Toolbox

If \( a_0, a_1, a_2, \ldots \) are convergents of a continued fraction, and

\[
c_n = \frac{p_n}{q_n} = [a_0 : a_1, a_2, \ldots, a_n]
\]

is the \( n \)th convergent, then

\[
p_0 = a_0, \quad p_1 = a_1a_0 + 1, \quad p_k = a_k \cdot p_{k-1} + p_{k-2};
\]
\[
q_0 = 1, \quad q_1 = a_1, \quad q_k = a_k \cdot q_{k-1} + q_{k-2}.
\]

Moreover,

\[
c_{k+1} - c_k = \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{p_{k+1}q_k - p_kq_{k+1}}{q_kq_{k+1}} = \frac{(-1)^k}{q_kq_{k+1}}
\]

hence the successive convergents of a number \( r \) look like this:

\[
\begin{array}{cccc}
   \bullet & c_0 & \bullet & \bullet & \bullet & c_2 & \bullet & c_4 & \bullet & c_3 & \bullet & c_1 & \bullet \\
   r & & & & & & & & & & & &
\end{array}
\]

Recall moreover that The Farey Sequence \( F_n \) is the list of all fractions between 0 and 1 in lowest terms with denominator \( \leq n \) in size order. These have the following two properties:

Property I: If \( \frac{a}{b} < \frac{c}{d} \) are consecutive in some Farey sequence \( F_n \), then \( bc - ad = 1 \).

Property II: If \( \frac{a}{b} < \frac{c}{d} < \frac{e}{f} \) are all consecutive, then \( \frac{c}{d} = \frac{a+e}{b+f} \).

The goal: Dirichlet’s Approximation Theorem

1. Prove that for any number \( \zeta \), there exists a rational number \( \frac{p}{q} \) such that \( \left| \frac{p}{q} - \zeta \right| < \frac{1}{q^2} \). Can you say more than that? Is there more than one such rational number? Are there infinitely many?

2. Using the results and the picture in the Toolbox above, what can you say about the approximations the convergents \( c_k \) give for a number \( r \)? How many “good” approximations of \( r \) can we obtain this way?

3. For any \( \zeta \) between 0 and 1, one can obtain approximations of \( \zeta \) by picking (one of) the Farey-neighbors of \( \zeta \) from each Farey-sequence. How well do these numbers approximate? How many “good” approximations of \( \zeta \) can we obtain this way?

4. Prove that if \( r \in \mathbb{Q} \) is rational, then there are only finitely many rational numbers \( \frac{p}{q} \) such that \( \left| \frac{p}{q} - r \right| < \frac{1}{q^2} \).

5. Prove that the Farey neighbors are the “only good approximations”, that is, if \( \left| \frac{p}{q} - \zeta \right| < \frac{1}{q^2} \), then there is an \( n \) such that \( \frac{p}{q} \in F_n \), and \( F_n \) has no elements between \( \frac{p}{q} \) and \( \zeta \).

6. Prove that the convergents of the continued fraction form of \( \zeta \) are the “only good approximations”, that is, if \( \left| \frac{p}{q} - \zeta \right| < \frac{1}{q^2} \), than \( \frac{p}{q} \) is a convergent of the continued fraction of \( \zeta \).
Common Sense and Pigeons

7. Prove that in a class of 40 people there is always at least two who have their birthday on the same day of a (possibly different) month.

8. There are 12 pairs of socks in a very untidy sock drawer, all of a different colour, all separated from their pairs. We would like to take three pairs of (matching) socks for a trip, but we are in a rush, so we just grab some socks and hurry away. At least how many socks should we grab to make sure we have three matching pairs?

9. Show that in a group of five people, there are two who have the same number of friends within the group. (Friendships are mutual.) What if there are more than five people in the group?

10. Show that the Fibonacci-sequence has an element that is divisible by 2011.

11. The points of the plane are coloured with red and blue. Show that there are two points, exactly one inch apart, that have the same colour. What if the points can be red, blue or green?

12. Prove that among 2011 integers there is always some of which the sum is divisible by 2011.

Dirichlet, once more

13. Prove that for any real number $\zeta$ and integer $n \geq 1$, there exist integers $a$ and $b$ such that $1 \leq a \leq n$ and

$$|a\zeta - b| < \frac{1}{n+1}.$$ 

14. Prove Dirichlet’s Approximation Theorem using the previous problem.