Assignment 5: Practice Test Solutions

1. Thursday Workshop/Practice Exam:

(a) Approximate \( \int_0^\pi x^2 \sin(x) \, dx \) using Simpson’s rule with \( n = 4 \). Find an upper bound for \( M_4 \), the maximum absolute value of the fourth derivative of \( x^2 \sin(x) \). Your upper bound does not need to be the actual maximum value, but you should explain why it is greater than the maximum. Use this to bound the Simpson’s rule error with \( n = 4 \).

What value of \( n \) would you need to make the error less than 1/1000?

We will use the \( y \)-values \( y_0 = 0, y_1 = \frac{\pi^2 \sqrt{2}}{32} = 0.436, y_2 = \frac{\pi^2}{4} = 2.467, y_3 = \frac{9\pi^2 \sqrt{2}}{32} = 3.926, y_4 = 0 \). The approximation is \( \frac{\pi}{4} \left( \frac{y_0 + 4y_1 + 2y_2 + 4y_3 + y_4}{3} \right) = 5.860 \). The fourth derivative of \( x^2 \sin(x) \) is \( x^2 \sin(x) - 12x^2 \sin(x) - 8x \cos(x) \). Using the triangle inequality, \( |x^2 \sin(x) - 12x^2 \sin(x) - 8x \cos(x)| \leq |x^2| |\sin(x)| + 12 |\sin(x)| + 8|x| |\cos(x)| \leq |x^2| + 8|x| + 12 \leq \pi^2 + 8\pi + 12 = 47.002 \). Using the error formula for Simpson’s rule, \( \frac{M_4(b-a)^5}{180n^4} \), we obtain \( \frac{47.002\pi^5}{180n^4} \). When \( n = 4 \), the error is less than \( .312 \). To obtain error less than 1/1000, we would need \( n > \sqrt[4]{\frac{1000 \times 47.002 \pi^5}{180}} = 16.8 \), so \( n \) is at least 18.

(b) The sequence \( a_n = \frac{5n+1}{4n-3} \) converges to \( \frac{5}{4} \). Given \( \epsilon > 0 \), find an integer \( N \) such that if \( n > N \) then \( \left| \frac{5}{4} - a_n \right| < \epsilon \).

We want to make \( \left| \frac{5}{4} - \frac{5n+1}{4n-3} \right| < \epsilon \). Simplifying, this is \( \left| \frac{-19}{16n-12} \right| < \epsilon \), or, since the denominator is always positive, \( \frac{19}{16n-12} < \epsilon \). This is achieved when \( n > \frac{3}{4} + \frac{19}{16\epsilon} \), so let \( N = \frac{3}{4} + \frac{19}{16\epsilon} \), rounded up.

(c) Show that \( a_n = \frac{2^n}{n\pi} \) converges to 0 using the squeeze theorem.

Certainly \( a_n > 0 \). We will squeeze \( a_n \) to 0 by giving an upper bound which approaches 0. Assuming that \( n \geq 3 \) (which is fine, since convergence is a statement about \( n \) sufficiently large), \( a_n \leq \frac{2^n}{3n\pi} = \left( \frac{2}{3} \right)^n \), and we have shown in class that \( \left( \frac{2}{3} \right)^n \to 0 \). So by the squeeze theorem, \( a_n \to 0 \).

(d) Find the limits as \( n \to \infty \) of \( \frac{\sqrt{2n+3}}{2\sqrt{n}} \) and \( \frac{100n^2 \cos(n^3 + \pi)}{2n^2 \ln(n)} \). Briefly justify each answer. You do not need to give an \( \epsilon \) argument.

The first expression simplifies to \( \frac{\sqrt{2} + 3}{4\sqrt{n}} \). As \( n \to \infty \), the \( \frac{3}{4n} \) term goes to 0, leaving a limit of \( \frac{\sqrt{2}}{4} \). In the second expression, the factor of \( \cos(n^3 + \pi) \) is between \(-1\) and 1, so the whole expression is squeezed between \( \frac{50}{\ln(n)} \) and \(-\frac{50}{\ln(n)} \). Thus it approaches 0.

(e) An error bound formula for approximation by rectangles is given by \( \frac{M_1(b-a)^2}{2n} \). For approximation by trapezoids, the error bound is \( \frac{M_2(b-a)^3}{12n^2} \). For Simpson’s rule, it is \( \frac{M_4(b-a)^5}{180n^4} \). Suppose you want to approximate \( \int_1^4 \sqrt{x} \, dx \) with error less than \( 10^{-6} \). What value of \( n \) do you need to choose for each method?

The first derivative of \( \sqrt{x} \) is given by \( \frac{1}{2} x^{-1/2} \), and \( |\frac{1}{2} x^{-1/2}| \) is maximized at the left endpoint \( x = 1 \), with value \( M_1 = \frac{1}{2} \). Thus the error bound for approximation by rectangles is \( \frac{\frac{1}{2}}{2n} \). To make this less than \( 10^{-6} \), you need \( n > 2,250,000 \). The second derivative of \( \sqrt{x} \) is given by \( -\frac{1}{4} x^{-3/2} \), so \( M_2 = \frac{1}{4} \). Thus the error bound for approximation by trapezoids is \( \frac{27}{384n^2} \). To make this less than \( 10^{-6} \), you need \( n > 750 \). The fourth derivative of \( \sqrt{x} \) is given by \( -\frac{15}{16} x^{-7/2} \), so \( M_4 = \frac{15}{16} \). Thus the error bound for Simpson’s rule is \( \frac{81}{64n^4} \). To make this less than \( 10^{-6} \), you need \( n > 33.54 \).
2. Practice Exam:

(a) Approximate $\int_0^1 x^3 - 2x^2 + 3x - 4 \, dx$ by trapezoids and using Simpson’s rule with $n = 4$. Find a bound for the error if trapezoids are used. Show that the Simpson’s rule approximation is exactly right, either using the error bound formula or direct computation of the integral.

We will use the values $y_0 = -4$, $y_1 = -2$, $y_2 = 2$, $y_3 = 14$, $y_4 = 40$. The approximation by trapezoids is $\frac{4}{3}(-2) + 2 + 14 + \frac{40}{2} = 32$. The approximation by Simpson’s rule is $\frac{-4}{3} + \frac{4(-2)}{3} + \frac{2(2)}{3} + \frac{4(14)}{3} + \frac{40}{3} = 29\frac{2}{3}$. The true answer is $\frac{2}{3} - \frac{2x^3}{3} + \frac{3x^2}{2} - 4x|_0^1 = 29\frac{2}{3}$. We could also see that Simpson’s rule is exactly right in this case because the error bound has a factor of $M_4$, which will be zero for any degree 3 polynomial.

(b) Use the squeeze theorem to find $\lim_{n \to \infty} \frac{3n^2 + (-1)^n \sin(\pi n/6)\sqrt{n}}{2n^2}$.

This sequence can be rewritten as $\frac{3}{2} + \frac{(-1)^n \sin(\pi n/6)}{2n^{3/2}}$. Since $(-1)^n \sin(\pi n/6)$ is between $-1$ and 1, the sequence is squeezed between $\frac{3}{2} - \frac{1}{2n^{3/2}}$ and $\frac{3}{2} + \frac{1}{2n^{3/2}}$, so it has a limit of $\frac{3}{2}$.

(c) Find a formula for the sum $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}$. These sums approach $\frac{1}{2}$ as $n \to \infty$. How large must you take $n$ for the sum to be within distance $\epsilon$ of $\frac{1}{2}$?

The first few terms of this sequence are $\frac{1}{3}$, $\frac{4}{9}$, $\frac{13}{27}$, $\frac{40}{81}$. The general term is $\frac{3^n - 1}{2 \cdot 3^n}$. We want to make $|\frac{1}{2} - \frac{3^n - 1}{2 \cdot 3^n}| < \epsilon$. This simplifies to $\frac{1}{2 \cdot 3^n} < \epsilon$, which holds if $n > \log_3(\frac{1}{2\epsilon})$.

(d) The sequence $a_n = \frac{10n}{n^2 - 3n + 3}$ converges to 0. For any $\epsilon > 0$, explain how to find $N$ such that if $n > N$ then $|a_n| < \epsilon$.

Given $\epsilon > 0$, we want to make $|\frac{10n}{n^2 - 3n + 3}| < \epsilon$. Since the numerator and denominator are always positive, we can remove the absolute value and show that $\frac{10n}{n^2 - 3n + 3} < \epsilon$. Rearranging this, we just need to show that $3n + \frac{10n}{\epsilon} - 3 < n^2$ for large enough $n$. Since $3n + \frac{10n}{\epsilon} - 3 < 3n + \frac{10n}{\epsilon}$, it suffices to show that $3n + \frac{10n}{\epsilon} < n^2$. This is true for $n > 3 + \frac{10}{\epsilon}$. Choose $N$ as $3 + \frac{10}{\epsilon}$, rounded up to an integer.

(e) No proof required. Find the limits as $n \to \infty$ of $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{n})$ and of $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{n})$. The terms of the first sequence simplify to $\frac{1}{n}$, so the limit is 0. The terms of the second sequence simplify to $\frac{n+1}{2n}$, so the limit is $\frac{1}{2}$.