Assignment Five

It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle—they are strictly limited in number, they require fresh horses, and must only be made at decisive moments.

-Alfred North Whitehead

For workshops this week, you will be assigned groups and given one of the next three problems. Work out each part in full, and write up your solutions. You will present your work to the class in workshop next Tuesday.

1. (Workshop Option 1)
   (a) Sketch graphs of \( f(x) = x^n \) for a variety of integers \( n \), positive and negative. What are the domain and range in each case?
   (b) In class we showed that \( \frac{d}{dx} x^n = nx^{n-1} \) for positive integers \( n \). Extend this to negative integers using the quotient rule.
   (c) Give a detailed definition of \( f(x) = x^n \) when \( n = \frac{p}{q} \), a rational number. Sketch graphs of several examples and give the domain and range for each.
   (d) Show that \( \frac{d}{dx} x^n = nx^{n-1} \) for rational numbers \( n \). (Hint: if \( f(x) = x^{\frac{p}{q}} \), then \( (f(x))^q = x^p \). Differentiate both sides.)
   (e) Give a detailed definition of \( f(x) = x^n \) when \( n \) is an irrational number. Sketch graphs of several examples and give the domain and range for each.
   (f) Show that \( \frac{d}{dx} x^n = nx^{n-1} \) for irrational numbers \( n \).

2. (Workshop Option 2) Recall that in last week’s assignment, we defined \( e \) as the unique constant such that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \), and then showed that \( \frac{d}{dx} e^x = e^x \).
   (a) Show that \( \frac{d}{dx} \ln(x) = \frac{1}{x} \). (Hint: \( e^{\ln(x)} = x \). Differentiate both sides.)
   (b) Sketch the graphs of \( a^x \) and \( \log_a(x) \) for a variety of positive constants \( a \), and give the domain and range of each. What happens if you try to take \( a \) negative?
   (c) Find the formulas for \( \frac{d}{dx} a^x \) and \( \frac{d}{dx} \log_a(x) \). (Hint: base change formulas.)
   (d) There are two other famous formulas for \( e^x \). One is \( \lim_{n \to \infty} (1 + \frac{x}{n})^n \) and the other is \( 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots \). We don’t yet have the tools to explain why these formulas make sense. But assuming that they do make sense, explain roughly why each formula is its own derivative.
   (e) Suppose that some function \( f(x) \) satisfies \( f'(x) = f(x) \). Show that \( f(x) \) is a constant multiple of \( e^x \). (Hint: differentiate \( \frac{f(x)}{e^x} \) using the quotient rule.)
3. (Workshop Option 3)

(a) Recall that we used a geometric argument to prove that \( \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \), i.e. that \( \frac{d}{dx} \sin(x) = \cos(x) \) at \( x = 0 \). Use a geometric argument to prove that \( \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0 \), i.e. that \( \frac{d}{dx} \cos(x) = -\sin(x) \) at \( x = 0 \).

(b) Prove that \( \frac{d}{dx} \sin(x) = \cos(x) \) using the limit definition and angle-sum formulas.

(c) Prove that \( \frac{d}{dx} \cos(x) = -\sin(x) \) for all \( x \).

(d) Suppose that you have two functions \( s(x), c(x) \) such that \( s'(x) = c(x), c'(x) = -s(x), s(x)^2 + c(x)^2 = 1 \), \( s(0) = 0 \) and \( c(0) = 1 \). Prove that \( s(x) = \sin(x) \) and \( c(x) = \cos(x) \).

(Hint: Differentiate \( (\sin(x) - s(x))^2 + (\cos(x) - c(x))^2 \).)

(e) Find formulas for the derivatives of the other trig functions, \( \tan(x), \sec(x), \csc(x), \) and \( \cot(x) \).

4. Section 3.3, problems 5, 8, 14. Section 3.4, problems 31, 32.

5. Section 4.5, problems 2, 4, 8, 10, 17.

6. Here are three graphs which do not define functions.

Hypocycloid \( x^{2/3} + y^{2/3} = 2^{2/3} \), Lemniscate \( (x^2 + y^2)^2 = x^2 - y^2 \), Hyperbola \( 4x^2 - 9y^2 = 1 \).

For each graph,

(a) Sketch the graph of an implicit function and state inequalities which define that function. You don’t need to solve for \( y \) in terms of \( x \), but in these examples it wouldn’t be too hard.

(b) Find the slope of a tangent line at point \( (x, y) \) by implicit differentiation.

(c) Where is the slope 0? where is it undefined? Make sure these answers make sense.

7. Section 3.5, problems 3, 4.

8. \( e^x \) is its own derivative. \( \sin(x) \) and \( \cos(x) \) are their own fourth derivatives. Find a function which takes exactly two derivatives to return to itself, and (harder) a function which takes exactly three.