Instructions Please answer all questions below. Write clearly and circle your final answers. Show your work for partial credit. You can use a calculator, but no notes or books. Good luck.

Useful power series formulas:

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]
\[
\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n+1 x^n}{n}
\]
\[
\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}
\]
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]
\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]
\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]
\[
(1+x)^p = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)x^n}{n!}
\]

Miscellaneous results:

Approximation by trapezoids: \( \int_a^b f(x) \, dx \approx \Delta x (\frac{1}{2} y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2} y_n) \) with error less than \( \frac{M_2(b-a)^3}{12n^2} \), where \( M_2 \) is the maximum value of \( |f''| \) in \([a, b]\).

Simpson’s rule: \( \int_a^b f(x) \, dx \approx \Delta x (\frac{1}{3} y_0 + \frac{4}{3} y_1 + \frac{2}{3} y_2 + \frac{4}{3} y_3 + \frac{2}{3} y_4 + \cdots + \frac{4}{3} y_{n-2} + \frac{2}{3} y_{n-1} + \frac{1}{3} y_n) \) with error less than \( \frac{M_4(b-a)^5}{180n^4} \), where \( M_4 \) is the maximum value of \( |f'''| \) in \([a, b]\).

Taylor’s formula and Lagrange’s error bound: \( f(x) \approx f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(N)}(0)}{N!} x^N \), with error less than \( \frac{M_{N+1}x^{N+1}}{(N+1)!} \), where \( M_{N+1} \) is the maximum value of \( |f^{N+1}| \) in \([0, x]\).

Curvature of parametric plane curves: \( k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \).
1. The cardioid \( r = \sin(\theta) + 1 \) is shown. Suppose you are standing at the point \( \theta = \frac{\pi}{4} \) and want to walk along the cardioid to the point \( \theta = \frac{5\pi}{4} \). Find the arc length in both directions.

The polar arc length formula is

\[
\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta
\]

To get the length clockwise from \( \frac{\pi}{4} \) to \( \frac{5\pi}{4} \), we must take \( \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \).

To get the length counterclockwise, we take \( \int_{\frac{5\pi}{4}}^{\frac{\pi}{4}} \).

The integral is

\[
\int \sqrt{(\sin(\theta) + 1)^2 + \cos(\theta)^2} \, d\theta
\]

\[
= \int \sqrt{1 + 2\sin \theta + \sin^2 \theta \cos^2 \theta} \, d\theta
\]

\[
= \int \sqrt{2 + 2\sin \theta} \, d\theta
\]

We must introduce \( \frac{\cos(\theta)}{\cos(\theta)} \) to make a \( u \)-substitution:

\[
= \int \sqrt{2 + 2\sin \theta} \frac{\cos \theta}{\cos \theta} \, d\theta = \int \frac{\sqrt{2 + 2\sin \theta}}{\cos \theta} \cos \theta \, d\theta
\]

\[
= \int \frac{\sqrt{2 + 2\sin \theta}}{1 - \sin^2 \theta} \cos \theta \, d\theta = \int \frac{\sqrt{2 + 2\sin \theta}}{1 - u^2} \, du \quad \text{where} \quad u = \sin \theta
\]

\[
= \int \frac{\sqrt{2(1+u)}}{(1+u)(1-u)} \, du = \int \frac{\sqrt{2}}{1-u} \, du = -2\sqrt{2}(1-u) = -2\sqrt{2} - 2\sin \theta
\]

From \( -\frac{3\pi}{4} \) to \( \frac{\pi}{4} \), we get

\[
+2\sqrt{2} - 2\sin \theta \bigg|_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} = 2 + 2\sqrt{2} - 2\sin \theta
\]

From \( \frac{\pi}{4} \) to \( \frac{5\pi}{4} \), we get

\[
-2\sqrt{2} - 2\sin \theta \bigg|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = 0 + 2\sqrt{2} - 2\sqrt{2} - 0
\]

The reason we break up the intervals this way is subtle. We want to make the arc length \(-2\sqrt{2} - 2\sin \theta\) an always increasing function of \( \theta \), and we must choose positive or negative square roots accordingly.
2. Find \( \lim_{x \to 0} \frac{\sin(x)(\cos(x) - 1)}{e^{x^2} - 1} \)

\[
\lim_{x \to 0} \frac{\sin(x)(\cos(x) - 1)}{e^{x^2} - 1}
\]

L'Hôpital's Rule

\[
= \frac{\cos^2 x - \cos(x) - \sin^2 x}{3x^2 e^{x^2}}
\]

L'Hôpital's Rule

\[
= \frac{-2\cos x \sin x + \sin(x)}{6x e^{x^2} + 9x^2 e^{x^2}}
\]

L'Hôpital's Rule

\[
= \frac{4\sin^3 x - 4\cos^2 x + \cos(x)}{6e^{x^2} + 18x^3 e^{x^2} + 36x^6 e^{x^2} + 27x^8 e^{x^2}}
\]

Finally, this is continuous at \( x = 0 \),

and the limit is

\[-\frac{1}{2}.
\]
3. The vector function \( \mathbf{r}(t) = \cos(t^2)i + \sin(t^2)j \) describes a particle in accelerating circular motion. Find the tangential and normal components of the particle's acceleration at time \( t = \sqrt{\frac{5\pi}{6}} \).

This particle is moving around the unit circle. Its position vector \(-\frac{\sqrt{2}}{2}i + \frac{1}{2}j\) at time \( t = \sqrt{\frac{5\pi}{6}} \) is shown.

The unit tangent vector \( \mathbf{T} = -\frac{1}{2} \mathbf{i} - \frac{\sqrt{2}}{2} \mathbf{j} \) and the unit normal vector \( \mathbf{N} = \frac{\sqrt{2}}{2} \mathbf{i} - \frac{1}{2} \mathbf{j} \).

\( \mathbf{r}'(t) = -2t \sin(t^2)i + 2t \cos(t^2)j \) is the velocity.

The speed \( |\mathbf{r}'(t)| \) is simply \( 2t \).

The rate of change of speed is \( 2 \).

Thus the tangential component of acceleration is \( 2 \mathbf{T} = -\mathbf{i} - \frac{\sqrt{2}}{2} \mathbf{j} \).

The normal component has magnitude curvature \( t \) speed.

and since this is on the unit circle, curvature = 1.

So the normal component is \( (2t)^2 \mathbf{N} = 4\left(\frac{5\pi}{6}\right)\left(\frac{\sqrt{2}}{2} \mathbf{i} - \frac{1}{2} \mathbf{j}\right) \)

\( = \frac{5\pi \sqrt{2}}{3} \mathbf{i} - \frac{5\pi}{3} \mathbf{j} \)
4. Give Taylor series (in sigma notation) for the following functions, by manipulating known series:

(a) \( \int xe^{2x+1} \, dx \)

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

\[
e^{2x} = \sum_{n=0}^{\infty} \frac{(2^n)x^n}{n!}
\]

\[
e^{2x+1} = \sum_{n=0}^{\infty} \frac{2^n e^x}{n!}
\]

\[
x e^{2x+1} = \sum_{n=0}^{\infty} \frac{2^n e^x x^{n+1}}{n!}
\]

\[
\int xe^{2x+1} \, dx = c + \sum_{n=0}^{\infty} \frac{2^n e^x x^{n+2}}{(n+2) n!}
\]

(b) \( \sqrt{9 + 4x^2} \)

\[
\sqrt{1 + x} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})(-\frac{1}{2})\cdots(\frac{1}{2} - \frac{n}{2} + 1)}{n!} x^n
\]

\[
\sqrt{1 + \frac{4}{9}x^2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})(-\frac{1}{2})\cdots(\frac{1}{2} - \frac{n}{2} + 1)}{n!} 4^n x^{2n}
\]

\[
3 \sqrt{1 + \frac{4}{9}x^2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})(-\frac{1}{2})\cdots(\frac{1}{2} - \frac{n}{2} + 1)}{n!} 3 \cdot 4^n x^{2n}
\]

(c) \( \int \frac{1}{8 - x^2} \, dx \)

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n
\]

\[
\frac{1}{1 - \frac{x^2}{8}} = \sum_{n=0}^{\infty} \frac{x^{2n}}{8^n}
\]

\[
\frac{1}{8 - x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{8^n + 1}
\]

\[
\int \frac{1}{8 - x^2} \, dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{8^n (3^n + 1)} + C
\]
5. Compute the following indefinite integrals. Show your work.

(a) \[ \int \frac{1}{x^2\sqrt{x^2-4}} \, dx \]

\[ x = 2 \sec u \]
\[ dx = 2 \sec u \tan u \, du \]

\[ \int \frac{1}{4 \sec^2 u \sqrt{4 \sec^2 u - 4}} \, 2 \sec u \tan u \, du = \int \frac{2 \sec u \tan u}{8 \sec^2 u} \, du \]

\[ = \frac{1}{4} \cos u \, du = \frac{1}{4} \sin (u) = \frac{1}{4} \sin \left( \arccos \left( \frac{x}{2} \right) \right) = \frac{1}{4} \frac{\sqrt{x^2 - 4}}{x} + C \]

(b) \[ \int x^4 \ln(x) \, dx \]

\[ f^2(x) = x^4, \quad g(x) = \ln(x) \]
\[ f(x) = \frac{x^5}{5}, \quad g'(x) = \frac{1}{x} \]

\[ \int x^4 \ln(x) \, dx = \frac{x^5}{5} \ln(x) - \int \frac{x^4}{5} \, dx = \frac{x^5}{5} \ln(x) - \frac{x^5}{25} + C \]

(c) \[ \int \frac{x+1}{x^2(x-1)} \, dx \]

\[ \frac{x+1}{x^2(x-1)} = \frac{a}{x-1} + \frac{b}{x^2} + \frac{c}{x} \]

\[ a(x^2) + b(x) + c(x-1) = x+1 \]

\[ \Rightarrow a = -1 \]
\[ b = c - 1 = -2 \]
\[ a = -b = 2 \]

5. \[ \int \frac{2}{x-1} + \frac{-2x-1}{x^2} \, dx \]

\[ = \int \frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2} \, dx = 2(\ln(x-1)) - 2(\ln(x)) + \frac{1}{x} + C \]
6. Approximate the following real numbers using Taylor series, with error less than 1/10. Do an error bound computation to justify each answer:

(a) \( \sqrt{103} \)

The series for \( \sqrt{100+x} \) is

\[
10 + \frac{1}{20} x - \frac{1}{4000} x^3 + \ldots
\]

This alternates after the first term, so we may plug in \( x = 3 \) to the first two terms and use the alternating series error bound:

\[
\sqrt{103} = 10 + \frac{3}{20} \quad \text{with error} \quad \frac{3^2}{4000} < \frac{1}{10}
\]

(b) \( e^x \)

The series for \( e^x \) is

\[
1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots
\]

For \( |x| \leq 1 \), Lagrange's error bound has

\[
M_{N+1} = \max_{0 \leq x \leq 1} |e^x| = e, \quad \text{so the error from} \quad N \text{ terms is at most} \quad \frac{e^{N+1}}{(N+1)!} \cdot \frac{x^{N+1}}{x^{N+1}}
\]

If we take \( N = 4 \), the error at \( x = 1 \) is less than \( \frac{e}{5!} = \frac{1}{120} \).

(c) \( \sin(\pi/5) \)

This approximation is \( e^{\pi/5} \approx \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{41}{24} \).

We can use \( \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \)

(or we could expand \( \sin(x) \) at a value closer to \( \pi/5 \), like \( \pi/6 \)).

From the first two terms, \( \sin \left( \frac{\pi}{5} \right) \approx \frac{\pi}{5} - \frac{\pi^3}{6 \cdot 5^3} \)

The error is \( \leq \frac{\pi^5}{5! \cdot 5^5} < \frac{1}{10} \).
7. The graph of the function \( y = \frac{1}{2}(e^x + e^{-x}) \) is called a catenary. Find a formula for its curvature at the point \((x, y)\).

Let \( x = t \), \( y = \frac{1}{2} (e^t + e^{-t}) \)

Then \( x^2 = 1 \), \( y^2 = \frac{1}{4} (e^t - e^{-t}) \)
\( x'' = 0 \), \( y'' = \frac{1}{2} (e^t + e^{-t}) \)

The curvature \( k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \)

\[ k = \frac{1 \cdot \frac{1}{2}(e^t + e^{-t}) - 0}{(1 + \frac{1}{4}(e^{2t} - 2 + e^{-2t}))^{3/2}} \]

This answer would be fine, but we can simplify a lot.

\[ = \frac{\frac{1}{2}(e^t + e^{-t})}{(\frac{1}{4}e^{2t} + \frac{1}{2} + \frac{1}{4}e^{-2t})^{3/2}} \]
\[ = \frac{\frac{1}{2}(e^t + e^{-t})}{\left(\frac{1}{2}(e^t + e^{-t})^2\right)^{3/2}} \]
\[ = \frac{1}{\left(\frac{1}{2}(e^t + e^{-t})^2\right)^{3/2}} = \frac{1}{y^2} \]
8. (a) Find the Taylor series, in sigma notation, of \( f(x) = \arctan(2x^3) \).

\[
\text{Known series} \quad \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}
\]

replace \( x \) with \( 2x^3 \):

\[
\arctan(2x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (2x^3)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{6n+3}
\]

(b) For which values of \( x \) does this series converge? Be sure to check the endpoints.

The ratio of two consecutive terms is

\[
\frac{\frac{(-1)^{n+1} 2^{n+3}}{2n+3} x^{6n+9}}{\frac{(-1)^n 2^{2n+1}}{2n+1} x^{6n+3}} = \frac{2n+1}{2n+3} \frac{(-1)^n}{(-1)^{n+1}} x^6
\]

As \( n \to \infty \), this ratio approaches \( -4x^6 \).

So we have convergence when \( 1-4x^6 < 1 \), i.e. when \( |x| \leq \frac{1}{\sqrt[6]{4}} = \frac{1}{\sqrt[3]{2}} \).

At the endpoint \( x = \frac{1}{\sqrt[3]{2}} \), the series is

\[
\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} \left(\frac{1}{\sqrt[3]{2}}\right)^{6n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} \left(\frac{2}{2^{2n+1}}\right)
\]

This converges by the alternating series test.

Similarly, at \( x = -\frac{1}{\sqrt[3]{2}} \), the series is

\[
\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} \left(-\frac{1}{\sqrt[3]{2}}\right)^{6n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} (-1) \left(\frac{2}{2^{2n+1}}\right)
\]

Since \( 6n+3 \) is always odd, there is always one extra minus sign. The series is still alternating and convergent.

The interval of convergence is \( -\frac{1}{\sqrt[3]{2}} \leq x \leq \frac{1}{\sqrt[3]{2}} \).
9. The graph of \( r = \cos\left(\frac{1}{3}\theta\right) \) is shown. Find each of the four enclosed areas.

We let \( A \) be the area of one half the larger region as shown, and let \( B \) be the area of one half the smaller region.

Between \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \), \( r \) goes from 1 to \( \frac{\sqrt{3}}{2} \). This is the outer loop of the curve in the first quadrant.

So \( A + B = \int_{0}^{\pi/2} \frac{1}{2} \cos\left(\frac{1}{3}\theta\right)^2 d\theta = \int_{0}^{\pi/2} \frac{\cos\theta + 1}{4} d\theta = \frac{\sin\theta + \Theta}{4} \bigg|_{0}^{\pi/2} = \frac{1}{4} + \frac{\pi}{8} \).

Between \( \theta = \frac{\pi}{2} \) and \( \theta = \pi \), \( r \) goes from \( \frac{\sqrt{3}}{2} \) to 0. This is the inner loop of the curve in the second quadrant.

So \( B = \int_{\pi/2}^{\pi} \frac{1}{2} \cos\left(\frac{1}{3}\theta\right)^2 d\theta = \int_{\pi/2}^{\pi} \frac{\sin\theta + \Theta}{4} d\theta = \frac{\pi}{8} - \frac{1}{4} \).

Thus \( A = \frac{1}{4} + \frac{\pi}{8} - \left( \frac{\pi}{8} - \frac{1}{4} \right) = \frac{1}{2} \).

The larger regions each have area 1.

The smaller regions each have area \( \frac{\pi}{4} - \frac{1}{2} \).
10. Evaluate the following. Briefly justify your answer.

(a) \[ \lim_{n \to \infty} \frac{n^2 \sqrt{n+1}}{\sqrt{n^2 + 1}} \]

\[ = \lim_{n \to \infty} \frac{n^2 + 1}{n^2 + 1} \]

\[ = 1 \]

Since \( \frac{n^2}{n^2} \to 0 \) as \( n \to \infty \), this limit is 0.

(b) \[ \frac{5}{2} - \frac{5}{4} + \frac{5}{8} - \frac{5}{16} + \cdots \]

\[ = \frac{5}{2} \left( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots \right) \]

\[ = \frac{5}{2} \left( \frac{1}{1 - (-\frac{1}{2})} \right) \]

\[ = \frac{5}{2} \left( \frac{2}{3} \right) = \frac{5}{3} \]

(c) \[ \frac{1000}{0!} + \frac{2000}{1!} + \frac{4000}{2!} + \frac{8000}{3!} + \frac{16000}{4!} + \cdots \]

\[ = 1000 \sum_{n=0}^{\infty} \frac{2^n}{n!} = 1000 e \]
11. Give examples of series of positive terms for which:

(a) The ratio test and nth term tests are inconclusive, but the series diverges by the integral test.

In order for the nth term test to be inconclusive, we must have \( \lim_{n \to \infty} a_n = 0 \).

In order for the ratio test to be inconclusive, we must have \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \).

An example of a divergent series with these properties is \( \sum_{n=1}^{\infty} \frac{1}{n} \). This series diverges by the integral test, since \( \int \frac{1}{x} \, dx \) diverges.

(b) The ratio test is inconclusive, but the series converges by the integral test.

If we take \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), the ratio test is still inconclusive because \( \lim_{n \to \infty} \frac{(n+1)^2}{n^2} \) does not exist.

In this case, the series converges by the integral test, since \( \int \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_1^\infty = 1 \).

(c) The ratio test or the integral test could be used to show that the series converges.

With a geometric series such as \( \sum_{n=0}^{\infty} e^{-n} \) (I chose e as the base to make the integral test calculation easier), the ratio test says that \( \lim_{n \to \infty} \frac{e^{-n}}{e^{-n}} = e^0 = 1 \), so the series converges. The integral test says that \( \int \frac{1}{x} \, dx = -e^{-x} \bigg|_0^\infty = e^{-0} = 1 \), so the series converges.
12. Suppose we want to approximate \( \int_1^7 \frac{1}{x} \, dx \).

(a) Do approximation by trapezoids with \( n = 6 \). Give an error bound for your answer.

We set \( x_0 = 1 \), \( x_1 = 2 \), \ldots, \( x_6 = 7 \), so \( \Delta x = 1 \).

Then \( y_0 = 1 \), \( y_1 = \frac{1}{2} \), \ldots, \( y_6 = \frac{1}{7} \).

The approximation by trapezoids of this integral is

\[
1 \left( \frac{1}{2} \left( 1 \right) + \frac{1}{3} \left( \frac{1}{2} \right) + \frac{1}{4} \left( \frac{1}{3} \right) + \frac{1}{5} \left( \frac{1}{4} \right) + \frac{1}{6} \left( \frac{1}{5} \right) + \frac{1}{7} \left( \frac{1}{6} \right) \right) = 2.02.
\]

The error bound is

\[
\frac{M_2 \cdot 6^2}{12 \cdot 6^2} = \frac{M_2}{2}.
\]

Here \( M_2 = \max \left| f''(x) \right| = \max \frac{2}{x^3} \quad \text{This is maximized at } x = 1 \),

where \( M_2 = 2 \).

Thus, the error bound is \( \frac{1}{6} \).

(b) Do Simpson's rule with \( n = 6 \). Again, give an error bound.

The Simpson's rule approximation is

\[
1 \left( \frac{1}{3} \left( 1 \right) + \frac{4}{3} \left( \frac{1}{2} \right) + \frac{2}{3} \left( \frac{1}{3} \right) + \frac{4}{3} \left( \frac{1}{4} \right) + \frac{2}{3} \left( \frac{1}{5} \right) + \frac{4}{3} \left( \frac{1}{6} \right) + \frac{1}{3} \left( \frac{1}{7} \right) \right) = 1.96.
\]

The error bound is

\[
\frac{M_4 \cdot 6^4}{180 \cdot 6^4} = \frac{M_4}{30}.
\]

Here \( M_4 = \max \left| f^{(4)}(x) \right| = \max \frac{24}{x^5} \quad \text{This is maximized at } x = 1 \),

where \( M_4 = 24 \).

Thus, the error bound is \( \frac{24}{30} = \frac{4}{5} \).