Euclidean Algorithm and Continued Fractions Practice

1. Use Euclid’s algorithm to write continued fractions for $23/17$, $180/139$, and $659/208$.

2. Does this still work if the fraction is not in lowest terms? Try replacing $23/17$ with $46/34$.

Not-Quite Uniqueness

3. Write down two different continued fractions for the number 1. Do the same for the numbers $1/2$ and $23/17$.

4. Suppose $[a_0, a_1, ...a_m] = [b_0, b_1, ...b_n]$ and $a_0 = b_0$. Prove that $[a_1, a_2, ...a_m] = [b_1, b_2, ...b_n]$.

5. Prove by induction: if $a_1, a_2, ... a_m$ are positive integers, then:

$$
\frac{1}{a_1} + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{... + \frac{1}{a_m}}}} > 0
$$

6. Deduce from problem 5 that

$$
\frac{1}{a_1} + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{... + \frac{1}{a_m}}}} \leq 1
$$

with equality if and only if the continued fraction is $\frac{1}{a_1}$ and $a_1 = 1$.

7. Suppose you have two distinct but equal continued fractions $[a_0, a_1, a_2, ...a_m] = [b_0, b_1, b_2, ...b_n]$. By repeated application of problem 4, this implies that $[a_k, a_{k+1}, a_{k+2}, ...a_m] = [b_k, b_{k+1}, b_{k+2}, ...b_n]$ where $k$ is the first entry with $a_k \neq b_k$. Suppose $a_k < b_k$. We know both are integers. What can you conclude about

$$
\frac{1}{a_k + \frac{1}{a_{k+1} + \frac{1}{a_{k+2} + \frac{1}{... + \frac{1}{a_m}}}}} \quad \text{and} \quad \frac{1}{b_{k+1} + \frac{1}{b_{k+2} + \frac{1}{... + \frac{1}{b_n}}}}
$$

Prove that every rational number has exactly two different continued fractions and describe them.

Above and Beyond

8. (Fibonacci Ratios) The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$ is defined by a recursive formula: $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$. Use the Euclidean algorithm to write continued fractions for ratios of consecutive Fibonacci numbers, e.g. $5/3, 8/5, 13/8$ etc. What is going on here?
9. Write a calculator or computer program that takes two integer inputs, performs the Euclidean algorithm, and outputs the greatest common divisor and the continued fraction.

10. *(Convergents)* Given a continued fraction \([a_0, a_1, a_2, \ldots a_m]\), the numbers \([a_0]\), \([a_0, a_1]\), \([a_0, a_1, a_2]\), \ldots \([a_0, a_1, a_{m-1}]\) are called convergents. Compute a list of convergents for each of the fractions in problem 1. What patterns do you see in the convergents?

11. Suppose we remove the requirement that \(a_1, a_2, \ldots a_n\) be positive, and allow them to be any integers. Write down many different continued fractions for the numbers 1, 1/2, and 23/17. Is there some way to describe all the continued fractions for a given number?