Assignment 5

Cady: Yeah, I like math.
Damian: Eww. Why?
Cady: Because it’s the same in every country.
-Mean Girls

1. Each of the following series $a_n$ converges to a limit $L$. For $\epsilon = \frac{1}{100}$, find an integer $N$ such that, if $n > N$, then $a_n$ is within distance $\epsilon$ of $L$. Then for an arbitrarily small $\epsilon$, explain how to find $N$.

   (a) $a_n = \frac{1}{n^2 + 3}$, $L = 0$.
   (b) $a_n = \frac{1}{n^2 - n - 3}$, $L = 0$.
   (c) $a_n = 3^{-n/2}$, $L = 0$.
   (d) $a_n = \frac{3n+2}{2n-1}$, $L = \frac{3}{2}$.
   (e) $a_n = \sqrt{n + 1} - \sqrt{n}$, $L = 0$.
   (f) $a_n = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{17}$, $L = \frac{1}{3}$.
   (g) $a_n = \frac{n^4}{n^4}$, $L = 0$.
   (h) $a_n = (-\frac{3}{4})^n$, $L = 0$.
   (i) $a_n = \frac{5n+6}{n+4}$, $L = 0$.
   (j) $a_n = \frac{n^2 + 2}{n^2 - 3}$, $L = 1$.


3. Thursday Workshop/Practice Exam:

   (a) Approximate $\int_0^\pi x^2 \sin(x) \, dx$ using Simpson’s rule with $n = 4$. Find an upper bound for $M_4$, the maximum absolute value of the fourth derivative of $x^2 \sin(x)$. Your upper bound does not need to be the actual maximum value, but you should explain why it is greater than the maximum. Use this to bound the Simpson’s rule error with $n = 4$. What value of $n$ would you need to make the error less than $1/1000$?

   (b) The sequence $a_n = \frac{5n+1}{4n-3}$ converges to $\frac{5}{4}$. Given $\epsilon > 0$, find an integer $N$ such that if $n > N$ then $|\frac{5}{4} - a_n| < \epsilon$.

   (c) Show that $a_n = \frac{2n}{n^2}$ converges to zero using the squeeze theorem.

   (d) Find the limits as $n \to \infty$ of $\frac{\sqrt{2n+1}}{2\sqrt{n}}$ and $\frac{100n^2 \cos(n^3 + \pi)}{2n^2 \ln(n)}$. Briefly justify each answer. You do not need to give an $\epsilon$ argument.

   (e) An error bound formula for approximation by rectangles is given by $M_1(b-a)^2$. For approximation by trapezoids, the error bound is $M_2(b-a)^3$. For Simpson’s rule, it is $\frac{M_3(b-a)^5}{180n^4}$. Suppose you want to approximate $\int_1^4 \sqrt{x} \, dx$ with error less than $10^{-6}$. What value of $n$ do you need to choose for each method?
4. Practice Exam:

(a) Approximate \( \int_{0}^{4} x^3 - 2x^2 + 3x - 4 \, dx \) by trapezoids and using Simpson’s rule with \( n = 4 \). Find a bound for the error if trapezoids are used. Show that the Simpson’s rule approximation is exactly right, either using the error bound formula or direct computation of the integral.

(b) Use the squeeze theorem to find \( \lim_{n \to \infty} \frac{3n^2 + (-1)^n \sin(\pi n/6)\sqrt{n}}{2n^2} \).

(c) Find a formula for the sum \( \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n} \). These sums approach \( \frac{1}{2} \) as \( n \to \infty \). How large must you take \( n \) for the sum to be within distance \( \epsilon \) of \( \frac{1}{2} \)?

(d) The sequence \( a_n = \frac{10n}{n^2 - 3n + 3} \) converges to 0. For any \( \epsilon > 0 \), explain how to find \( N \) such that if \( n > N \) then \( |a_n| < \epsilon \).

(e) No proof required. Find the limits as \( n \to \infty \) of \( (1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{n}) \) and of \( (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) \cdots (1 - \frac{1}{n^2}) \).