1 Introduction

My research aims to attack longstanding problems in analytic number theory with cutting-edge techniques in the representation theory of Kac-Moody groups. Number theorists have used symmetry since Dirichlet’s theorem on primes in arithmetic progressions; in the theory of automorphic forms, symmetry is encoded in finite-dimensional algebraic groups. There has long been speculation about how infinite-dimensional Kac-Moody groups might play a role. Now the infinite-dimensional theory is beginning to arrive. My work is primarily in Kac-Moody Weyl group multiple Dirichlet series. These are functions of several complex variables with infinitely many hyperplanes of symmetry. Intricate and beautiful, they contain rich representation-theoretic data, and unexplored possibilities for number theory.

One of the great achievements of representation theory is the classification of semisimple Lie groups by Dynkin diagrams or Cartan matrices–one begins with a group and a maximal torus, then passes to a Lie algebra with a Cartan subalgebra, and finally to an abstract system of Chevalley generators with relations encoded in the Cartan matrix. This process can be reversed, so that we begin with a Cartan matrix and produce a group. The group is finite-dimensional if and only if the matrix is positive-definite; otherwise it is an infinite-dimensional Kac-Moody group. The last decade has seen automorphic constructions on infinite-dimensional groups including Eisenstein series [18, 19, 20], Hecke algebras [4, 5, 21, 1], the Gindikin-Karpelevich formula [3], and the Casselman-Shalika formula [30]. These are all carried out for affine Kac-Moody groups, the most approachable case. In my Ph.D. thesis, I construct very particular automorphic objects: Kac-Moody multiple Dirichlet series. Beginning with three natural axioms based on finite field algebraic geometry, I prove the following [37]:

Theorem 1. For each simply-laced affine Kac-Moody group $G$ of rank $r$, there exists a unique multivariable Dirichlet series $Z(s_1, \ldots, s_r)$ satisfying the axioms. It has meromorphic continuation to a half-space and functional equations corresponding to the Weyl group $W$ of $G$.

This leads to some of the first direct applications of Kac-Moody theory to number theory. Diaconu and Pasol have recently proven the same result for another class of Kac-Moody groups going beyond affine [15]. Remarkably, the axioms take the Cartan matrix as an input, and do not distinguish between finite and Kac-Moody types. We hope that this method will lead the way toward general automorphic Kac-Moody theory.

As a postdoc, I’ve been led from analytic number theory toward infinite-dimensional representation theory, automorphic forms, and algebraic geometry. Much of my research has become experimental and computational, using Sage or Mathematica to explore finite field algebraic geometry and generating function combinatorics. Because my research is in a very new area, there is still low-hanging fruit for an interested undergraduate researcher—I will describe below how an undergraduate could use my work to conjecture a multivariable analogue of the Riemann hypothesis. At the same time, my work lies at an emerging nexus of research, and raises questions that will last for years to come.
2 Multiple Dirichlet Series and the Axiomatic Approach

Let $\Gamma$ be a finite graph with vertices labeled 1 to $r$. Write $i \sim j$ for adjacent vertices. Then a multiple Dirichlet series in variables $s_1, \ldots, s_r \in \mathbb{C}$ can be defined, very roughly, as

$$Z(s_1, \ldots, s_r) = \sum_{m_1, \ldots, m_r \in \mathbb{N}} \prod_{i \sim j} \left( \frac{m_i}{m_j} \right) m_1^{-s_1} \cdots m_r^{-s_r}$$

where $\left( \frac{z}{m} \right)$ denotes the quadratic residue symbol. When $\Gamma$ is just a single vertex, there are no residue symbols, and $Z(s_1) = \zeta(s_1)$, the Riemann zeta function. Can we generalize the famous properties of the zeta function—the Euler product over primes, the meromorphic continuation and functional equation—to all multiple Dirichlet series $Z$?

When $\Gamma$ is the $A_2$ Dynkin diagram, with two adjacent vertices, $Z$ takes the form

$$Z(s_1, s_2) = \sum_{m_1, m_2} \left( \frac{m_1}{m_2} \right) m_1^{-s_1} m_2^{-s_2} = \sum_{m_2} L(s_1, \chi_{m_2}) m_2^{-s_2}$$

where, in the latter formula, we have expressed the sum over $m_1$ as a Dirichlet L-function with quadratic character $\chi_{m_2}(\cdot) = \left( \frac{\cdot}{m_2} \right)$. Goldfeld and Hoffstein analyze this series, and used it to give asymptotics for the average value of $L(s, \chi_{m_2})$ [22]. From its expression in terms of $L(s_1, \chi_{m_2})$, $Z$ inherits a functional equation in $s_1 \mapsto 1 - s_1, s_2 \mapsto s_2 + s_1 - \frac{1}{2}$. But by quadratic reciprocity, $Z$ can also be written in terms of $L(s_2, \chi_{m_1})$, yielding a functional equation in $s_2 \mapsto 1 - s_2, s_1 \mapsto s_1 + s_2 - \frac{1}{2}$. Together, these generate $S_3$, the Weyl group of $A_2$. In fact the group of functional equations is always the Weyl group of the diagram, and something much deeper is true: the multiple Dirichlet series is a Whittaker function for an Eisenstein series on a metaplectic cover of the algebraic group defined by $\Gamma$.

When $\Gamma$ is the “star-shaped” graph:

with $r + 1$ vertices, the series $Z$ can be written as

$$Z(s_1, \ldots, s_{r+1}) = \sum_{m_{r+1}} L(s_1, \chi_{m_{r+1}}) \cdots L(s_r, \chi_{m_{r+1}}) m_{r+1}^{-s_{r+1}}.$$

This has special importance for analytic number theory: it generates the $r$th moment of quadratic Dirichlet L-functions. Quadratic L-functions are a symplectic family in the sense of Katz and Sarnak [24]; there is great interest in their distribution, and several methods of computing statistics. The first three moments were computed by Jutila and Soundarajan [23, 33] using approximate functional equations and related techniques, and by Diaconu, Goldfeld, and Hoffstein [22, 14] using multiple Dirichlet series. The fourth moment was computed by Soundarajan and Young, assuming the generalized Riemann hypothesis [34], and by Bucur and Diaconu [8] over the rational function field using multiple Dirichlet series. The fifth and higher moments are completely open. All the competing strategies seem to reach the same obstacle. For multiple Dirichlet series, this
obstacle is the transition to Kac-Moody groups: the first three moments correspond to root systems $A_2, A_3, D_4$, the fourth corresponds to the affine root system $\tilde{D}_4$, and higher moments correspond to more general Kac-Moody root systems.

The problem is that infinitely many functional equations imply infinitely many poles; moreover, the poles accumulate at essential singularities along the boundary of the Tits cone. The same phenomenon occurs in the Kac-Moody Eisenstein series constructed by Garland [18]. Meromorphic continuation past this boundary is likely impossible, but any continuation toward the boundary has number theoretic significance. Moreover, Kac-Moody root systems include imaginary roots, whose subtle and interesting combinatorial properties are reflected in the multiple Dirichlet series. In my thesis, I make the definition of (1) precise when $\Gamma$ is a simply-laced affine Dynkin diagram:

Furthermore, I show that the multiple Dirichlet series are uniquely determined by four natural axioms concerning their coefficients. The axioms, which arise from considering the coefficients as point counts on varieties in the function field case, seem to have little to do with L-functions and their functional equations. But I have shown that they imply the desired interpretation in terms of L-functions, and the desired Coxeter group of symmetries, for all graphs $\Gamma$. The remaining issue is meromorphic continuation: over the rational function field $\mathbb{F}_q(t)$, I obtain meromorphic continuation to the boundary in type $\tilde{A}$ [35], and a nontrivial meromorphic continuation in all types. The project of meromorphic continuation to the boundary for other function fields and other affine types is ongoing.

Several steps are required to make formula (1) more than a heuristic. It is best to replace $\prod_{i \sim j} \left( \frac{n_i}{n_j} \right)$ with a function $H(n_1, \ldots, n_r)$. The essential property of $H$ is called twisted multiplicativity: if $\gcd(n_1 \cdots n_r, n'_1 \cdots n'_r) = 1$, then

$$H(n_1n'_1, \ldots, n_rn'_r) = H(n_1, \ldots, n_r)H(n'_1, \ldots, n'_r) \prod_{i \sim j} \left( \frac{n'_i}{n'_j} \right).$$

(4)

The multiple Dirichlet series is not an Euler product, but its coefficients are multiplicative up to signs, which come from the quadratic characters. We then define the $p$-part

$$Z_p(s_1, \ldots, s_r) = \sum_{a_1, \ldots, a_r \geq 0} H(p^{a_1}, \ldots, p^{a_r})|p|^{-a_1s_1-\cdots-a_rs_r}.$$

(5)

The theory of these $p$-parts is largely complete when $\Gamma$ is the Dynkin diagram of a finite-dimensional group. Combinatorial constructions appeared simultaneously in papers of Brubaker, Bump and Friedberg [6, 7], and Chinta and Gunnells [11, 12]. In this case, $Z_p$ is a rational function. It can be described as an average over the Weyl group or a sum over a highest-weight crystal. $Z_p$ is the Whittaker function of a local metaplectic representation, and formulas for $Z_p$ are metaplectic analogues of the Casselman-Shalika formula [9]. This fact was proven first in types $A$ [6, 13] and $B$ [17]. A proof for all finite root systems was recently given by McNamara [29]. The literature has focused on the local theory; important questions about the global object $Z$ remain open.

The axiomatic method, which is developed in my work and in a preprint of Diaconu and Pasol [15], represents a new approach to the construction of $Z_p$. It is geometric rather than
combinatorial in origin, and extends beyond the finite cases. Since the Casselman-Shalika formula and its generalizations carry over unchanged to function fields, it suffices to take the base field as \( \mathbb{F}_q(t) \). Then the multiple Dirichlet series is a power series:

\[
Z(s_1, \ldots, s_r) = \sum_{a_1, \ldots, a_r \geq 0} c_{a_1, \ldots, a_r}(q) q^{-a_1 s_1 - \cdots - a_r s_r}
\]  

with coefficients

\[
c_{a_1, \ldots, a_r}(q) = \sum_{f_i \in \mathbb{F}_q[t], \deg(f_i) = a_i} H(f_1, \ldots, f_r).
\]

These coefficients can be interpreted as statistics relating to point counts on curves. For example, when \( Z \) is the \( r \)th moment multiple Dirichlet series of \( (3) \), the coefficient \( c_{1, \ldots, 1, 3} \) gives the \( r \)th moment of the trace of Frobenius on elliptic curves \( y^2 = f(x) \), with \( \deg(f) = 3 \) [8, 2].

When \( Z \) is the series of \( (3) \), Diaconu and Pasol interpret its coefficients as cohomological invariants of the moduli of hyperelliptic curves of given genus. There are two fundamental properties:

- first, a Poincaré duality statement relating local coefficients to global:

\[
H(p^{a_1}, \ldots, p^{a_r}) = |p|^{a_1 + \cdots + a_r} c_{a_1, \ldots, a_r}(1/|p|)
\]

and, second, a cohomological purity condition: \( c_{a_1, \ldots, a_r}(q) \) has nonzero terms only in degrees \( (a_1 + \cdots + a_r)/2 \) to \( a_1 + \cdots + a_r \), so the degree of \( H(p^{a_1}, \ldots, p^{a_r}) \) is less than \( (a_1 + \cdots + a_r)/2 \). These properties, along with \( (4) \), are the axioms. They make sense when the coefficients are polynomials, or, more generally, \( q \)-Weil numbers, such as the Fourier coefficients of modular forms.

Much of my work has taken these axioms as a starting point. I have shown that in all cases, even where no geometric interpretation is currently available, the axioms imply the expected connection to \( L \)-functions, and the expected group of functional equations. For finite root systems, I show that the axiomatic construction recovers the same \( p \)-parts constructed by Chinta-Gunnells and Brubaker-Bump-Friedberg. For affine root systems, I show that the axioms produce a unique series, different from what the naive generalization of the finite-type constructions would produce. I explain the contribution of imaginary roots to this series, giving some preliminary evidence that it is the correct one from the standpoint of metaplectic Kac-Moody representations.

3 Projects in Kac-Moody Groups

Metaplectic Kac-Moody groups have not yet been constructed, but are likely to appear soon in my work or that of Patnaik and Puskás. The construction of Matsumoto [28] is already in terms of Chevalley generators, and should carry over to the infinite-dimensional setting without great difficulty. Garland’s construction [18] of affine Borel Eisenstein series must be carried out in the metaplectic context—this is the natural first step in the field of metaplectic Kac-Moody automorphic forms. I hope to collaborate with experts in the area on this project. Further projects could include extending the cuspidal Kac-Moody Eisenstein series of [20] to the metaplectic setting. From my perspective, there are two enormous challenges ahead: one is to develop a theory of unipotent integration on infinite-dimensional spaces, in order to define the Archimedean Kac-Moody Whittaker functions. The second is to develop tools robust enough to move beyond affine Kac-Moody groups. I will focus on this question below, because the axiomatic method affords a unique
point of view. Since non-affine Kac-Moody groups are a new frontier, I am exploring them at the level of combinatorial representation theory before setting any number-theoretic goals. This is where computer experiments with power series, Kac-Weyl characters, and generating functions enter my work, and where a motivated undergraduate researcher could make interesting discoveries without needing to understand the whole theory.

Patnaik has proven the (non-metaplectic) affine Casselman-Shalika formula [30]. Like the finite-type Casselman-Shalika formula, it consists of a Weyl character and a twisted Weyl denominator. But an additional factor appears in Kac-Moody types: a power series \( m \) with terms corresponding to imaginary roots. The same series appears in the Macdonald constant term conjecture [27, 10] and the affine Gindikin-Karpelevich formula [3]. One of my current projects, joint with Brubaker and Diaconu, is to write the axiomatic multiple Dirichlet series as a metaplectic analogue of a Weyl character, multiplied by some imaginary power series. This series will likely be related to the \( m \) factor after a good choice of normalization. If the long-term goal is to construct and analyze Whittaker functions for all metaplectic Kac-Moody groups, then it will be vital to understand the interplay between our construction and Patnaik’s. Further, we must guess the imaginary correction factor which will appear in more complicated Kac-Moody types. Developing combinatorial algorithms and computer programs to do this is a very interesting and approachable problem.

Currently, with Muthiah and Puskás, I am investigating analogues of the Macdonald constant term conjecture beyond affine type. Define the power series

\[
\Delta_q(x) = \prod_{\alpha \in \Phi^+} (1 - qx^{\alpha})^{\text{mult}(\alpha)},
\]

(9)
a product over positive roots including imaginary roots. In affine types, Macdonald studies the Weyl group symmetrizer of \( \Delta_q \):

\[
\sum_{w \in W} w \left( \frac{\Delta_q(x)}{\Delta_1(x)} \right) = P(q)m_q(x).
\]

(10)

Here \( P(q) \) is the Poincaré polynomial \( \sum_w (-q)^{\ell(w)} \) and \( m_q(x) \) is the \( m \) factor described above, a product over imaginary roots only. The Macdonald constant term conjecture is equivalent to an explicit formula for \( m_q \) in terms of invariants of the underlying finite root system [27]. The identity (10) must generalize to all Kac-Moody types, but the issue is defining \( m_q \). Muthiah, Puska, and I have found a very general characterization of \( m_q \): it is the unique power series supported only on imaginary roots such that \( \Delta_q m_q \) is supported only on real roots. Using this characterization, we have computed power series expansions of \( m_q \) in rank 2 hyperbolic types. It is still a product of over imaginary roots, but now the factors grow in complexity with the length of the roots. We have observed patterns which we intend to prove combinatorially in certain rank 2 cases, and conjecture in greater generality.

4 Projects in Automorphic Forms

Let \( k \) be a global field and \( G \) a split reductive algebraic group, of rank \( r \), with Borel subgroup \( B = TU \). Let \( E(g) \) be the Borel Eisenstein series on \( G \), and \( \psi \) a character of \( U(k) \setminus U(A_k) \). Then the integral which produces a Fourier-Whittaker coefficient of \( E \) is:

\[
W(g) = \int_{U(k) \setminus U(A_k)} E(ug)\bar{\psi}(u)du.
\]

(11)
Since $E$ is the sum of a test function over $B(k) \setminus G(k)$, using the Bruhat decomposition, this integral unfolds to a finite sum of integrals on $U(\mathbb{A}_k)$. This unfolding is a foundational argument in automorphic forms. When the test function is a pure tensor, the integral splits into an Euler product, giving rise to $L$-functions in the coefficients of Eisenstein series—this is the Langlands-Shahidi method [26, 32]. Kazhdan and Patterson develop Eisenstein series on metaplectic covers $\widetilde{G}$, and give the unfolding computation [25]. They remark that the Whittaker coefficients are Dirichlet series with Gauss sums as coefficients—what we now know as multiple Dirichlet series. Recent progress in multiple Dirichlet series can shed new light on metaplectic Eisenstein series and their unfolding.

We now understand that these Dirichlet series are not Eulerian, but that they satisfy a twisted multiplicativity property (4) which allows all their coefficients to be expressed in terms of $p$-parts. Twisted multiplicativity has become even more important in the axiomatic point of view. It should be directly observable from the metaplectic unfolding argument. This argument would simplify the whole theory of metaplectic Whittaker functions, reducing many global questions directly to local ones. The challenge is to understand induced representations from metaplectic tori. If $T(\mathbb{A})$ is the torus in $G(\mathbb{A})$, its preimage $\widetilde{T}(\mathbb{A})$ in $\widetilde{G}(\mathbb{A})$ is generally nonabelian. This is why multiplicity one theorems fail for some metaplectic groups. However, $\widetilde{T}(\mathbb{A})$ contains two notable maximal abelian subgroups: one which is roughly $\widetilde{T}(\mathbb{A})^n \widetilde{T}(k)$, and the other roughly $\widetilde{T}(\mathbb{A})^n \prod_v \widetilde{T}(\mathbb{O}_v)$. Induction from the first subgroup gives a natural construction of Eisenstein series, but induction from the second naturally yields Euler products. The interplay between them creates twisted multiplicativity. This proof is not yet in the literature, but many of the ingredients already appear in the work of Kazhdan and Patterson. I intend to collaborate with others in the field of multiple Dirichlet series who have already given the problem serious thought.

A larger goal is to justify all the axioms directly from the adelic integral. The present geometric construction seems very distant from the methods of automorphic forms. But the integral of (11), over $k = \mathbb{F}_q(t)$, must satisfy the Poincaré duality and cohomological purity conditions; this should be observable directly from its definition. The simplest possible case is the function field zeta function, $\zeta(s) = (1 - q^{-1-s})^{-1}$, with Euler factor $\zeta_p(s) = (1 - |p|^{-s})^{-1}$. Here Poincaré duality says that $\zeta_p(s)$ is obtained from $\zeta(s)$ after the change of variables $q \mapsto \frac{1}{|p|}$, $s \mapsto \deg(p)(s-1)$, and cohomological purity says that the coefficient of $q^{-as}$ in $\zeta(s)$ has degree $a$. Writing $\zeta(s)$ as an adelic integral, as in Tate’s thesis, these properties follow from the strong approximation theorem on $\mathbb{A}^*_k$. If this argument applies to metaplectic Whittaker integrals, the result would be very illuminating. If it extends to the Kac-Moody setting, then it would completely determine the Whittaker functions for all affine and star-shaped Dynkin diagrams, thanks to uniqueness theorems proven by Diaconu, Pasol, and myself. Our work thus far has focused on constructing multiple Dirichlet series satisfying certain natural, expected properties of the Whittaker function. Now that automorphic constructions are being carried out in Kac-Moody cases, it may be possible to make the connection directly.

**Problems in Algebraic Geometry**

An important and ongoing project is to investigate the geometric meaning of the coefficients $c_{a_1,\ldots,a_r}(q)$ of function field multiple Dirichlet series. One particularly nice example pertains to the star-shaped Dynkin diagram with $r + 1$ vertices. Its coefficients include moments up to $r$ for the trace of Frobenius in the family of elliptic curves over $\mathbb{F}_q$. Birch studies all such moments via the Eichler-Selberg trace formula [2]. The first 9 are polynomials in $q$, but the 10th is not,
because of the appearance of the Ramanujan cusp form. This means that Kac-Moody multiple Dirichlet series, unlike the finite-type versions, contain deep geometric information about higher cohomology. The geometric approach is a radical departure from the existing theory of multiple Dirichlet series, and raises as many questions as it answers. It would be extremely valuable to have concrete examples. I have written about the simplest example, the $A_2$ multiple Dirichlet series, over arbitrary function fields [36], and I am currently at work on a paper explaining its geometry from the axiomatic point of view.

Diaconu and Pasol use an induction to define $c_{a_1,...,a_r}(q)$ in terms of the cohomology of $H_g$, the moduli space of hyperelliptic curves [15]. By definition, this sum can be understood as a sum over curves $y^2 = f(x)$ over $F_q$, for $f$ of fixed degree. The induction is simultaneously on the degree of the curve and the singularity type. The base case uses information about genus 0 curves, and also deep results about the moduli of nonsingular hyperelliptic curves in arbitrary genus. The inductive step defines the global coefficient $c_{a_1,...,a_r}(q)$, and the local coefficients $H$ for the highly singular curves $y^2 = (x - \alpha)^d$. The induction makes use of a rich combinatorics of factorization type. I have used a similar induction to prove that the axioms imply the Weyl group of functional equations. I am currently attempting to combine the inductive setup with a sieve to count squarefree polynomials $f$ only. This would allow results on the moments of L-functions with squarefree conductor

$$\sum_{\begin{subarray}{c} \text{deg}(f)=d \\ f \text{ squarefree} \end{subarray}} L(s, \chi_f)^r$$

both in the limit as $q \to \infty$, as studied in [24], and $d \to \infty$, as in [16]. Rubinstein and Wu have recently been able to evaluate these sums exactly for certain small values of $r$ and $d$ [31]; I hope to compute more.

In finite and affine types, the coefficients $c_{a_1,...,a_r}(q)$ are polynomials in $q$, of degree at most $a_1 + \cdots + a_r$. After multiplying the series $Z$ by a ratio of Weyl denominators like the $\Delta_1$ discussed above, the new coefficients should have degree roughly $\frac{1}{2}(a_1 + \cdots + a_r)$. This statement is a multivariable analogue of the Hasse-Weil bound for coefficients of the zeta functions of algebraic varieties. An undergraduate could study the finite-type multiple Dirichlet series over $F_q(t)$, which are rational functions, to state the precise conjecture on the coefficients, and explain what it means about the zeroes of $Z$. This is a multivariable version of the Riemann hypothesis! And the induction on degree and singularity type is well-suited to prove it, using Deligne’s proof of the Riemann hypothesis for algebraic varieties over $F_q$ as a starting point.

Finally, the coefficients $c_{a_1,...,a_r}(q)$ are all polynomials in finite and affine types, but the argument of Birch [2] shows that in very complicated Kac-Moody types, they are not. Does polynomiality hold for hyperbolic types? What is the simplest example of a Kac-Moody Whittaker function in which non-polynomial coefficients appear? Cherednik has given similar polynomiality arguments for coefficients of formal sums over affine Weyl groups, via the double affine Hecke algebra, in work related to the Macdonald constant term conjecture [10]. So this question is significant from the standpoint of Kac-Moody groups and representations as well as algebraic geometry. What does polynomiality of the coefficients say about the structure of the underlying Eisenstein series? If there is a classification of all Kac-Moody groups whose Whittaker functions have polynomial coefficients, this seems like a natural class on which to define automorphic forms. I hope this will be the future direction of my work.
References Cited


