Pure Periods

1. Calculate the continued fraction form of \( \sqrt{11} \) and \( 3 + \sqrt{11} \).

2. What are the numbers \([1, 2, 3, 4]\) and \([4, 3, 2, 1]\)? Compute all the convergents.

3. Find a quadratic equation \( ax^2 + bx + c = 0 \) with integer coefficients \( a, b, c \) such that \( x = [1, 2, 3, 4] \) is a solution. What is the other solution? Do the same for \([4, 3, 2, 1]\).

4. For the convergents \( \frac{p_0}{q_0}, \ldots, \frac{p_3}{q_3} \) of \([1, 2, 3, 4]\) compute \( \frac{p_k}{p_{k-1}} \). Are the numbers familiar?

5. For positive integers \( a_0, \ldots, a_n \), let \( x = [a_0, \ldots, a_n] \).
   a) Try to express \( x \) in a finite “continued fraction” form \( x = [b_0, \ldots, b_n, b_{n+1}] \), where \( b_0, \ldots, b_n \) are integers, but \( b_{n+1} \) can be anything.
   b) Find positive integers \( a, b, c \) such that \( ax^2 + bx + c = 0 \).
   c) What can you say about the size of \( x \) and \( -\frac{1}{x} \)?

6. Prove that if \( \frac{p_k}{q_k} \) are the convergents of \([a_0, \ldots, a_n]\), and \( \frac{p'_k}{q'_k} \) are the convergents of \([a_n, \ldots, a_0]\), then \( \frac{p_n}{q_n-1} = \frac{p'_n}{q'_n} \) and \( \frac{q_n}{q_n-1} = \frac{p'_n}{q'_n} \). (Hint: first try to compute the continued fraction form of \( \frac{p_n}{q_n} \) using only the recursive formula.)

7. Using the previous problem, what can you conclude about \( p_n, p_{n-1}, p'_n, q'_n \)? (Hint: remember that the recursive formula for convergents produces them in a simplified form.)

8. Using problems 5 and 7, look at the quadratic equations of which \([a_0, \ldots, a_n]\) and \([a_n, \ldots, a_0]\) are roots. What is the relationship between these?

Theorem
If \( a_0, \ldots, a_n \), are positive integers, then the purely periodic continued fraction

\[ \alpha = [a_0, \ldots, a_n] \]

is greater than 1 and is the root of a quadratic equation with integer coefficients. The same polynomial has an other root, namely \( \alpha' = -\frac{1}{\beta} \), where \( 0 > \alpha' > -1 \) and

\[ \beta = [a_n, \ldots, a_0], \]

the continued fraction with the period reversed.

9. What can you say about continued fractions that are periodic, but not purely periodic; i. e. have the form \([a_0, \ldots, a_k, a_{k+1}, \ldots, a_{k+l}]\)?
**Quadratic Irrationals**

In the following problems, we examine properties of numbers of the form \(A + B\sqrt{D}\), where \(A\) and \(B\) are rational numbers, and \(D\) is a positive integer that is not a perfect square. These are called quadratic irrationals. The conjugate of a quadratic irrational \(\alpha = A + B\sqrt{D}\) is \(\alpha' = A - B\sqrt{D}\). Notice that \(\alpha\) is irrational exactly if \(B \neq 0\). The quadratic irrational \(\alpha\) is called reduced if \(\alpha > 1\) and \(0 > \alpha' > -1\).

10. If \(\alpha = A_1 + B_1\sqrt{D}\) and \(\beta = A_1 + B_1\sqrt{D}\) are quadratic irrationals, prove that \(\alpha + \beta, \alpha \cdot \beta\) and \(\alpha / \beta\) are also quadratic irrationals.

11. Prove that \(\alpha = A + B\sqrt{D}\) is the solution of a quadratic equation \(ax^2 + bx + c = 0\), with integer coefficients. Express the coefficients \(a, b, c\) in terms of \(A, B\) and \(D\). Show that the other root of the same equation is \(\alpha'\).

12. Prove that if \(\alpha_i\) is a quadratic irrational and \(\alpha_i'\) is its conjugate \((i = 1, 2)\), then

\[
\begin{align*}
(\alpha_1 + \alpha_2)' &= \alpha_1' + \alpha_2' \\
(\alpha_1 - \alpha_2)' &= \alpha_1' - \alpha_2' \\
(\alpha_1 \cdot \alpha_2)' &= \alpha_1' \cdot \alpha_2' \\
\left(\frac{\alpha_1}{\alpha_2}\right)' &= \frac{\alpha_1'}{\alpha_2'}
\end{align*}
\]

13. Consider the equation \(ax^2 + bx + c\) with \(a, b, c\) integers. The solutions of this can be written as \(P \pm \sqrt{D} / Q\), where \(P, Q, D\) are integers. Express \(P, Q, D\) in terms of \(a, b, c\), and try to give necessary and sufficient conditions on \(P, Q\) and \(D\) for \(P + \sqrt{D} / Q\) to be a reduced quadratic irrational.

14. Conclude from the previous problem that for any given \(D\) there are only a finite number of reduced quadratic irrationals of the form \(P + \sqrt{D} / Q\).

15. Prove that if \(\alpha\) is a reduced quadratic irrational, and \(a_0\) is the largest integer less than \(\alpha\), then \(\alpha = a_0 + 1 / \alpha_1\), where \(\alpha_1\) is also a reduced quadratic irrational. What is the relationship between the irrational parts of \(\alpha\) and \(\alpha_1\)?

**Theorem**

If \(\alpha\) is a reduced quadratic irrational, so that \(\alpha > 1\) is the root of a quadratic equation with integer coefficients whose conjugate root \(\alpha'\) lies between \(-1\) and \(0\), then the continued fraction form of \(\alpha\) is purely periodic.