1) The area of the large rectangle is \((a+c)(b+d) = ab + ad + bc + cd\).
Hence the area of the central parallelogram is 
\[(ab + ad + bc + cd) - (2 \cdot \frac{ab}{2} - 2 \cdot \frac{cd}{2} \cdot 2 \cdot bc) = ad - bc\]
The area of the triangle \((0, 0), (ab), (c, d)\) is \(\frac{1}{2} (ad - bc)\).
This diagram assumes that \(a, b, c, d,\) and \(ad - bc\) are all \(> 0\).
However, similar arguments will cover all other cases and the general formula for the area is \(\frac{1}{2} |ad - bc|\).

2) The area is \(13 + \frac{23}{2} - 1 = \frac{47}{2}\)
3) Suppose \( \frac{a}{b} < \frac{c}{d} < \frac{e}{f} \) are consecutive in some Farey sequence.

Assuming Property I, we know

\[
bc - ad = 1 = de - cf
\]

\[
bc + cf = ad + de
\]

\[
c(b + f) = d(a + e)
\]

\[
\frac{c}{d} = \frac{a + e}{b + f}
\]

This is Property II.

4) Assume Property II: If \( \frac{a}{b} < \frac{c}{d} < \frac{e}{f} \) are consecutive, then \( \frac{c}{d} = \frac{a + e}{b + f} \).

Doing the algebra of problem 3 backwards proves that \( bc - ad = de - cf \); all we need is to show that these quantities are equal to 1.

Notice that the first pair of consecutive terms in \( F_n \) is always \( \frac{0}{1} < \frac{1}{n} \), with \( 1 - 0 = n \). This implies that the next pair \( \frac{1}{n} < \frac{1}{n+1} \) has \( n \cdot 1 - (n+1) = 1 \) also, since \( bc - ad = de - cf \) for all triples of consecutive terms. This in turn implies that the next pair \( \frac{1}{n+1} < \frac{1}{n+2} \) has \( (n+1) \cdot 1 - (n+2) = 1 \).

Proceeding inductively on pairs in this way, we see that all pairs \( \frac{a}{b} < \frac{c}{d} \) of consecutive terms have \( bc - ad = 1 \).
Since Property I implies Property II, as in problem 3, we may focus on proving Property I by induction on $n$.

**Base Case:** $F_1$, which is simply $\frac{0}{1}$, clearly has Property I.

**Inductive Step:** Suppose that $F_{n-1}$ has property I, and we must add in the fractions with denominator $n$.

Let $\frac{a}{b} < \frac{c}{d}$ be consecutive in $F_{n-1}$, but let $\frac{m}{n}$ lie between them in $F_n$.

Let $\frac{a}{b} < \frac{m}{n} < \frac{c}{d}$. By inductive hypothesis, $bc - ad = 1$. We would like to prove that $\frac{m}{n}$ is the only term in $F_n$ between $\frac{a}{b}$ and $\frac{c}{d}$, and that $bn - an = cn - dm = 1$.

$$\frac{1}{bd} = \frac{c - a}{d} = \left(\frac{c}{d} - \frac{m}{n}\right) + \left(\frac{m}{n} - \frac{a}{b}\right)$$

Notice that $\frac{c}{d} - \frac{m}{n} = \frac{cn - dm}{dn} \geq \frac{1}{dn}$ since $cn - dm$ is a positive integer.

Similarly $\frac{m}{n} - \frac{a}{b} = \frac{bm - an}{bn} \geq \frac{1}{bn}$.

So

$$\left(\frac{c}{d} - \frac{m}{n}\right) + \left(\frac{m}{n} - \frac{a}{b}\right) \geq \frac{1}{dn} + \frac{1}{bn} = \frac{b + d}{bdn}$$

We now have $\frac{1}{bd} \geq \frac{b + d}{bdn}$, or equivalently $n \geq b + d$.

However, if $n > b + d$, then there would be an additional term in $F_{n-1}$, $\frac{a + c}{b + d}$, between $\frac{a}{b}$ and $\frac{c}{d}$. But we chose $\frac{a}{b}, \frac{c}{d}$ consecutive.

Hence $n = b + d$.

$$\frac{1}{bd} = \frac{c - a}{d} = \left(\frac{c}{d} - \frac{m}{n}\right) + \left(\frac{m}{n} - \frac{a}{b}\right) \geq \frac{1}{dn} + \frac{1}{bn} = \frac{b + d}{bdn} = \frac{1}{bd}$$

The inequality $\left(\frac{c}{d} - \frac{m}{n}\right) + \left(\frac{m}{n} - \frac{a}{b}\right) \geq \frac{1}{dn} + \frac{1}{bn}$ is actually an equality, so $cn - dm = bn - an = 1$.

We also have $n = b + d$, and substituting this in $cn - dm = bn - an$ shows $m = a + c$, so $\frac{m}{n}$ is uniquely determined by its neighbors.
6) Ask in class!

7) \( \log_2 \frac{3}{2} = [0; 1, 1, 2, 2, 3, ...] \)

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<th>2</th>
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The well-tempered scales which contain the closest approximations to a perfect fifth are the denominators above. A twelve-tone scale contains a very close approximation at 7 steps. 5-tone or 41-tone scales would also work, but they are impractical for various reasons.
8) \( x^2 - y^2 = 1 \) has no nontrivial solutions (the only consecutive squares are 0 and 1)
\( x^2 - 2y^2 = 1 \) has nontrivial solutions, e.g. \( x = 3, \ y = 2 \)
\( x^2 - 3y^2 = 1 \) has nontrivial solutions, e.g. \( x = 2, \ y = 1 \)
\( x^2 - 4y^2 = x^2 - (2y)^2 = 1 \) has no nontrivial solutions
\( x^2 - 5y^2 = 1 \) has nontrivial solutions, e.g. \( x = 9, \ y = 4 \)
\( x^2 - 6y^2 = 1 \) has nontrivial solutions, e.g. \( x = 5, \ y = 2 \)
\( x^2 - 7y^2 = 1 \) has nontrivial solutions, e.g. \( x = 8, \ y = 3 \)
\( x^2 - 8y^2 = 1 \) has nontrivial solutions, e.g. \( x = 3, \ y = 1 \)
\( x^2 - 4y^2 = x^2 - (3y)^2 = 1 \) has no nontrivial solutions

The right guess to make is \( x^2 - Dy^2 \) has nontrivial solutions as long as \( D \) is not a perfect square.

9) Ask in class!