nu - \mu) = \pi_J(\nu) - \pi_J(\mu) = \pi_J(\mu_{\text{dom}}) - \pi_J(\mu) = \pi_J(\mu_{\text{dom}} - \mu) = \mu_{\text{dom}} - \mu,
\]

which means that \( \nu = \mu_{\text{dom}} \), as required.

The proof of Claim II involves a careful analysis of combinatorial properties of the Dynkin diagrams and inverse Cartan matrices in the classical types. These arguments
are quite technical and tedious, and for that reason we have relegated the proof of this claim to the final section in this thesis, Section 6.5

Now finally we turn to the exceptional types. To handle the exceptional types, we use an exhaustive computer verification. This is possible because there are only finitely many exceptional types. However, note that, even for a fixed root system \( \Phi \), Claim I is a priori not a finite statement: there are finitely many choices for \( I \subseteq [n] \), but there are potentially infinitely many \( \mu \) that enter into the statement of the claim. Thus, we need to modify this claim slightly so that we need only consider finitely many \( \mu \). The intuition as to why this is possible is that increasing the \( \omega_i \) coefficients of \( \mu \) should only make the claim easier to establish, and so to verify the claim we can restrict to \( \mu \) with \( \omega_i \) coefficients less than some upper bound (in fact, the upper bound will be 1); but also, these \( \omega_i \) coefficients get a lower bound from the fact that \( v \in P^R_{\geq 0} \).

**Claim III.** Suppose that \( \Phi \) is of exceptional type (i.e., of Type \( G_2 \), \( F_4 \), \( E_6 \), \( E_7 \), or \( E_8 \)). Let \( I \subseteq [n] \). Let \( \mu \in P \) be such that \( \mu - \pi_I(\mu) \in P^R_{\geq 0} \) and \( \pi_I(\mu) \) is zero or a minuscule weight of \( \Phi_I \). Suppose moreover that we have \( \langle \mu, \alpha^\vee_i \rangle \leq 1 \) for all \( i \notin I \). Then:

- \( \{ i \in [n]: \langle \mu, \omega_i^\vee \rangle \neq \langle \mu_{\text{dom}}, \omega_i^\vee \rangle \} \subseteq I \cup \{ i \in [n]: \langle \mu, \alpha^\vee_i \rangle \leq 0 \} \);

- if \( \mu \leq \nu \leq \mu_{\text{dom}} \) and \( \nu \in P_{\geq 0} \), then \( \nu = \mu_{\text{dom}} \).

First let us explain why Claim III is a finite statement for each exceptional root system \( \Phi \). There are only finitely many choices of \( I \subseteq [n] \). Let \( \mu \) be as in the statement of the Claim III. Thus we can write \( \mu = v + \sum_{i \in I} a_i \alpha_i \) for some \( v \in P^R_{\geq 0} \) with \( \pi_I(v) = 0 \), and such that \( \pi_I(\mu) = \sum_{i=1}^n a_i \alpha_i \) is zero or a minuscule weight of \( \Phi_I \) (hence \( a_i \in \mathbb{R}_{\geq 0} \)). Let us also write \( \mu = \sum_{i=1}^n c_i \omega_i \) for \( c_i \in \mathbb{Z} \). There are only finitely many choices for the coefficients \( a_i \) because there are only finitely many zero-or-minuscule weights of \( \Phi_I \). And, since \( \langle v, \alpha^\vee_i \rangle = 0 \) for any \( i \in I \), the coefficients \( c_i \) for \( i \in I \) are determined by \( \pi_I(\mu) \). But the fact that \( v \in P^R_{\geq 0} \) implies that \( c_i \geq \langle \sum_{j \in I} a_j \alpha_j, \alpha^\vee_i \rangle \) for any \( i \in [n] \). So for \( j \notin I \), we have \( \langle \sum_{j \in I} a_j \alpha_j, \alpha^\vee_i \rangle \leq c_j \leq 1 \), which, together with the fact that \( c_j \in \mathbb{Z} \), means for any fixed choice of the \( a_i, i \in I \) we have only finitely many choices for \( c_j \). Finally, it is also clear that for any fixed \( \mu \) there any only finitely
many \( \nu \in P \) with \( \mu \leq \nu \leq \mu_{\text{dom}} \) whose (lack of) dominance we need to check. In this way, Claim \( \text{III} \) is indeed a finite statement for each exceptional root system \( \Phi \).

We verified Claim \( \text{III} \) via exhaustive computation using Sage \[67\] \[66\]. The code used to verify Claim \( \text{III} \) is available upon request from the first author.

Lastly, let us explain why Claim \( \text{III} \) implies Claim \( \text{I} \) for \( \Phi \) of exceptional type. So assume \( \Phi \) is of exceptional type. Any \( \mu \in P \) satisfying the conditions of Claim \( \text{I} \) can be written as \( \mu = \mu' + \sum_{i \in J} c_i \omega_i \) for some coefficients \( c_i \in \mathbb{N} \), where \( \mu' \in P \) satisfies the stronger condition of Claim \( \text{III} \) and where \( J := \{ i \in [n] : i \notin I, \langle \mu', \alpha_i^\vee \rangle = 1 \} \). By the first bulleted item in Claim \( \text{III} \) we have \( J \cap \{ i \in [n] : \langle \mu', \alpha_i^\vee \rangle \neq \langle \mu_{\text{dom}}', \omega_i^\vee \rangle \} = \emptyset \), which implies \( \mu_{\text{dom}} = \mu_{\text{dom}}' + \sum_{i \in J} c_i \omega_i \). So now let \( \nu \in P_{\geq 0} \) satisfy \( \mu \leq \nu \leq \mu_{\text{dom}} \). Then \( \nu = \nu' + \sum_{i \in J} c_i \omega_i \) for some \( \nu' \in P_{\geq 0} \). Since \( \mu' \leq \nu' \leq \mu_{\text{dom}}' \), the second bulleted item of Claim \( \text{III} \) implies \( \nu' = \mu_{\text{dom}}' \), which means that \( \nu = \mu_{\text{dom}} \), as required.

\textbf{Remark 6.2.8.} Claim \( \text{I} \) in the proof of Lemma \[6.2.3\] may fail to hold for the exceptional types. To see this, suppose that \( \Phi = E_8 \). Recall that we follow the numbering of the simple roots in Figure \[3-2\]. Let \( I := \{ 1, 2, 3, 5, 6, 7, 8 \} \), and let \( \mu := \omega_1 + \omega_2 - \omega_4 + \omega_5 \).

It is easy to see that \( \pi_I(\mu) \) is minuscule, and we can also check \( \mu - \pi_I(\mu) = \frac{1}{30} \omega_4 \in P^E_{\geq 0} \). However, \( \mu_{\text{dom}} = \omega_8 = \mu + (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8) \).

So if we set \( J := \{ i \in [8] : \langle \mu, \alpha_i^\vee \rangle \neq \langle \mu_{\text{dom}}, \omega_i^\vee \rangle \} \), then \( J = [8] \). And hence \( \pi_I(\mu_{\text{dom}}) = \omega_8 \), which is not a minuscule weight of \( \Phi_J = \Phi \) (because there are no minuscule weights in Type \( E_8 \)). Note that Claim \( \text{I} \) remains true in this case: \( \mu_{\text{dom}} \) is the unique minimal (in root order) dominant weight greater (in root order) than \( \mu \).

\textbf{Remark 6.2.9.} Let \( I \subseteq [n] \). Then,

\[
\sum_{i \in I} a_i \alpha_i \text{ is a minuscule weight of } \Phi_I \Rightarrow \langle u, \alpha_i^\vee \rangle > -2 \text{ for all } i \in [n]. \tag{6.3}
\]

(Note crucially the the \( u \) in (6.3) is not assumed to be a weight of \( \Phi \); in general it is just some vector in \( \text{Span}_R(\Phi_I) \) which is a minuscule weight of \( \Phi_I \), i.e., which is nonzero but has \( \langle u, \alpha^\vee \rangle \in \{ 0, 1 \} \) for all \( \alpha \in \Phi_I^\vee \).) The arguments given in Section 6.5 establish (6.3) for \( \Phi \) of classical type. For the exceptional root systems we verified (6.3) via exhaustive computation with Sage \[67\] \[66\]. However, we know of no simple,
uniform proof that (6.3) holds for all $\Phi$. This statement feels like the kind of result which should have a uniform proof. A uniform proof of (6.3) might serve as the first step towards a uniform proof of Claim I in the proof of Lemma 6.2.5. Indeed, if (6.3) were not the case, then it is easy to see that Claim I would fail, and in turn Lemma 6.2.5 would be false.

**Remark 6.2.10.** For any root system $\Phi'$ (irreducible or not), define the polytope

$$
\mathcal{P}_{\Phi'} := \{ v \in \text{Span}_R(\Phi') : \langle v, \alpha^\vee \rangle \leq 1 \text{ for all } \alpha \in \Phi \}.
$$

If $\Phi'$ is irreducible then $\mathcal{P}_{\Phi'}$ is an *alcoved polytope* in the sense of Lam and Postnikov [49]. Note that $\mathcal{P}_\Phi$ is *not* a $W$-permutohedron, but it is $W$-invariant and hence each vertex of $\mathcal{P}_\Phi$ has a Weyl group orbit representative in $P_{\geq 0}$. The minuscule weights of $\Phi$ are a subset of the vertices of $\mathcal{P}_\Phi$ in $P_{\geq 0}$, but in general there are other such vertices which are fractional scalar multiples of the other fundamental weights of $\Phi$. Based on the discussion in Remark 6.2.9 we were lead to consider the following assertion, which computational evidence suggests is true:

$$
\mathcal{P}_{\Phi'} \subseteq 2 \cdot \text{interior}(\mathcal{P}_\Phi) \text{ for any sub-root system } \Phi' \text{ of } \Phi. \quad (6.4)
$$

In (6.4) we mean by sub-root system that $\Phi' = \Phi \cap U$ for a subspace $U \subseteq V$. Observe that (6.4) would imply (6.3), because as mentioned the minuscule weights of $\Phi'$ are vertices of $\mathcal{P}_{\Phi'}$. We believe that a similar case-by-case analysis could be used to establish (6.4), but we are more interested in a uniform proof of this assertion.

Let us now restate Lemma 6.2.5 in the language of quotients from the previous section:

**Corollary 6.2.11.** Let $\lambda \in P_{\geq 0}$ and $X \subseteq \Phi^\vee$. Then,

$$
\text{quot}_X(\Pi(\lambda)) \cap \text{quot}_X(Q + \lambda) = \text{quot}_X(\Pi^Q(\lambda)).
$$

**Proof.** A point in $\text{quot}_X(\Pi(\lambda)) \cap \text{quot}_X(Q + \lambda)$ is an affine subspace of $V$ of the form.
\( \mu + \text{Span}_R(X) \) for some \( \mu \in Q + \lambda \) satisfying \( (\mu + \text{Span}_R(X)) \cap \Pi(\lambda) \neq \emptyset \), while a point in \( \text{quot}_X(\Pi^Q(\lambda)) \) is an affine subspace of \( V \) of the form \( \mu + \text{Span}_R(X) \) for \( \mu \in \Pi^Q(\lambda) \). These two kinds of affine subspaces coincide thanks to Lemma 6.2.5.

Now we can prove the formula for the number of lattice points in a permutohedron plus dilating regular permutohedron.

**Theorem 6.2.12.** Let \( \lambda \in P_{\geq 0} \) and \( k \in \mathbb{N}[\Phi]^W \). Then

\[
\#\Pi^Q(\lambda + \rho_k) = \sum_{X \subseteq \Phi^+} \#\text{quot}_X(\Pi^Q(\lambda)) \cdot r\text{Vol}_Q(X) k^X,
\]

where \( k^X := \prod_{\alpha \in X} k(\alpha) \).

**Proof.** This follows from Theorem 6.1.1 together with Corollary 6.2.11. \( \square \)

### 6.3 Formula for symmetric Ehrhart-like polynomials

The first step in our proof of the positivity of the coefficients of \( L^\text{sym}_\lambda(k) \) is to directly relate the fibers \( #(s^\text{sym}_k)^{-1}(\lambda) \) to polytopes of the form \( \Pi^Q(\lambda + \rho_k) \). In fact, the permutohedron non-escaping lemma already does this:

**Proposition 6.3.1.** For \( \lambda \in P_{\geq 0} \) and good \( k \in \mathbb{N}[\Phi]^W \),

\[
\bigcup_{w \in W^\lambda_{\alpha, 1}} (s^\text{sym}_k)^{-1}(w\lambda) = \Pi^Q(\lambda + \rho_k) \setminus \bigcup_{\mu \neq \lambda \in P_{\geq 0}, \mu \leq \lambda} \Pi^Q(\mu + \rho_k).
\]

**Proof.** By Lemma 4.5.2 no \( \nu \in P \) with \( \nu \notin \Pi^Q(\lambda + \rho_k) \) could possibly \( \lambda \xrightarrow{\text{sym}, k} -\)stabilize to a weight in \( \Pi^Q(\lambda + \rho_k) \), so certainly \( \bigcup_{w \in W^\lambda_{\alpha, 1}} (s^\text{sym}_k)^{-1}(w\lambda) \subseteq \Pi^Q(\lambda + \rho_k) \).

On the other hand, if \( \nu \in \Pi^Q(\mu + \rho_k) \) for some \( \mu \leq \lambda \in P_{\geq 0} \) with \( \nu \neq \lambda \), then, again by Lemma 4.5.2 the \( \lambda \xrightarrow{\text{sym}, k} -\)stabilization of \( \nu \) must still belong to \( \Pi^Q(\mu + \rho_k) \) and so cannot be equal to \( \eta_k(w\lambda) \) for any \( w \in W^\lambda_{\alpha, 1} \). Finally, it is an easy exercise to check (for instance using Proposition 3.1.2) that \( \eta_k(P) \cap \Pi^Q(\lambda + \rho_k) = \eta_k(\Pi^Q(\lambda)) \). Hence,
if \( \nu \in \Pi^Q(\lambda + \rho_k) \) and \( \nu \notin \Pi^Q(\mu + \rho_k) \) for any \( \mu \leq \lambda \in P_{\geq 0} \) with \( \mu \neq \lambda \), then, once more by Lemma 4.5.2, the \( \lambda \overset{\text{sym}_k}{\longrightarrow} \)-stabilization of \( \nu \) cannot belong to \( \Pi^Q(\mu + \rho_k) \) for any \( \mu \leq \lambda \in P_{\geq 0} \) with \( \mu \neq \lambda \), but must belong to \( \Pi^Q(\lambda + \rho_k) \), so the only possibility is that this stabilization is equal to \( \eta_k(w\lambda) \) for some \( w \in W_{\lambda,1}^{\alpha} \).

Remark 6.3.2. If \( \lambda \in P_{\geq 0} \) satisfies \( I_{\lambda}^{0,1} = [n] \), then Proposition 6.3.1 says that for any good \( k \in \mathbb{N}[\Phi]^W \) we have

\[
(s_k^{\text{sym}})^{-1}(\lambda) = \Pi^Q(\lambda + \rho_k) \setminus \bigcup_{\mu \neq \lambda \in P_{\geq 0}, \mu \leq \lambda} \Pi^Q(\mu + \rho_k). 
\]

(6.5)

Theorem 4.6.1 shows that for any \( \lambda \in P \) with \( \langle \lambda, \alpha^\vee \rangle \neq -1 \) for all \( \alpha \in \Phi^+ \), the set \( (s_k^{\text{sym}})^{-1}(\lambda) \) belongs to the affine subspace \( \lambda + \text{Span}_{\mathbb{R}}(w_{\lambda,\Phi_{\lambda,1}^{0,1}}) \). Thus, for weights belonging to \( (s_k^{\text{sym}})^{-1}(\lambda) \), symmetric interval-firing is the same as the corresponding process with respect to the sub-root system \( w_{\lambda,\Phi_{\lambda,1}^{0,1}} \). In this way, every \( (s_k^{\text{sym}})^{-1}(\lambda) \) can be written as a difference of permutohedra as in (6.5), except that we may first need to project to a sub-root system. Hence by induction on the rank of our root system, to understand the \( (s_k^{\text{sym}})^{-1}(\lambda) \) and \( L_{\lambda}^{\text{sym}}(k) \) for arbitrary \( \lambda \in P \), it is enough to just consider those \( \lambda \in P_{\geq 0} \) with \( I_{\lambda}^{0,1} = [n] \). However, we will not invoke these kind of inductive arguments in this section because we can easily avoid them.

With Proposition 6.3.1 in hand, the strategy to understand \( L_{\lambda}^{\text{sym}}(k) \) is now just to use inclusion-exclusion on our formula for \( \Pi^Q(\lambda + \rho_k) \) (Theorem 6.2.12). The following series of propositions will prepare us for applying this inclusion-exclusion.

Proposition 6.3.3. Let \( \lambda, \mu \in P_{\geq 0} \) with \( \lambda - \mu \in Q \). Then,

- \( \Pi(\lambda + \rho_k) \cap \Pi(\mu + \rho_k) = \Pi(\lambda \land \mu + \rho_k) \) for any \( k \in \mathbb{N}[\Phi]^W \);
- \( \Pi^Q(\lambda + \rho_k) \cap \Pi^Q(\mu + \rho_k) = \Pi^Q(\lambda \land \mu + \rho_k) \) for any \( k \in \mathbb{N}[\Phi]^W \).

Proof. Both of these statements follow immediately from Proposition 3.1.2.

Proposition 6.3.4. Let \( \lambda, \mu \in P_{\geq 0} \) with \( \lambda - \mu \in Q \). Then,
\( \text{quot}_X(\Pi(\lambda)) \cap \text{quot}_X(\Pi(\mu)) = \text{quot}_X(\Pi(\lambda \land \mu)) \) for any \( X \subseteq \Phi^+ \);

\( \text{quot}_X(\Pi^Q(\lambda)) \cap \text{quot}_X(\Pi^Q(\mu)) = \text{quot}_X(\Pi^Q(\lambda \land \mu)) \) for any \( X \subseteq \Phi^+ \).

**Proof.** We begin with the first bulleted item. Since by Proposition 6.3.3 we have that \( \Pi(\lambda) \cap \Pi(\mu) = \Pi(\lambda \land \mu) \), it is clear that \( \text{quot}_X(\Pi(\lambda \land \mu)) \subseteq \text{quot}_X(\Pi(\lambda)) \cap \text{quot}_X(\Pi(\mu)) \).

Let us show the other containment. A point in \( \text{quot}_X(\Pi(\lambda)) \cap \text{quot}_X(\Pi(\mu)) \) is an affine subspace \( v + \text{Span}_R(X) \) for some \( v \in V \) which satisfies \( (v + \text{Span}_R(X)) \cap \Pi(\lambda) \neq \emptyset \) and \( (v + \text{Span}_R(X)) \cap \Pi(\mu) \neq \emptyset \). Let \( v^0_X \) denote the unique point in \( v + \text{Span}_R(X) \cap \Pi(\lambda) \) for which \( \pi_X(v^0_X) = 0 \). As described in the beginning of the proof of Lemma 6.2.5, the fact that \( v + \text{Span}_R(X) \cap \Pi(\lambda) \neq \emptyset \) implies that \( v^0_X \in \Pi(\lambda) \). Similarly, we have that \( v^0_X \in \Pi(\mu) \). So by Proposition 6.3.3 we get \( v^0_X \in \Pi(\lambda \land \mu) \). This means that we have \( v + \text{Span}_R(X) \in \text{quot}_X(\Pi(\lambda \land \mu)) \).

The second bulleted item follows from the first by intersecting with \((Q + \lambda)\) and applying the integrality property of slices of permutohedra (Lemma 6.2.5). \( \square \)

**Proposition 6.3.5.** For \( \lambda \in P_{\geq 0} \),

\[
\# \left( \text{quot}_X(\Pi^Q(\lambda)) \setminus \bigcup_{\substack{\mu \neq \lambda \in P_{\geq 0}, \\ \mu \leq \lambda}} \text{quot}_X(\Pi^Q(\mu)) \right) = \# \left\{ \mu \in W(\lambda) : \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_R(X) \right\}.
\]

**Proof.** First observe that

\[
\text{quot}_X(\Pi^Q(\lambda)) \setminus \bigcup_{\substack{\mu \neq \lambda \in P_{\geq 0}, \\ \mu \leq \lambda}} \text{quot}_X(\Pi^Q(\mu)) = \text{quot}_X(W(\lambda)) \setminus \bigcup_{\substack{\mu \neq \lambda \in P_{\geq 0}, \\ \mu \leq \lambda}} \text{quot}_X(\Pi^Q(\mu)).
\]

Indeed, suppose that \( \nu \in \Pi^Q(\lambda) \) and \( \nu \not\in W(\lambda) \). Then \( \nu_{\text{dom}} \leq \lambda \) (by Proposition 3.1.2) but \( \nu \neq \lambda_{\text{dom}} \). And of course \( \nu \in \Pi^Q(\nu_{\text{dom}}) \). Therefore \( \text{quot}_X(\nu) \) does not belong to the set we are interested in counting.

So now suppose \( \nu \in W(\lambda) \). Let \( W' \subseteq W \) denote the Weyl group of \( \Phi \cap \text{Span}_R(X) \), and observe that \( (\nu + \text{Span}_R(X)) \cap \Pi^Q(\lambda) = \nu + \text{ConvexHull}(\pi_X(\nu)) \). Let \( \nu' \) be the unique element in \( W'(\nu) \) for which \( \pi_X(\nu') \) is a dominant weight of \( \Phi \cap \text{Span}_R(X) \). First suppose that \( \pi_X(\nu') \) is not zero or a minuscule weight of \( \Phi \cap \text{Span}_R(X) \), i.e., that we
have $\langle \nu', \alpha^\vee \rangle \notin \{0, 1\}$ for some $\alpha \in \Phi^+ \cap \text{Span}_\mathbb{R}(X)$. In this case we will have that we have that ConvexHull $W'(\pi_X(\nu')) \cap (Q + \pi_X(\nu')) \neq W'(\pi_X(\nu'))$ by a characterization of zero-or-minuscule weights mentioned above. So there is some $\mu \in \Pi^Q(\lambda)$ with $\mu_{\text{dom}} \neq \lambda$ for which $\mu \in (\nu + \text{Span}_\mathbb{R}(X)) \cap \Pi^Q(\lambda)$, and thus $\text{quot}_X(\nu)$ does not belong to the set we are interested in counting. Now suppose that $\langle \nu', \alpha^\vee \rangle \in \{0, 1\}$ for all $\alpha \in \Phi^+ \cap \text{Span}_\mathbb{R}(X)$. Then, by that same characterization of zero-or-minuscule weights mentioned above, there can be no $\mu \in \Pi^Q(\lambda)$ with $\mu_{\text{dom}} \neq \lambda$ for which $\mu \in (\nu + \text{Span}_\mathbb{R}(X)) \cap \Pi^Q(\lambda)$. Thus in this case $\text{quot}_X(\nu)$ does in fact belong to the set we are interested in counting. But we would overcount if we counted two different elements of $W(\lambda)$ which become equal after quotienting by $X$. Therefore, we only count a given $\nu$ if $\nu = \nu'$, i.e., if $\langle \nu, \alpha^\vee \rangle \in \{0, 1\}$ for all $\alpha \in \Phi^+ \cap \text{Span}_\mathbb{R}(X)$. In this way we obtain the claimed formula.

We now apply inclusion-exclusion on the formula for $\Pi^Q(\lambda + \rho_k)$ (Theorem 6.2.12).

**Corollary 6.3.6.** For $\lambda \in P_{\geq 0}$ and good $k \in \mathbb{N}[\Phi]^W$,

$$
\sum_{w \in W_{\lambda}^{\rho, 1}} L_{w, \lambda}^{\text{sym}}(k) = \sum_{X \subseteq \Phi^+ \text{ linearly independent}} # \left\{ \mu \in W(\lambda): \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_\mathbb{R}(X) \right\} \cdot \text{rVol}_Q(X) k^X.
$$

**Proof.** First note that $\sum_{w \in W_{\lambda}^{\rho, 1}} L_{w, \lambda}^{\text{sym}}(k) = \# \left( \bigcup_{w \in W_{\lambda}^{\rho, 1}} (s_k^{\text{sym}})^{-1}(w\lambda) \right)$ because all the fibers of $s_k^{\text{sym}}$ are disjoint. Thus, by Proposition 6.3.1 we have

$$
\sum_{w \in W_{\lambda}^{\rho, 1}} L_{w, \lambda}^{\text{sym}}(k) = \# \left( \Pi^Q(\lambda + \rho_k) \setminus \bigcup_{\mu \neq \lambda \in P_{\geq 0}, \mu \leq \lambda} \Pi^Q(\mu + \rho_k) \right) \quad (6.6)
$$

Let $(\mathcal{L}, \leq)$ be a meet semi-lattice, and let $F: \mathcal{L} \to 2^S$ be a function which associates to every $p \in \mathcal{L}$ some finite subset $F(p)$ of a set $S$ such that $F(p) \cap F(q) = F(p \wedge q)$, where $\wedge$ is the meet operation of $\mathcal{L}$. Then a simple application of the Möbius
inversion formula (see e.g. [73, §3.7]) says that

\[
\# \left( F(p) \setminus \bigcup_{q \leq p, q \neq p} F(q) \right) = \sum_{q \leq p} \mu_{\mathcal{L}}(q, p) \cdot \# F(q),
\]

where \( \mu_{\mathcal{L}}(q, p) \) is the Möbius function of \( \mathcal{L} \). (Do not confuse this Möbius function with a weight \( \mu \in P \).)

Hence, by Proposition 6.3.3 we have

\[
\# \left( \Pi^Q(\lambda + \rho_k) \setminus \bigcup_{\mu \neq \lambda \in P_{\geq 0}, \mu \leq \lambda} \Pi^Q(\mu + \rho_k) \right) = \sum_{\nu \leq \lambda \in P_{\geq 0}} \mu_{(P_{\geq 0}, \leq)}(\nu, \lambda) \cdot \# \Pi^Q(\nu + \rho_k),
\]

where \( \mu_{(P_{\geq 0}, \leq)} \) is the Möbius function of the poset \( (P_{\geq 0}, \leq) \) of dominant weights with respect to root order (here we are using Stembridge’s result, stated as Proposition 6.2.3 above, that each connected component of this poset is a meet semi-lattice).

Then by Theorem 6.2.12, we have

\[
\text{RHS of (6.7)} = \sum_{\nu \leq \lambda \in P_{\geq 0}} \mu_{(P_{\geq 0}, \leq)}(\nu, \lambda) \cdot \left( \sum_{X \subseteq \Phi^+, X \text{ is linearly independent}} \# \text{quot}_X(\Pi^Q(\nu)) \cdot r\text{Vol}_Q(X) k^X \right)
\]

\[
= \sum_{X \subseteq \Phi^+, X \text{ is linearly independent}} \left( \sum_{\nu \leq \lambda \in P_{\geq 0}} \mu_{(P_{\geq 0}, \leq)}(\nu, \lambda) \cdot \# \text{quot}_X(\Pi^Q(\nu)) \right) \cdot r\text{Vol}_Q(X) k^X
\]

\[
= \sum_{X \subseteq \Phi^+, X \text{ is linearly independent}} \# \left( \text{quot}_X(\Pi^Q(\lambda)) \setminus \bigcup_{\mu \neq \lambda \in P_{\geq 0}, \mu \leq \lambda} \text{quot}_X(\Pi^Q(\mu)) \right) \cdot r\text{Vol}_Q(X) k^X,
\]

\[
= \sum_{X \subseteq \Phi^+, X \text{ is linearly independent}} \# \left\{ \mu \in W(\lambda) : \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_R(X) \right\} \cdot r\text{Vol}_Q(X) k^X
\]

(6.8)
where in the third line we applied Proposition 6.3.4 together with Möbius inversion, and in the last line we applied Proposition 6.3.5. Putting together equations (6.6), (6.7), and (6.8) proves the corollary.

**Proposition 6.3.7.** Let \( \lambda \in P_{\geq 0} \). Let \( w \in W_{\lambda}^{0,1} \). Let \( k \in \mathbb{N}[\Phi]^W \) be good. Then we have \( L_{w,\lambda}^{\text{sym}}(k) = L_{\lambda}^{\text{sym}}(k) \). Consequently, \( L_{w,\lambda}^{\text{sym}}(k) = \frac{1}{[W : W_{\lambda}^{0,1}]} \sum_{w' \in W_{\lambda}^{0,1}} L_{w',\lambda}^{\text{sym}}(k) \).

**Proof.** This is an immediate consequence of the \( W \)-symmetry property of the symmetric interval-firing process (Theorem 4.2.1).

**Proposition 6.3.8.** Let \( \lambda \in P_{\geq 0} \) and \( w \in W_{\lambda}^{0,1} \). Then for any \( X \subseteq \Phi^+ \), the quantity \( \#\{\mu \in wW_{\lambda}^{0,1}(\lambda) : \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_R(X)\} \) is nonzero only if \( w^{-1}(X) \subseteq \Phi_{\lambda}^{0,1} \). Consequently,

\[
\sum_{X \subseteq \Phi^+ \atop X \text{ is linearly independent}} \# \left\{ \mu \in wW_{\lambda}^{0,1}(\lambda) : \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_R(X) \right\} \cdot r\text{Vol}_Q(X) k^X
\]

\[
= \frac{1}{[W : W_{\lambda}^{0,1}]} \sum_{X \subseteq \Phi^+ \atop X \text{ is linearly independent}} \# \left\{ \mu \in W(\lambda) : \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_R(X) \right\} \cdot r\text{Vol}_Q(X) k^X.
\]

**Proof.** For the first claim: let \( X \subseteq \Phi^+ \) for which \( w^{-1}(X) \not\subseteq \Phi_{\lambda}^{0,1} \). This means there is some \( \alpha \in X \) and \( i \not\in I_{\lambda}^{0,1} \) for which \( w^{-1}(\alpha)^\vee \) has a nonzero coefficient in front of \( \alpha_i^\vee \) in its expansion in terms of simple co-roots. By definition of \( I_{\lambda}^{0,1} \) we have \( \langle \lambda, \alpha_i^\vee \rangle \geq 2 \). Since the coefficients expressing \( w^{-1}(\alpha)^\vee \) in terms of simple co-roots are either all nonnegative integers or all nonpositive integers, we have \( |\langle \lambda, w^{-1}(\alpha)^\vee \rangle| \geq 2 \).

Thus \( |\langle \lambda, \alpha_i^\vee \rangle| \geq 2 \), which means that the claimed quantity is indeed zero.

For the “consequently,” statement: observe that the prior statement implies that in the expression

\[
\sum_{X \subseteq \Phi^+ \atop X \text{ is linearly independent}} \# \left\{ \mu \in wW_{\lambda}^{0,1}(\lambda) : \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_R(X) \right\} \cdot r\text{Vol}_Q(X) k^X \tag{6.9}
\]
we need only sum over \( X \subseteq w(\Phi_{\lambda}) \cap \Phi^+ \). But recall that, because \( w \in W_{\lambda}^g \), we have \( w(\Phi_{\lambda}) \cap \Phi^+ = w(\Phi_{\lambda} \cap \Phi^+) \). Then making the substitution \( X \mapsto w^{-1}(X) \), we see that the quantity in (6.9) is equal to

\[
\sum_{X \subseteq \Phi_{\lambda} \cap \Phi^+, \text{X is linearly independent}} \# \left\{ \mu \in W_{\lambda}^g(\lambda) \colon \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_R(X) \right\} \cdot r\text{Vol}_Q(X) k^X. \tag{6.10}
\]

That the expression (6.10) does not depend on \( w \) then gives the claimed formula. \( \square \)

**Theorem 6.3.9.** Let \( \lambda \in P \). Let \( k \in \mathbb{N}[\Phi]^W \) be good. Then if \( \langle \lambda, \alpha^\vee \rangle = -1 \) for some positive root \( \alpha \in \Phi^+ \) we have \( L_{\lambda}^\text{sym}(k) = 0 \), and otherwise

\[
L_{\lambda}^\text{sym}(k) = \sum_{X \subseteq \Phi^+, X \text{ is linearly independent}} \# \left\{ \mu \in w_{\lambda}W_{\lambda}^g(\lambda_{\text{dom}}) \colon \langle \mu, \alpha^\vee \rangle \in \{0, 1\} \text{ for all } \alpha \in \Phi^+ \cap \text{Span}_R(X) \right\} \cdot r\text{Vol}_Q(X) k^X.
\]

**Proof.** This follows immediately by combining Corollary 6.3.6 with Propositions 6.3.7 and 6.3.8. \( \square \)

**Remark 6.3.10.** If we only cared about proving the positivity of the coefficients of the symmetric Ehrhart-like polynomials, we could have actually avoided the use of the subtle integrality property of slices of permutohedra. This is because the same inclusion-exclusion strategy as above but invoking only Theorem 6.1.1 and the first bulleted item in Proposition 6.3.4 would yield the formula

\[
L_{\lambda}^\text{sym}(k) = \sum_{X \subseteq \Phi^+, X \text{ is linearly independent}} \# \left( \left( \text{quot}_X(\Pi(\lambda)) \setminus \bigcup_{\mu \neq \lambda \in P_{\geq 0}, \mu \leq \lambda} \text{quot}_X(\Pi(\mu)) \right) \cap \text{quot}_X(Q + \lambda) \right) \cdot r\text{Vol}_Q(X) k^X, \tag{6.11}
\]

for \( \lambda \in P_{\geq 0} \) with \( \ell_{\lambda}^g = [n] \). As mentioned in Remark 6.3.2 by induction on the rank of the root system it is enough to consider \( \lambda \) of this form. However, the formula in (6.11) is not ideal from a combinatorial perspective because the coefficients poten-
tially involve checking every rational point in \( \Pi(\lambda) \). The formula in Theorem 6.3.9 is much more combinatorial, and makes clear the significance of minuscule weights.

### 6.4 Truncated Ehrhart-like polynomials and other future directions

In this section we discuss some open questions and future directions, starting with the truncated Ehrhart-like polynomials.

#### 6.4.1 Truncated Ehrhart-like polynomials

One might hope that the formula for the symmetric Ehrhart-like polynomials could suggest a formula for the truncated polynomials. Lemma 5.3.1 implies that for any good \( k \in \mathbb{N}[\Phi]^W \) and any \( \lambda \in P \) with \( \langle \lambda, \alpha^\vee \rangle \neq -1 \) for all \( \alpha \in \Phi^+ \) we have

\[
(s_k^{\text{sym}})^{-1}(\lambda) = \bigcup_{\mu \in w_\lambda W_{\lambda,1} (\lambda_{\text{dom}})} (s_k^{\text{tr}})^{-1}(\mu),
\]

or at the level of Ehrhart-like polynomials,

\[
L^{\text{sym}}(k) = \sum_{\mu \in w_\lambda W_{\lambda,1} (\lambda_{\text{dom}})} I^{\text{tr}}(\mu).
\]

Hence, the formula in Theorem 6.3.9 very naturally suggests the following conjecture:

**Conjecture 6.4.1.** Let \( \lambda \in P \) be any weight. Then for any good \( k \in \mathbb{N}[\Phi]^W \) we have

\[
L^{\text{tr}}(k) = \sum_X r\text{Vol}_Q(X) k^X,
\]

where the sum is over all \( X \subseteq \Phi^+ \) such that:

- \( X \) is linearly independent;

- \( \langle \lambda, \alpha^\vee \rangle \in \{0, 1\} \) for all \( \alpha \in \Phi^+ \cap \text{Span}_\mathbb{R}(X) \).
However, in fact Conjecture 6.4.1 is false in general! The smallest counterexample is in Type $G_2$. Table 6.1 records, for $\Phi = A_1, A_2, B_2, G_2, A_3, B_3, C_3, A_4, D_4$, all counterexamples to Conjecture 6.4.1 among those $\lambda \in P_{\geq 0}$ with $I_{\text{dom}}^{0,1} = [n]$ (recall that, as in Remark 6.3.2, for other $\lambda$ we can understand the Ehrhart-like polynomials by projecting to a smaller sub-root system). These counterexamples were found using Sage [67]. There are counterexamples in $G_2, C_3,$ and $D_4$. Observe that whenever there is a counterexample $\lambda$ to Conjecture 6.4.1 there must be also be another $\mu \in W(\lambda)$ which is a counterexample because we know, thanks to Theorem 6.3.9, that if we sum the left- and right-hand sides of the formula in Conjecture 6.4.1 along Weyl group orbits they agree. Furthermore, although for larger root systems $\Phi$ we could not carry out an exhaustive search for counterexamples, for $\Phi = B_4, C_4,$ and $A_5$ we were able to check whether the left- and right-hand sides of Conjecture 6.4.1 agree when we plug in $k = 1$; this information is recorded in Table 6.2.

Another remark about the wrong formula in Conjecture 6.4.1: it at least has the symmetry that we expect. Namely, recall there is a copy of the abelian group $P/Q \simeq C \subseteq W$ which acts in a natural way on (the symmetric closure of) the truncated interval-firing process. We believe that the truncated polynomials should be invariant under this action of $P/Q$ (see Remark 5.4.4). This action of $P/Q$ preserves the $\Phi^\vee$-Shi arrangement, and hence the right-hand side of the formula appearing in Conjecture 6.4.1 is indeed invariant under this action of $P/Q$.

Looking at Tables 6.1 and 6.2 we see that no counterexamples to Conjecture 6.4.1 are known when $\Phi$ is of either Type A or Type B. Therefore we are prompted to ask:

**Question 6.4.2.** Is Conjecture 6.4.1 true when $\Phi$ is of either Type A or Type B?

Note that Lemma 5.3.9 that if $\Phi$ is simply laced, then for any $k \in \mathbb{N}$ and any $\lambda \in P$ we have

$$(s_k^{\text{tr}})^{-1}(\lambda) = \bigcup_{\mu \in (s_k^{\text{tr}})^{-1}(\lambda)} (s_{k-1}^{\text{sym}})^{-1}(\mu).$$
Counterexamples $\lambda$ to Conjecture [6.4.1]

| $\Phi$ | $\#\{\lambda: r_{\lambda_{dum}}^{0,1} = [n]\}$ | $\omega_1$: LHS = $4k_l + 2k_s + 1$  
RHS = $3k_l + 2k_s + 1$  
$\omega_2$: LHS = $2k_l + k_s + 1$  
RHS = $3k_l + k_s + 1$  
$\omega_2$: LHS = $4k_l^2 + 14k_lk_s + 8k_s^2 + 3k_l + 5k_s + 1$  
RHS = $4k_l^2 + 13k_lk_s + 8k_s^2 + 3k_l + 5k_s + 1$  
$\omega_2$: LHS = $2k_l^2 + 7k_lk_s + 4k_s^2 + 3k_l + 4k_s + k_s + 1$  
RHS = $2k_l^2 + 8k_lk_s + 4k_s^2 + 3k_l + 4k_s + k_s + 1$  
$\omega_1 - \omega_2 + \omega_3$: LHS = $2k_l^2 + 7k_lk_s + 4k_s^2 + 3k_l + 4k_s + k_s + 1$  
RHS = $2k_l^2 + 8k_lk_s + 4k_s^2 + 3k_l + 4k_s + k_s + 1$  
$\omega_2$: LHS = $106k^3 + 51k^2 + 11k + 1$  
RHS = $105k^3 + 51k^2 + 11k + 1$  
$\omega_2$: LHS = $53k^3 + 39k^2 + 10k + 1$  
RHS = $54k^3 + 39k^2 + 10k + 1$  
$\omega_1 - \omega_2 + \omega_3 + \omega_4$: LHS = $53k^3 + 39k^2 + 10k + 1$  
RHS = $54k^3 + 39k^2 + 10k + 1$  
$\omega_2$: LHS = $106k^3 + 51k^2 + 11k + 1$  
RHS = $105k^3 + 51k^2 + 11k + 1$  
| $A_1$ | 3 | (None) |
| $A_2$ | 13 | (None) |
| $B_1$ | 17 | (None) |
| $C_3$ | 147 | $\omega_1$: LHS = $4k_l + 2k_s + 1$  
RHS = $3k_l + 2k_s + 1$  
$\omega_2$: LHS = $2k_l^2 + 7k_lk_s + 4k_s^2 + 3k_l + 4k_s + k_s + 1$  
RHS = $2k_l^2 + 8k_lk_s + 4k_s^2 + 3k_l + 4k_s + k_s + 1$  
$\omega_1 - \omega_2 + \omega_3$: LHS = $2k_l^2 + 7k_lk_s + 4k_s^2 + 3k_l + 4k_s + k_s + 1$  
RHS = $2k_l^2 + 8k_lk_s + 4k_s^2 + 3k_l + 4k_s + k_s + 1$  
$\omega_2$: LHS = $106k^3 + 51k^2 + 11k + 1$  
RHS = $105k^3 + 51k^2 + 11k + 1$  
$\omega_2$: LHS = $53k^3 + 39k^2 + 10k + 1$  
RHS = $54k^3 + 39k^2 + 10k + 1$  
$\omega_1 - \omega_2 + \omega_3 + \omega_4$: LHS = $53k^3 + 39k^2 + 10k + 1$  
RHS = $54k^3 + 39k^2 + 10k + 1$  
| $A_4$ | 541 | (None) |
| $B_3$ | 147 | (None) |
| $D_4$ | 865 | (None) |

Table 6.1: Counterexamples to Conjecture [6.4.1] for small $\Phi$. 
\[ \Phi \quad \# \{ \lambda : I_{\lambda_{\text{dom}}}^{0,1} = [n] \} \quad \text{Number of } \lambda \text{ for which the LHS and RHS of Conjecture 6.4.1 disagree when } k = 1 \]

<table>
<thead>
<tr>
<th>( B_4 )</th>
<th>1697</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_4 )</td>
<td>1697</td>
<td>60</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>4683</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.2: Counterexamples to Conjecture [6.4.1] with \( k = 1 \) for small \( \Phi \).

or at the level of Ehrhart-like polynomials,

\[
L^{\text{tr}}_{\lambda}(k) = \sum_{\mu \in (s^{\text{tr}})\setminus^{-1}(\lambda)} L_{\mu}^{\text{sym}}(k - 1). \tag{6.12}
\]

Indeed, equation (6.12) is precisely how the polynomiality of \( L^{\text{tr}}_{\lambda}(k) \) was established. But the fact that \( k - 1 \) appears on the right-hand side of (6.12) means that it is totally unclear how to deduce positivity for the truncated polynomials from the positivity for the symmetric polynomials.

The truncated Ehrhart-like polynomials remain largely mysterious, but still seem very much worthy of further investigation.

### 6.4.2 Lattice point formulas via tilings

The way we obtained the formula for the symmetric Ehrhart-like polynomials was via a miraculous transfer from inclusion-exclusion at the level of polynomials to inclusion-exclusion at the level of coefficients. We could ask for a more geometric proof via tilings which better “explains” why these polynomials have the form they do. Let us describe what we have in mind.

Recall that for a linearly independent set \( X = \{v_1, \ldots, v_m\} \) of lattice vectors \( v_1, \ldots, v_m \in \mathbb{Z}^n \), a \textit{half-open parallelepiped} with edge set \( X \) is a convex set \( \mathcal{Z}_{X}^{h.o.} \) of
the form
\[
\mathcal{Z}_X^{h,o} = \sum_{i=1}^{m} \begin{cases} 
(0, v_i) & \text{if } \epsilon_i = 1; \\
(0, v_i) & \text{if } \epsilon_i = -1,
\end{cases}
\]
for some choice of signs \((\epsilon_1, \epsilon_2, \ldots, \epsilon_m) \in \{-1, 1\}^m\). For such a half-open parallelepiped, we always have that \(#(\mathcal{Z}_X^{h,o} \cap \mathbb{Z}^n) = r\text{Vol}(X)\) (see e.g. [12, Lemma 9.8]).

As mentioned in the proof of Theorem 6.1.1, it is well-known that the zonotope \(\mathcal{Z} := \sum_{i=1}^{m} [0, v_i]\) can be decomposed into pieces which are (up to translation) of the form \(\mathcal{Z}_X^{h,o}\) for linearly independent subsets \(X \subseteq \{v_1, \ldots, v_m\}\), with each such subset \(X\) contributing exactly one piece. In fact, we can decompose the \(k\)th dilate \(k\mathcal{Z}\) into pieces of the form \(\mathcal{Z}_{kX}^{h,o}\) in a manner consistent across all \(k \in \mathbb{N}\) (here we use the notation \(kX := \{kv: v \in X\}\)).

Given the form of the formula in Theorem 6.3.9, we can ask whether a similar decomposition into half-open parallelepipeds exists for the symmetric Ehrhart-like polynomials. Here for \(X \subseteq \Phi^+\) and \(k \in \mathbb{N}[\Phi]^W\) we use \(kX := \{k(\alpha)\alpha: \alpha \in X\}\).

**Question 6.4.3.** For good \(k \in \mathbb{N}[\Phi]^W\) and \(\lambda \in P \geq 0\) with \(I_{\lambda}^{0,1} = [n]\), can we decompose \((s_k^{sym})^{-1}(\lambda)\) into pieces which are (up to translation) of the form \(\mathcal{Z}_{kX}^{h,o} \cap Q\) for linearly independent \(X \subseteq \Phi^+\), with each such subset \(X\) contributing
\[
\#\{\mu \in W(\lambda): \langle \mu, \alpha^\vee \rangle \in \{0, 1\}\} \text{ for all } \alpha \in \text{Span}_R(X) \cap \Phi^+
\]
many pieces? (Of course we also want the decomposition to be consistent across \(k\).)

In light of Question 6.4.2 above, we can even ambitiously ask whether the same could be done for truncated Ehrhart-like polynomials in Types A and B.

**Question 6.4.4.** Suppose \(\Phi\) is of Type A or B. For good \(k \in \mathbb{N}[\Phi]^W\) and \(\lambda \in P\), can we decompose \((s_k^{tr})^{-1}(\lambda)\) into pieces which are (up to translation) of the form \(\mathcal{Z}_{kX}^{h,o} \cap Q\) for linearly independent \(X \subseteq \Phi^+\) which satisfy \(\langle \lambda, \alpha^\vee \rangle \in \{0, 1\}\) for all \(\alpha \in \text{Span}_R(X) \cap \Phi^+\), with each such subset \(X\) contributing exactly one piece? (Of course we also want the decomposition to be consistent across \(k\).)
6.4.3 Reciprocity for Ehrhart-like polynomials

It is well-known that an Ehrhart polynomial $L_P$ of a lattice polytope $P$ satisfies a \textit{reciprocity theorem}, which says that the evaluation $L_P(-k)$ of this polynomial at a negative number $-k$ has a predictable sign and up to this sign counts the number of lattice points in the \textit{(relative) interior} of the $k$th dilate $kP$ of the polytope (see [53] or [73, Theorem 4.6.9]). It is thus reasonable to ask if the Ehrhart-like polynomials also satisfy any kind of reciprocity. Unfortunately, we have not been able to discover anything interesting along these lines. In fact, we actually have some negative examples: Table 6.3 shows the evaluation at $k = -1$ for some symmetric and truncated Ehrhart-like polynomials in the case $\Phi = B_3$ (of course these should be bivariate polynomials because $B_3$ is not simply laced, but we reduced to the univariate case $k_l = k_s = k$ for simplicity). There is no particular pattern to the sign of this evaluation. In particular it does not just depend on the rank of $\Phi$ and the degree of the polynomial. We have no conjectures about reciprocity at the moment.

6.4.4 Uniform proof of integrality property

We would obviously prefer to have a completely uniform proof of the subtle integrality property of slices of permutohedra (Lemma 6.2.5). Remarks 6.2.9 and 6.2.10 suggest a possible place to start looking for such a proof.

Another place to start looking for such a proof might be representation theory. The most direct connection to representation theory is the following: let $g$ be a simple Lie algebra over the complex numbers whose corresponding root system is $\Phi$; then for a dominant weight $\lambda \in P_{\geq 0}$, the discrete permutohedron $\Pi^Q(\lambda)$ consists of those weights $\mu \in P$ appearing with nonzero multiplicity in the irreducible representation $V^\lambda$ of $g$ with highest weight $\lambda$ (see e.g. [74]). Hence it is not unreasonable to think that the slices of permutohedra, and the corresponding integrality property, could have some representation-theoretic meaning.
<table>
<thead>
<tr>
<th>Weight $\lambda$</th>
<th>$L^\text{sym}_\lambda(k)$</th>
<th>$L^\text{sym}_\lambda(-1)$</th>
<th>$L^\text{tr}_\lambda(k)$</th>
<th>$L^\text{tr}_\lambda(-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$87k^3 + 39k^2 + 9k + 1$</td>
<td>$-56$</td>
<td>$87k^3 + 39k^2 + 9k + 1$</td>
<td>$-56$</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$78k^2 + 36k + 6$</td>
<td>48</td>
<td>$23k^2 + 8k + 1$</td>
<td>16</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>$36k^2 + 48k + 12k$</td>
<td>0</td>
<td>$7k^2 + 6k + 1$</td>
<td>2</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>$87k^3 + 108k^2 + 48k + 8$</td>
<td>$-19$</td>
<td>$87k^3 + 39k^2 + 9k + 1$</td>
<td>$-56$</td>
</tr>
<tr>
<td>$\omega_1 + \omega_2$</td>
<td>$12k^2 + 60k + 24$</td>
<td>$-24$</td>
<td>$k^2 + 4k + 1$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$\omega_1 + \omega_3$</td>
<td>$78k^2 + 84k + 24$</td>
<td>18</td>
<td>$12k^2 + 6k + 1$</td>
<td>7</td>
</tr>
<tr>
<td>$\omega_2 + \omega_3$</td>
<td>$36k^2 + 60k + 24$</td>
<td>0</td>
<td>$4k^2 + 4k + 1$</td>
<td>1</td>
</tr>
<tr>
<td>$\omega_1 + \omega_2 + \omega_3$</td>
<td>$12k^2 + 72k + 48$</td>
<td>$-12$</td>
<td>$k^2 + 3k + 1$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Table 6.3: Evaluation of symmetric and truncated polynomials at $-1$ for $\Phi = B_3$. 
Table 6.4: Formulas for the simple root coefficients of minuscule weights.

<table>
<thead>
<tr>
<th>Type</th>
<th>Minuscule weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\omega_j = \sum_{i=1}^{n} \min \left( \frac{(n+1-j)}{n+1}, \frac{j(n+1-i)}{n+1} \right) \alpha_i$, for all $j \in [n]$.</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\omega_n = \sum_{i=1}^{n} \frac{i}{2} \alpha_i$.</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\omega_1 = \left( \sum_{i=1}^{n-1} \alpha_i \right) + \frac{1}{2} \alpha_n$.</td>
</tr>
</tbody>
</table>
| $D_n$  | $\omega_1 = \left( \sum_{i=1}^{n-2} \alpha_i \right) + \frac{1}{2} \alpha_{n-1} + \frac{1}{2} \alpha_n$;  
|        | $\omega_{n-1} = \left( \sum_{i=1}^{n-2} \frac{i}{2} \alpha_i \right) + \frac{n}{4} \alpha_{n-1} + \frac{n-2}{4} \alpha_n$;  
|        | $\omega_n = \left( \sum_{i=1}^{n-2} \frac{i}{2} \alpha_i \right) + \frac{n-2}{4} \alpha_{n-1} + \frac{n}{4} \alpha_n$. |

6.4.5 Slices of permutohedra

It would be interesting to further study the slices of permutohedra which appear in the integrality lemma (Lemma 6.2.5). By slices we of course mean the polytopes of the form $\left( \mu + \text{Span}_R(X) \right) \cap \Pi(\lambda)$ for $\lambda \in P_{\geq 0}$, $\mu \in Q + \lambda$, and $X \subseteq \Phi^+$. Note that these polytopes are certainly invariant under the Weyl group $W' \subseteq W$ of the sub-root system $\text{Span}_R(X) \cap \Phi$, but they need not be $W'$-permutohedra. The quotients $\text{quot}_X(\Pi(\lambda))$ of permutohedra are generalized permutohedra (at least in Type A; but we believe the corresponding statement should be true in all types). However, these slices $(\mu + \text{Span}_R(X)) \cap \Pi(\lambda)$ need not even be generalized permutohedra: for instance, in $\Phi = A_3$ the slices can have more vertices than would be possible for a generalized permutohedron. Nevertheless we could still ask to understand more about their combinatorial structure. Note that in general these slices do not have integer vertices; however in Type A these slices actually do have integer vertices, which ultimately has to do with the fact that the roots in Type A are totally unimodular. Also, our Lemma 6.2.5 says that, even though these slices might not have integer vertices, they do have integer points as long as they are nonempty.
6.5 Proof of integrality property for classical types

In this final section we prove Claim [II] in the proof of Lemma 6.2.5. A few preliminaries are in order. We follow the numbering of the simple roots from Figure 3-2. We use $D$ to denote the Dynkin diagram of $\Phi$. For any $I \subseteq [n]$ we use $D[I]$ to denote the subdiagram obtained by restricting $D$ to the vertices in $I$. Any subdiagram $D[I]$ of a Dynkin diagram $D$ is a Dynkin diagram; hence any such $D[I]$ is a disjoint union of connected Dynkin diagrams, which we refer to as the connected components of $D[I]$, or just the components for short. Here is some nonstandard terminology for Dynkin diagrams we will find convenient: we refer to a vertex of Dynkin diagram that is adjacent to one or fewer other vertices as being an endpoint of that Dynkin diagram; and for a Dynkin diagram of Type $B_n$, $C_n$, or $D_n$, we refer to the vertex 1 as the left endpoint of this Dynkin diagram. Another convenient resource for us will be Table 6.4, which records formulas for the simple root coefficients of minuscule weights in the classical types. These coefficients appear also in [46, §13, Table 1]. Note that for any $j \in [n]$ we have $\omega_j = \sum_{i=1}^n a_i \alpha_i$ where $(a_1, \ldots, a_n)$ is the $j$th row of the inverse of the Cartan matrix of $\Phi$. Now we proceed to prove Claim [II]

**Proof.** Let $I \subseteq [n]$ and let $\mu \in P$ be as in Claim [II]. Write $\mu = v + \sum_{i \in I} a_i \alpha_i$ for some $v \in P_{\geq 0}$ with $\pi_I(v) = 0$, and such that $\pi_I(\mu) = \sum_{i=1}^n a_i \alpha_i$ is zero or a minuscule weight of $\Phi_I$ (hence $a_i \in \mathbb{R}_{\geq 0}$). Let us also write $\mu = \sum_{i=1}^n c_i \omega_i$ for $c_i \in \mathbb{Z}$. To prove Claim [II] we consider each of the classical types in turn.

**Type A:** Suppose $\Phi = A_n$. We want to understand what the coefficients $c_i$ can be. To understand these coefficients, it is helpful to label the vertex $i$ of the Dynkin diagram $D$ of $\Phi$ by the coefficient $c_i$. Note that because $\pi_I(\mu)$ is zero or a minuscule weight of $\Phi_I$, in each connected component of $D[I]$ we have at most one nonzero $c_i$-label, which must be equal to 1. In other words, coloring in red the vertices belonging to $I$ and edges between these vertices, our $c_i$-labeled Dynkin diagram $D$ might be:

```
1  ???  0  1  0  ???  1  0  ???  ???  1  ???  1
```

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Here the “????” represent unknown coefficients. What are these coefficients? We claim that \( c_i \geq -1 \) for all \( i \in [n] \). As mentioned, we have \( c_i \in \{0, 1\} \) for \( i \in I \) because \( \pi_I(\mu) \) is zero or a minuscule weight of \( \Phi_I \). So we need to check that \( c_i \geq -1 \) only for \( i \notin I \).

Let \( i \in [n] \) with \( i \notin I \). Observe that \( v \in P_{\geq 0}^R \) implies that \( c_i \geq \langle \sum_{j \in I} a_j \alpha_j, \alpha_i^\vee \rangle \) for all \( i \in [n] \). Looking at the Dynkin diagram, we see that \( \langle \sum_{j \in I} a_j \alpha_j, \alpha_i^\vee \rangle = -a_{i-1} - a_{i+1} \) (where we use the convention \( a_j := 0 \) for \( j \notin I \)). But for \( j \in \{i-1, i+1\} \): either \( j \notin I \), in which case \( a_j = 0 \); or \( j \in I \) is an endpoint of a Type A connected component of \( D[I] \), in which case we can verify that \( a_j < 1 \) by looking at Table 6.4. Hence \( a_{i-1}, a_{i+1} < 1 \). So \( c_i > -2 \) for all \( i \in [n] \). Since all \( c_i \in \mathbb{Z} \) are integers, this indeed means that \( c_i \geq -1 \) for all \( i \in [n] \). Moreover, for any \( i \in [n] \) with \( c_i = -1 \) we must have the following: \( i \notin I \); \( i-1, i+1 \in I \); the connected component of \( D[I] \) which contains \( i-1 \) has a vertex labeled by 1 (because otherwise we would have that \( a_{i-1} = 0 \)); and ditto for the connected component which contains \( i+1 \). That is to say, at a vertex \( i \in [n] \) with \( c_i = -1 \) we have a local picture like the following:

\[
\begin{array}{cccccccc}
\cdot & \cdot & 1 & \cdot & -1 & \cdot & 1 & \cdot \\
\end{array}
\]  

(6.13)

So our \( c_i \)-labeled Dynkin diagram \( D \) could for example be the following:

\[
\begin{array}{ccccccccccc}
1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 3 & 0 & 1 & -1 & 1 \\
\end{array}
\]

Now set \( K := I \cup \{i \in [n]: c_i = -1\} \). For instance, coloring in blue the vertices in \( K \) and edges between these vertices, the previous example looks like:

\[
\begin{array}{ccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 0 & 3 & 0 & 1 & -1 & 1 \\
\end{array}
\]

Let \( D[K'] \) be a connected component of \( D[K] \) for \( K' \subseteq K \). The local picture in (6.13) implies that if we read the \( c_i \)-labels of the vertices in \( D[K'] \) left-to-right, the nonzero labels follow a pattern of \( 1, -1, 1, -1, 1, \ldots, -1, 1 \). For instance, choosing \( D[K'] \) to be the left connected component from the example above, the labeled \( D[K'] \) looks
Let us consider making such a weight \( \mu = \sum_{i=1}^{n} c_i \omega_i \) dominant via a series of reflections. Suppose that as we apply reflections, the labels of the Dynkin diagram get updated with the new \( \omega_i \) coefficients of the weight to which we reflect. We refer to applying the reflection \( s_{\alpha_i} \) as reflecting at vertex \( i \). We will always reflect only at vertices with negative labels. So let \( i \) be any vertex with label \(-1\). How does reflecting at \( i \) affect the labels? It turns the \(-1\) into a \( 1 \) and subtracts one from labels of the neighboring vertices. Observe that if \( i \) belongs to the connected component \( D[K'] \) of \( D[K] \), then reflecting at \( i \) will preserve the property that the nonzero labels of vertices of \( D[K'] \) follow a left-to-right pattern of \( 1, -1, 1, \ldots, -1, 1 \). And hence by repeatedly reflecting at vertices in \( D[K'] \) with label \(-1\), we can make the labels of all the vertices within any given connected component \( D[K'] \) of \( D[K] \) nonnegative. For instance, for the labeled component \( D[K'] \) depicted in (6.14), the following series of reflections makes the labels of vertices in \( D[K'] \) nonnegative (at each step we circle the vertex at which we are reflecting):

Because we maintain the \( 1, -1, \ldots, 1 \) pattern of nonzero labels at every step, after carrying out all reflections necessary to make these labels nonnegative, \( D[K'] \) will have at most one vertex not labeled by \( 0 \) and the label of that vertex has to be \( 1 \).
Also, note that in this reflection process we will never reflect at an endpoint of $D[K']$. Therefore, these reflections will not alter the labels of vertices outside of $D[K']$.

Thus to obtain $\mu_{\text{dom}}$ from $\mu$ we need only reflect at vertices $i \in K$, which means that $\mu_{\text{dom}} - \mu \in \text{Span}_\mathbb{Z}(\{\alpha_i : i \in K\})$, i.e., that $J \subseteq K$. Since, after carrying out the appropriate reflections to make all labels nonnegative, any connected component $D[K']$ of $D[K]$ has at most one nonzero label, which must be a label of 1, and since any such component is a Type A diagram, we see that $\pi_{K'}(\mu_{\text{dom}})$ is zero or a minuscule weight of $\Phi_{K'}$ for all such $K' \subseteq K$. Because the projection of $\pi_K(\mu_{\text{dom}})$ to every irreducible sub-root system of $\Phi_K$ is zero or minuscule, we conclude that $\pi_K(\mu_{\text{dom}})$ is zero or a minuscule weight of $\Phi_K$. That $J \subseteq K$ thus implies that $\pi_J(\mu_{\text{dom}})$ is zero or a minuscule weight, as required.

**Type B:** Suppose $\Phi = B_n$. The argument will be a variant of the argument from the Type A case above. We claim that once again we have $c_i \geq -1$ for all $i \in [n]$. We need to check this only for $i \notin I$. As with the Type A argument, the fact that $v \in P_{\geq 0}^\mathbb{R}$ implies that $c_i \geq \langle \sum_{j \in I} a_j \alpha_j, \alpha_i^\vee \rangle$ for all $i \in [n]$. First consider the case $i \in [n - 1]$ and $i \notin I$. Then $c_i \geq -a_{i-1} - a_{i+1}$. What could these $a_{i-1}$ and $a_{i+1}$ be? If $i - 1 \in I$ then it is an endpoint of a Type A component of $D[I]$ so that $a_{i-1} < 1$. Meanwhile, if $i + 1 \in I$, then it is either an endpoint of a Type A component, so that $a_{i+1} < 1$, or it is the left endpoint of a Type B component, in which case we can look at Table 6.4 to verify that $a_{i+1} \leq \frac{1}{2}$. So in any event we get $c_i > -2$ and thus indeed $c_i \geq -1$. Moreover, in the case where we do have $c_i = -1$ the local picture at $i$ is of one of the following three forms (where again we use red to denote the elements of $I$):

\begin{center}
\begin{tikzpicture}
\draw[very thick] (0,0) -- (3,0);
\draw[very thick,red] (3,0) -- (4,0);
\draw[very thick] (4,0) -- (5,0);
\draw[very thick] (5,0) -- (7,0);
\draw[very thick,red] (7,0) -- (8,0);
\node at (1.5,0) {1};
\node at (4,0) {-1};
\node at (6.5,0) {1};
\end{tikzpicture}
\end{center}

(6.15)

\begin{center}
\begin{tikzpicture}
\draw[very thick] (0,0) -- (3,0);
\draw[very thick] (3,0) -- (4,0);
\draw[very thick] (4,0) -- (5,0);
\draw[very thick] (5,0) -- (6,0);
\draw[very thick] (6,0) -- (7,0);
\draw[very thick] (7,0) -- (8,0);
\node at (1.5,0) {1};
\node at (4,0) {-1};
\node at (5,0) {0};
\node at (6,0) {0};
\node at (7.5,0) {1};
\end{tikzpicture}
\end{center}

(6.16)
Finally, consider $i = n$ and $i \notin I$. Then we have $c_n \geq -2a_{n-1}$. If $i-1 \in I$ then it is an endpoint of a Type A component of $D[I]$ so that $a_{i-1} < 1$. This means that $c_n > -2$, i.e., that $c_n \geq -1$. Moreover, in the case where $c_n = -1$ the local picture at $n$ is of the following form:

$\begin{align*}
\begin{array}{cccc}
\text{1} & \text{1} & \text{1} & \text{1} \\
\end{array}
\end{align*}$

(6.17)

Now set $K := I \cup \{i \in [n] : c_i = -1\}$. Let $D[K']$ be a connected component of $D[K]$ for $K' \subseteq K$. If $D[K']$ is a Type A diagram, then it follows by considering the possible local pictures (6.15)-(6.18) that the nonzero labels among vertices in $D[K']$ have a left-to-right pattern of $1, -1, 1, -1, \ldots, 1$. So we already understand, from the Type A case above, how reflections work for such a component: as long as no reflections at vertices outside of $D[K']$ alter the labels of vertices in $D[K']$, we will have that $\pi_{K'}(\mu_{\text{dom}})$ is zero or a minuscule weight of $\pi_{K'}(\mu_{\text{dom}})$.

The new possibility, which we now consider, is that $D[K']$ is a Type B subdiagram. This means that $n \in K'$. The local pictures (6.15)-(6.18) imply that if $c_n = 0$ then $c_i = 0$ for all vertices $i \in K'$. We do not need to perform any reflections at vertices in $D[K']$ if all the labels of all vertices in $D[K']$ are zero, so let us assume that not all these labels are zero. Thus $c_n \in \{-1, 1\}$. Moreover, again by considering (6.15)-(6.18), we see that the nonzero labels among vertices in $D[K']$ follow a left-to-right pattern of $1, -1, 1, -1, \ldots, \pm 1$, where that last $\pm 1$ is the label of vertex $n$. That is to say, $D[K']$ looks like one of the following:

$\begin{align*}
\begin{array}{cccc}
\text{1} & \text{1} & \text{0} & \text{1} \\
\end{array}
\end{align*}$

(6.18)

$\begin{align*}
\begin{array}{cccc}
\text{1} & \text{1} & \text{0} & \text{1} \\
\end{array}
\end{align*}$

(6.19)
If $D[K']$ is in the form of (6.19), then we can always reflect at $n$ to make $c_n = 1$ and bring it into the form of (6.20). So we may as well assume that $D[K']$ is in the form of (6.20). What happens as we apply reflections to make the labels nonnegative? Consider this series of reflections for the example (6.20):

Observe how at every step we maintain the property that the nonzero labels among vertices in $D[K']$ follow a left-to-right pattern of $1, -1, 1, -1 \ldots, \pm 1$, with the last $\pm 1$ being the label of vertex $n$. This is easy to see for any reflection at a vertex other than $n - 1$. And when we reflect at vertex $n - 1$, we turn the label of vertex $n - 1$ into 1 (from $-1$), we subtract one from the label of vertex $n - 1$, and we turn the label of vertex $n - 1$ into 1 (from $-1$). So reflection at vertex $n - 1$ indeed preserves the requisite property. Thus it follows that after applying reflections to make all vertex labels nonnegative, the labels of all vertices in $D[K']$ will be zero except for vertex $n$ which will have a label of 1. This terminal labeling corresponds to $\omega_n$, which is a minuscule weight of the Type B sub-root system $\Phi_{K'}$. Also note that to make the labels of vertices in $D[K']$ nonnegative we never need to reflect at the left endpoint of $D[K']$. 

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Thus, we again can obtain $\mu_{\text{dom}}$ from $\mu$ by reflecting only at vertices $i \in K$, and thus $J \subseteq K$. And as we have seen, $\pi_{K'}(\mu_{\text{dom}})$ is zero or a minuscule weight of $\Phi_{K'}$ for any connected component $D[K']$ of $D[K]$. Because the projection of $\pi_K(\mu_{\text{dom}})$ to every irreducible sub-root system of $\Phi_K$ is zero or minuscule, we conclude that $\pi_K(\mu_{\text{dom}})$ is zero or a minuscule weight of $\Phi_K$. Thus, $\pi_J(\mu_{\text{dom}})$ is zero or a minuscule weight of $\Phi_J$, as required.

**Type C:** Suppose $\Phi = C_n$. The argument will again be a variant of the argument from the Type A case. Yet again the first step is to establish that $c_i \geq -1$ for all $i \in [i]$. We need to check this only for $i \notin I$. As before we have $c_i \geq \langle \sum_{j \in I} a_j \alpha_j, \alpha_i^\vee \rangle$ for all $i \in [n]$. First consider the case $i \in [n-2]$ and $i \notin I$. Then we have $c_i \geq -a_{i-1} - a_{i+1}$. What could these $a_{i-1}$ and $a_{i+1}$ be? If $i - 1 \in I$ then it is an endpoint of a Type A component of $D[I]$ so that $a_{i-1} < 1$. Meanwhile, if $i + 1 \in I$, then it is either an endpoint of a Type A component, so that $a_{i+1} < 1$, or it is the left endpoint of a Type C component, in which case we can look at Table 6.4 to verify that $a_{i+1} \leq 1$. So in any event we get $c_i > -2$ and thus indeed $c_i \geq -1$. Moreover, in the case where we do have $c_i = -1$ the local picture at $i$ is of one of the following two forms:

![Diagram](6.21)

Next, consider $i = n - 1$ and $i \notin I$. We have $c_{n-1} \geq -a_{n-2} - 2a_n$. If $n - 2 \in I$ then $n - 2$ is an endpoint of a Type A connected component of $D[I]$ and so $a_{n-2} < 1$. Meanwhile, if $n \in I$ then it forms a Type $A_1$ connected component of $D[I]$ so that $a_n \leq \frac{1}{2}$. So again we conclude that $c_{n-1} > -2$ and hence $c_{n-1} \geq -1$. Moreover, the

\[\text{If } i = n - 2 \text{ here, then by convention we view } D[[n-1, n]] \text{ as forming a } "C_2" = B_2 \text{ subdiagram; one can easily verify in this case that } a_{n-1} \leq 1 \text{ and that the local picture in (6.22) applies.}\]
local picture $n - 1$ in the case $c_{n-1} = -1$ looks like the following:

\[ \begin{array}{c}
-1 \\
\hline
1
\end{array} \]  

(6.23)

Finally, consider $i = n$ and $i \notin I$. We have $c_n \geq -a_{n-1}$. If $a_{n-1} \in I$ then it is an endpoint of a Type A connected component of $D[I]$ and so $a_{n-1} < 1$. This means that $c_n > -1$, and thus in fact $c_n \geq 0$.

Unlike in the cases of Type A or Type B, it is possible that there is $i \in [n]$ which has $c_i = -1$ but $a_{i-1} = 0$. Therefore it is appropriate in this case to define

\[ K := I \cup \{ i \in [n] : \text{there is } k \geq i \text{ with } c_k = -1 \text{ and } c_j = 0 \text{ for all } i \leq j < k \}. \]

(Note that all $i$ with $c_i = -1$ belong to $K$.) Let $D[K']$ be a connected component of $D[K]$ for $K' \subseteq K$. If $D[K']$ is a Type A component, then by considering the possible local pictures (6.21)-(6.23), we see that the nonzero labels among vertices in $D[K']$ follow a left-to-right pattern of $1, -1, 1, -1, \ldots, 1$. We already understand, from the Type A case above, how reflections work for such a component: as long as no reflections at vertices outside of $D[K']$ alter the labels of vertices in $D[K']$, we will have that $\pi_{K'}(\mu_{dom})$ is zero or a minuscule weight of $\pi_{K'}(\mu_{dom})$.

So now consider the possibility that $D[K']$ is a Type C component (or the degenerate “$C_2 = B_2$” subdiagram $D([n-1, n])$). This means that $n \in K'$. It could be that the nonzero $c_i$-labels of vertices in $D[K']$ follow a left-to-right pattern of $1, -1, 1, -1, \ldots, 1$, like the following:

\[ \begin{array}{c}
0 & 1 & -1 & 0 & 1 & 0 & -1 & 1
\end{array} \]  

(6.24)

Reflections for a component in the form of (6.24) behave exactly the same as with a Type A component: we will never reflect at either endpoint of $D[K']$; and we will eventually terminate at a labeling with at most one nonzero label, which has to be 1. However, observe that in this situation the terminal labeling might not correspond
to a minuscule weight of $\Phi_{K'}$, because $D[K']$ is a Type C diagram. Nevertheless, we can get around this issue by noting that $D[K' \setminus \{n\}]$ is a Type A diagram and that we never have to reflect at vertex $n$. Thus what we can say is that the terminal labeling of $D[K' \setminus \{n\}]$ corresponds to zero or a minuscule weight of $\Phi_{K' \setminus \{n\}}$. By considering the local pictures (6.21)-(6.23), we see that the only other possibility for the $c_i$-coefficients of $D[K']$ beyond the form depicted in (6.24) is that the nonzero $c_i$-labels of $D[K']$ are one $-1$ and one 1, with vertex labeled $-1$ adjacent and to the left of the vertex labeled 1. This looks like the following:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

(6.25)

For a labeling in the form of (6.25), we can always carry a series of reflections like this:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
\Rightarrow & & & & & & & \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
\Rightarrow & & & & & & & \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\Rightarrow & & & & & & & \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Rightarrow & & & & & & & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Let $i$ be the left endpoint of $D[K']$. In the above series of reflection, we reflected, once, at vertex $i$. Therefore this series of reflections will subtract one from the label of vertex $i - 1$, and so we have to be concerned that we will make this label negative. (Of course this is assuming $i \neq 1$; if $i = 1$ there is nothing to worry about.) However, by definition of $K$ and because $D[K']$ is a connected component of $D[K]$, we have that $i - 1 \notin K$ and $c_{i-1} \geq 1$. Since all the other components of $D[K]$ are Type A, their reflections will not alter the label of $i - 1$, and so $i - 1$ will maintain a nonnegative label even though we reflected at $i$. Note also the following about the above series of reflections: we did not have to reflect at vertex $n$; and we will end with exactly
one nonzero label among vertices in \( D[K'] \), which will be a 1. Therefore whether the \( c_i \)-labels of vertices in \( D[K'] \) are in the form of (6.24) or (6.25), the vertices at which we need to reflect in order to make these labels nonnegative are a subset of \( K' \setminus \{ n \} \) and the terminal labeling of \( D[K' \setminus \{ n \}] \) corresponds to zero or a minuscule weight of \( \Phi_{K' \setminus \{ n \}} \).

Therefore, in order to make \( \mu \) dominant we need only reflect at vertices in \( K \setminus \{ n \} \), i.e., we have \( J \subseteq K \setminus \{ n \} \). And because the projection of \( \pi_{K \setminus \{ n \}}(\mu_{\text{dom}}) \) to every irreducible sub-root system of \( \Phi_{K \setminus \{ n \}} \) is zero or minuscule, we conclude that \( \pi_{K \setminus \{ n \}}(\mu_{\text{dom}}) \) is zero or a minuscule weight of \( \Phi_{K \setminus \{ n \}} \). Thus, \( \pi_J(\mu_{\text{dom}}) \) is zero or a minuscule weight of \( \Phi_J \), as required.

**Type D:** Suppose \( \Phi = D_n \). The argument will again be a variant of the argument from the Type A case. It turns out that this last case is the most involved. Yet again the first step is to establish that \( c_i \geq -1 \) for all \( i \in [n] \). We need to check this only for \( i \notin I \). As before, we have \( c_i \geq \langle \sum_{j \in I} a_j \alpha_j, \alpha_i \rangle \) for all \( i \in [n] \). First consider the case \( i \in [n - 3] \) and \( i \notin I \). Then we have \( c_i \geq -a_{i-1} - a_{i+1} \). What could these \( a_{i-1} \) and \( a_{i+1} \) be? If \( i - 1 \in I \) then it is an endpoint of a Type A component of \( D[I] \) so that \( a_{i-1} < 1 \). Meanwhile, if \( i + 1 \in I \), then it is either an endpoint of a Type A component, so that \( a_{i+1} < 1 \), or it is the left endpoint of a Type D component, in which case we can look at Table 6.4 to verify that \( a_{i+1} \leq 1 \). So in any event we get \( c_i > -2 \) and thus indeed \( c_i \geq -1 \). Moreover, in the case where we do have \( c_i = -1 \) the local picture at \( i \) is of one of the following possible forms:

\[
\begin{align*}
&1 \quad -1 \quad 1 \\
\quad \cdots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots
\end{align*}
\]

(6.26)

\[
\begin{align*}
&-1 \quad 1 \quad 0 \quad 0 \quad 0 \\
\quad \cdots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots
\end{align*}
\]

(6.27)

\[\text{If } i = n - 3 \text{ here, then by convention we view } D([n - 2, n - 1, n]) \text{ as forming a } D_3 \text{ subdiagram; one can easily verify in this case that } a_{n-1} \leq 1 \text{ and that the local picture in (6.27) or (6.25) applies.}\]
Next, consider \( i = n - 2 \) and \( i \notin I \). We have \( c_{n-2} \geq -a_{n-3} - a_{n-1} - a_n \). If \( n - 3 \in I \) then it is an endpoint of a Type A connected component of \( D[I] \), so \( a_{n-3} < 1 \). Meanwhile, for \( j \in \{n - 1, n\} \), if \( j \in I \) then it forms a Type \( A_1 \) connected component of \( D[I] \), so \( a_j \leq \frac{1}{2} \). Hence we see that \( c_{n-2} > -2 \), i.e., \( c_{n-2} \geq -1 \). Moreover, in the case where we do have \( c_{n-2} = -1 \) the local picture at \( n - 2 \) is of one of the following forms:

Next consider \( i = n - 1 \) and \( i \notin I \). Thus \( c_{n-1} \geq -a_{n-2} \). If \( n - 2 \in I \), then either it is the endpoint of a Type A connected component of \( D[I] \), in which case \( a_{n-2} < 1 \), or it is not an endpoint but is rather adjacent to an endpoint of a Type A connected component of \( D[I] \), in which case we can look at Table 6.4 to verify that \( a_{n-2} < 2 \). So we see again that \( c_{n-1} > -2 \), i.e., that \( c_{n-1} \geq -1 \). Moreover, in the case where we do have \( c_{n-1} = -1 \) the local picture at \( n - 1 \) looks as follows:

Note crucially, as is depicted in (6.31), that if \( c_{n-1} = -1 \) then \( c_n = 0 \): this is because
if the connected component of $D[I]$ containing $n - 2$ had a label of 1 at one of its endpoints, then we would get $a_{n-2} < 1$, as can be verified by checking Table 6.4. Finally, the case of $i = n$ is completely symmetric to the case of $i = n - 1$; we will get a mirror-image picture of (6.31).

Now set

$$K := I \cup \{i \in [n] : \text{there is } k \geq i \text{ with } c_k = -1 \text{ and } c_j = 0 \text{ for all } i \leq j < k\} \cup \{n-1, n\}.$$ 

Let $D[K']$ be a connected component of $D[K]$ for $K' \subseteq K$. If $\{n - 1, n\} \not\subseteq K'$, then $D[K']$ is a Type A component and the possible local pictures (6.26)-(6.31) imply that the nonzero labels among vertices in $D[K']$ follow a left-to-right pattern of $1, -1, 1, -1, \ldots, 1$. We already understand, from the Type A case above, how reflections work for such a component: as long as no reflections at vertices outside of $D[K']$ alter the labels of vertices in $D[K']$, we will have that $\pi_{K'}(\mu_{dom})$ is zero or a minuscule weight of $\pi_{K'}(\mu_{dom})$.

So now consider the possibility that $\{n - 1, n\} \subseteq K'$. Thus $D[K']$ is a Type D component (or the degenerate “$D_3$” = $A_3$ subdiagram $D[\{n - 2, n - 1, n\}]$). Let $i$ be the left endpoint of $D[K']$ (with $i := n - 2$ for the degenerate $D_3$ case). There are several possible cases to consider based on the possible $c_i$-labels of vertices in $D[K']$. First let us consider the case where $c_{n-1}, c_n \geq 0$ and the left-to-right pattern of the nonzero $c_i$-labels of vertices in $D[K \setminus \{n - 1, n\}]$ is $1, -1, 1, -1, \ldots, 1$. This looks like the following:

$$0 \quad 1 \quad -1 \quad 0 \quad 1 \quad -1 \quad 0 \quad 1 \quad \geq 0 \quad \geq 0$$

(6.32)

Reflections for a component in the form of (6.32) behave exactly the same as with a Type A component: we will never reflect at any endpoints of $D[K']$; and we will eventually terminate at a labeling with at most one nonzero label among the vertices in $D[K' \setminus \{n - 1, n\}]$, which has to be 1. Hence the terminal labeling of $D[K' \setminus \{n - 1, n\}]$ corresponds to zero or a minuscule weight of $\Phi_{K' \setminus \{n-1, n\}}$. Next consider the case
where $c_{n-1}, c_n = 0$ and the $c_i$-labels of vertices in $D[K' \setminus \{n-1, n\}]$ are all zero except for one $-1$ and one $1$, with the vertex labeled $-1$ adjacent and to the left of the vertex labeled $1$. This looks like the following:

\begin{equation}
\begin{array}{cccccccccc}
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\end{equation}

This case is similar to (6.24) from Type C. For a labeling as in (6.33) we can always carry out a series of reflections as follows:

\begin{equation}
\begin{array}{cccccccccc}
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{cccccccccc}
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{cccccccccc}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\end{equation}

In the above series of reflection, we reflected, once, at vertex $i$. Therefore this series of reflections will subtract one from the label of vertex $i - 1$, and so we have to be concerned that we will make this label negative. (Of course this is assuming $i \neq 1$; if $i = 1$ there is nothing to worry about.) However, by definition of $K$ and because $D[K']$ is a connected component of $D[K]$, we have that $i - 1 \notin K$ and $c_{i-1} \geq 1$. Since all the other components of $D[K]$ are Type A, their reflections will not alter the label of $i - 1$, and thus we see that $i - 1$ will maintain a nonnegative label even though we reflected at $i$. Note also the following about the above series of reflections: we did not have to reflect at vertices $n - 1$ or $n$; and again the terminal labeling of $D[K' \setminus \{n - 1, n\}]$ corresponds to a minuscule weight of $\Phi_{K' \setminus \{n-1, n\}}$. Next let us consider the possibility $c_{n-2} = -1$ and $c_{n-1}, c_n \geq 1$. The local pictures (6.26)-(6.31), imply that if the $c_i$-labels of all vertices in $D[K' \setminus \{n - 2, n - 1, n\}]$ are zero, then
\(c_{n-1} = c_n = 1\) and the situation looks like:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \swarrow 1 \\
& & & & & & & 1 \\
\end{array}
\]  
(6.34)

By reflecting at \(n - 2\), we can bring a labeling in the form of (6.34) into the form of (6.33). Therefore the arguments above imply that: the reflections needed to make all labels in \(D[K']\) nonnegative will not make negative the labels of any vertices outside \(D[K']\); we will not have to reflect at vertices \(n - 1\) or \(n\); and the terminal the terminal labeling of \(D[K' \setminus \{n-1, n\}]\) will correspond to a a minuscule weight of \(\Phi_{K' \setminus \{n-1, n\}}\). Still assuming that \(c_{n-2} = -1\) and \(c_{n-1}, c_n \geq 1\), by looking at the local pictures (6.26)-(6.31) we see that the only other possibility beyond the form of (6.34) is that the \(c_i\)-labels of vertices in \(D[K' \setminus \{n - 2, n - 1, n\}]\) follow a left-to-right pattern of \(1, -1, 1, -1, \ldots, 1\). This looks like the following:

\[
\begin{array}{cccccccc}
1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \swarrow 1 \\
& & & & & & & 1 \\
\end{array}
\]  
(6.35)

By reflecting at \(n - 2\), we can bring a labeling in the form of (6.35) into the form of (6.32). Therefore the arguments above imply that: to make the labels of vertices in \(D[K']\) nonnegative we will not need to reflect at an endpoint of \(D[K']\); and the terminal labeling of \(D[K' \setminus \{n-1, n\}]\) will correspond to a minuscule weight of \(\Phi_{K' \setminus \{n-1, n\}}\). Finally, we can see from the local pictures (6.26)-(6.31) that the only remaining possibility for the \(c_i\)-labels of vertices in \(D[K']\) is that: the nonzero \(c_i\)-labels of vertices in \(D[K' \setminus \{n - 1, n\}]\) follow a left-to-right pattern of \(1, -1, 1, -1, \ldots, \pm 1\); the label of one of the vertices \(n - 1\) or \(n\) is \(\mp 1\); and the label of the other of these two vertices is 0. This will look like one of the following two cases:

\[
\begin{array}{cccccccc}
1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \swarrow 1 \\
& & & & & & & 1 \\
\end{array}
\]  
(6.36)
These cases are similar to (6.19) and (6.20) from Type B. First observe that if $D[K']$ is in the form of (6.36), where the nonzero label among $n$ or $n-1$ is $-1$, then we can always reflect at that vertex to make the nonzero label among these two vertices 1 and bring it into the form of (6.37). So we may as well assume that $D[K']$ is in the form of (6.37). Consider the following series of reflections for the example (6.37):

Observe how at every step we maintain the property that the nonzero labels among vertices in $D[K' \setminus \{n-1, n\}]$ follow a left-to-right pattern of $1, -1, 1, -1, \ldots, \pm 1$ and that one of the labels among vertices $n-1$ and $n$ is $\mp 1$ while the other is 0. This is easy to see for any reflection at a vertex other than $n-2$. And when we reflect at vertex $n-2$, we turn the label of vertex $n-2$ into 1 (from $-1$), we subtract one from the label of vertex $n-3$, and we turn the label of one of $n$ or $n-1$ into 0 (from 1), and turn the label of the other of these two into $-1$ (from 0). So reflection at vertex $n-2$ indeed preserves the requisite property. Thus it follows that after applying reflections to make all vertex labels nonnegative, the labels of all vertices in $D[K']$ will be zero.
except for one of the vertices \( n - 1 \) or \( n \) which will have a label of 1. This terminal labeling corresponds to either \( \omega_{n-1} \) or \( \omega_n \), which is a minuscule weight of the Type D sub-root system \( \Phi_{\bar{K}'} \). So define

\[
\tilde{K}' := \begin{cases} 
K' & \text{if we have to reflect at vertex } n - 1 \text{ or } n \text{ to make } \mu \text{ dominant}; \\
K' \setminus \{n, n - 1\} & \text{otherwise}.
\end{cases}
\]

No matter which of the forms \((6.32)-(6.37)\) the \( c_i \)-labeling of \( D[K'] \) is in, the terminal labeling of \( D[\tilde{K}'] \) will correspond to zero or a minuscule weight of \( \Phi_{\bar{K}'} \).

Set

\[
\tilde{K} := \begin{cases} 
K & \text{if we have to reflect at vertex } n - 1 \text{ or } n \text{ to make } \mu \text{ dominant}; \\
K \setminus \{n, n - 1\} & \text{otherwise}.
\end{cases}
\]

To make \( \mu \) dominant we only need to reflect at vertices in \( \tilde{K} \), i.e., we have \( J \subseteq \tilde{K} \).

Furthermore, the projection of \( \pi_{\tilde{K}}(\mu_{\text{dom}}) \) to every irreducible sub-root system of \( \Phi_{\tilde{K}} \) is zero or minuscule, and hence we conclude that \( \pi_{\tilde{K}}(\mu_{\text{dom}}) \) is zero or a minuscule weight of \( \Phi_{\tilde{K}} \). That \( J \subseteq \tilde{K} \) thus implies that \( \pi_J(\mu_{\text{dom}}) \) is zero a minuscule weight of \( \Phi_J \), as required. This completes the type-by-type verification of Claim \( \Pi \).
Bibliography


