Examples

1. Use the Euclidean algorithm and the "magic box" to write down the continued fraction of 2011/1776, find all convergents, and solve the linear Diophantine equation 2011x + 1776y = 1.

2. Write down the first five convergents for √2, √3, √5, and π.

Big Questions

3. (The Fibonacci Sequence) Remember that the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13,... is defined by \( F_0 = F_1 = 1 \), and the recursive formula \( F_{n+1} = F_n + F_{n-1} \).

   a) If you haven’t done it before, write down continued fractions for 2/1, 3/2, 5/3, 8/5, 13/8, etc. until you see the pattern. What is the continued fraction for \( F_n/F_{n-1} \)?

   b) Write down an infinite continued fraction for the limit of the Fibonacci ratios \( \lim_{n \to \infty} F_n/F_{n-1} \). We call this number \( \phi \) (phi).

   c) Using nothing but the continued fraction for \( \phi \) and a little cleverness, prove that \( \phi^2 = \phi + 1 \) and solve for \( \phi \).

   d) Using \( \phi^2 = \phi + 1 \), expand \( \phi^3, \phi^4, \phi^5, \) etc. until you can write down a formula for \( \phi^n \) (as an integer plus an integer multiple of \( \phi \)).

   e) Let \( \bar{\phi} \) be the other solution to \( x^2 = x + 1 \). What is \( \phi - \bar{\phi} \)? What is \( \phi^n - \bar{\phi}^n \)?

   f) Write down an explicit, non-recursive formula for \( F_n \).

   g) Some generalizations: the Lucas numbers 2, 1, 3, 4, 7, 11, 18,... are defined just like the Fibonacci numbers except that \( L_0 = 2 \). Prove that Lucas ratios tend toward the same limit \( \phi \) and find a formula for \( L_n \). You could also change the recursive formula, for example 1, 2, 5, 12, 29, 70, 169,... is given by \( G_{n+1} = 2G_n + G_{n-1} \) and 1, 1, 2, 4, 7, 13, 24, 44, 81,... is given by \( H_{n+1} = H_n + H_{n-1} + H_{n-2} \). Can you find formulas for these?

4. (Music) Pythagoras and his followers experimented with the sound of vibrating strings of various lengths. They discovered that musical notes sound harmonious together if their frequencies are related by simple integer ratios. For example, notes an octave apart have a ratio of 2 : 1. On a modern piano, this means that each "half-step" between adjacent keys represents a ratio of \( \sqrt[12]{2} : 1 \). Write out the first four convergents of \( 2^{1/12}, 2^{2/12}, 2^{3/12}, 2^{4/12}, 2^{5/12}, 2^{6/12}, 2^{7/12}, 2^{8/12}, 2^{9/12}, 2^{10/12}, \) and \( 2^{11/12} \). Which notes would you expect to sound best together?

5. (For programmers) Write a calculator or computer program (or update your program from two weeks ago) to take two integer inputs \( p \) and \( q \), perform the Euclidean algorithm and the "magic box", and output the continued fraction along with a complete list of convergents.
6. (For those who know calculus) This exercise is adapted from a paper by Henry Cohn. You will prove that the continued fraction of e is \([2 : 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots] \) or more elegantly \([1 : 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots] \)

Let \(p_n/q_n\) represent the nth convergent of this continued fraction. You must show that \(\lim_{n \to \infty} p_n - q_n e = 0\).

Define the following quantities:

\[
A_n = -\int_0^1 \frac{x^n(x - 1)^n}{n!} e^x \, dx
\]
\[
B_n = \int_0^1 \frac{x^{n+1}(x - 1)^n}{n!} e^x \, dx
\]
\[
C_n = \int_0^1 \frac{x^n(x - 1)^{n+1}}{n!} e^x \, dx
\]

Claim: \(A_n = p_{3n} - q_{3n} e, \ B_n = p_{3n+1} - q_{3n+1} e, \) and \(C_n = p_{3n+2} - q_{3n+2} e\).

a) The first three convergents are \(p_0/q_0 = 1/1, \ p_1/q_1 = 1/0\) (don’t worry about it), and \(p_0/q_0 = 2/1\). Prove three base cases of the claim:

\[
A_0 = p_0 - q_0 e
\]
\[
B_0 = p_1 - q_1 e
\]
\[
C_0 = p_2 - q_2 e
\]

b) Now prove recursive formulas to verify the claim for all \(n\):

\[
A_n = C_{n-1} + B_{n-1}
\]
\[
B_n = 2nA_n + C_{n-1}
\]
\[
C_n = B_n + A_n
\]

(Hint: the third formula is easy. For the first two, use integration by parts.)

c) Finally, prove that \(A_n, \ B_n, \) and \(C_n \to 0\) as \(n \to \infty\).