Surface integrals

Surface given by \( \mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \)

To compute a surface integral, cut surface up into many little pieces; each piece, measure its contribution.

Then add up. "Pieces" are little \( u \)-by-\( v \) rectangles — contribution is \( f(\mathbf{r}(u,v)) \, dA \)

\[ dA = | \mathbf{u} \times \mathbf{v} | \, du \, dv \]

**Surface integral is**

\[ \iint_S f(x,y,z) \, dA = \iint_D f(\mathbf{r}(u,v)) \left| \mathbf{u} \times \mathbf{v} \right| \, dA \]

**Example.** Compute \( \iint_S x^2 \, dS \) where \( S \) is the sphere \( x^2 + y^2 + z^2 = 2 \).
As with other integrals, interest comes b/c we can measure things with them. But have interest for other reasons, e.g., Flux: given a surface (which we think of as a netting) & a field \( \vec{F} \) (fluid flow), what is the flux of fluid through the surface?

Ans: \[ \iint_S \vec{F} \cdot \vec{n} \, dS, \] where \( \vec{n} \) is the unit normal with \[ \vec{n} = \frac{\vec{a} \times \vec{v}}{|\vec{a} \times \vec{v}|}. \]

Side note: orientable vs. non-orientable surfaces like Möbius strip, what we're interested in.

E.g. What is the flux of the vector field \( \langle x, y, z \rangle \) through the surface of the cylinder bounded by \( z = 0, z = b, x^2 + y^2 = a \)?

E.g. What is the flux of the vector field \( \vec{F}(x, y, z) = z\vec{i} + y\vec{j} + x\vec{k} \) through the sphere \( x^2 + y^2 + z^2 = 1 \)?
True form: \[ \hat{r}(\phi, \theta) = \left< \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \right> \]

\[ r \times \hat{r} = \left< \sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi \right> \]

\[ |r \times \hat{r}| = \sin \gamma \]

A surface is **closed** if it encloses a region (e.g., the sphere or the cylinder); in this case, we take \( \mathbf{n} \) to be the **outwards** normal.

Now suppose we have a surface \( S \) enclosing a solid region \( R \). Cut the region into many little boxes, and consider the field flowing through this region. The flux out of the entire region is \[ \oiint \mathbf{F} \cdot \mathbf{n} \, dS \]. Also, the flux is the sum over every little box of the flux out of the box (since flux on adjacent...
boxes cancel). For each of these boxes, the flux is

\[
\text{div} \mathbf{F} \, dV_{	ext{box}} = \text{div} \mathbf{F} \, dV \quad \text{in } \mathbb{R}^n
\]

(flux density)

\[
\lim_{\text{box size} \to 0} \frac{\int_{S_{\text{box}}} \mathbf{F} \cdot \hat{n} \, dS}{\int_{V_{\text{box}}} \text{div} \mathbf{F} \, dV} = \text{div} \mathbf{F}
\]

Limit \quad \text{Gauss' Theorem} / \text{Divergence Theorem}

\[
\int_{S} \mathbf{F} \cdot \hat{n} \, dS = \int_{V} \text{div} \mathbf{F} \, dV
\]

E.g. Evaluate these integrals from earlier using Gauss' Divergence Theorem