(The numerators of these fractions are the moments of mass of the thin plate w.r.t. the x- and y-axes, respectively.)

Note: When $S = \text{const}$, we've already discussed (both when we did Pappus' theorem).

\[ \text{Inset here: note that if } \int f(x)g(y) \, dx \, dy = \frac{1}{2} \left( \frac{d}{dx} \int g(y) \, dy \right) \]

Next: polar coordinates.

Idea: locate point by distance from the origin $(r)$ and angle with positive $x$-axis $(\theta)$

\[ r \theta \]

First observation: not unique

\[ (r, \theta) = (-r, \theta + \pi) = (r, \theta + 2\pi) = (-r, \theta - \pi), \ldots \]

Also, $(0, \theta)$ is the origin for every angle $\theta$.

Translating:

\[ x = r \cos \theta \quad \quad \quad \quad y = r \sin \theta \quad \quad \quad \quad r^2 = x^2 + y^2 \quad \quad \quad \quad \quad \theta = \arctan \frac{y}{x} \]

The graph of an equation in polar coordinates is just the set of points that satisfy the equation.

Examples:

- Graph: $r = 1$
  - $\theta = \pi/4$
  - $r = 2 \cos \theta$

**Question:** Which of the following points lie on the curve with equation $r = \sin^2 \theta$?

- $(2, \pi/4)$
- $(4, \pi/4)$
- $(2, 3\pi/2)$ (truly!)
Question: Identify the graph $r = 2 \csc \theta$.

Question: Give a polar equation for the rectangular equation $y = x^2$.

Use: Some things are much easier to describe in polar coordinates than in rectangular coordinates: a simple example is the 4-leaf rose

$r = \sin 2\theta$

Imagine trying to give it in rectangular coordinates!

(Question: Can we give it parametric coordinates?)

Remark: Finding intersections of polar curves can be very tricky:

Consider the curves $r = 1 + \cos^2 \theta$ and $r = -1 - \cos^2 \theta$.

Their graphs are identical (Note: Not an ellipse)

Next, let's consider some of the classical problems of calculus for polar coordinates. Arclength:

\[
\begin{align*}
\dot{x} &= r \cos \theta \Rightarrow \dot{x} = dr \cos \theta - r \sin \theta \, d\theta \\
\dot{y} &= r \sin \theta \Rightarrow \dot{y} = dr \sin \theta + r \cos \theta \, d\theta
\end{align*}
\]

\[
\Rightarrow \quad \dot{s}^2 = \dot{x}^2 + \dot{y}^2
\]

\[
= \left( dr^2 \cos^2 \theta - 2 r \sin \theta \cos \theta \, dr \, d\theta + r^2 \sin^2 \theta \, d\theta^2 \right) \\
+ \left( dr^2 \sin^2 \theta + 2 r \sin \theta \cos \theta \, dr \, d\theta + r^2 \cos^2 \theta \, d\theta^2 \right)
\]

\[
= dr^2 + r^2 \, d\theta^2
\]

So, for example, arclength of the spiral $r = \theta^2$?
Areas in polar coordinates

Given a curve "around the origin," how much area a region sector it defines?

Not a case

Alternative:

\[
\text{Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta
\]

\[
dA = \frac{1}{2} r^2 \, d\theta
\]

Some care is necessary in choosing the right bounds.

Area inside one petal of four-bit case \( r = \sin 2\theta \)?

\[
\int_{0}^{\pi/2} \frac{1}{2} (\sin 2\theta)^2 \, d\theta
\]

\[
= \int_{0}^{\pi/2} \left( \frac{1 - \cos 4\theta}{2} \right) \, d\theta
\]

\[
= \text{...} \quad \text{indicated here}
\]

Sometimes polar coordinates make double integrals easier to compute (e.g., if region is most naturally described in polar).

In this case, our element of area is