1. Required problems

- Exercises from the text: 8.5.8, 8.5.11, 8.5.12, 9.2.3, 9.2.4

1. (a) In class we counted labeled unicyclic graphs (i.e., connected graphs on \( n \) vertices with \( n \) edges) and gave an asymptotic formula for their growth. Give an exact formula for the number of labeled graphs on \( n \) vertices with \( n \) edges (that is, both connected and disconnected).

(b) Roughly what fraction of labeled graphs on \( n \) vertices with \( n \) edges are connected when \( n \) is large? (Your answer probably will depend on \( n \). Let’s agree that we don’t care about constant factors. You might want to use the following fact (which can be shown using arguments similar to those we’ve used in computing other asymptotics): for \( N, k \) large, if \( k \) is of the same order as \( \sqrt{N} \) then \( \binom{N}{k} \) is of the same order as \( N^k/k! \).)

[Not part of the assignment: recall that you proved on the previous p-set that for every graph \( G \) at least one of \( G \) and \( \overline{G} \) is connected, so a randomly selected graph has probability at least \( \frac{1}{2} \) of being connected. (In fact the probability is very close to 1 for large \( n \).) This seems somewhat in conflict with the preceding result. But a random graph probably has something like \( n^2/4 \) edges, whereas the graphs we are looking at in this question are very sparse, and so much less likely to be connected.]

(2) Call a graph on \( n \) vertices bicyclic if it is connected and has \( n + 1 \) edges. If one repeatedly deletes all degree-1 vertices from such a graph, one is eventually left with a graph (called the 2-core) in one of two forms: either a “barbell” or a “banana.” (We are not defining these terms; do some examples to figure out what they mean! Note that this “barbell” is completely different from the barbell in 8.5.12.)

(a) Give an example of a bicyclic graph on 15 vertices such that this process results in a barbell.

(b) Draw the barbell with edge set \{13, 36, 61, 17, 75, 52, 28, 84, 45\} and suppose \( n \geq 8 \). How many bicyclic graphs on \( n \) vertices have this barbell as 2-core?

(c) How many bananas are there with 5 labeled vertices?

[Not part of the assignment: this sort of analysis could be used to compute the number of bicyclic graphs in the same way that we computed the number of unicyclic graphs. In the end, we would pick up another factor of \( C \cdot n^{3/2} \).]

(3) Recall from the handout the definitions of \( O, o, \Theta, \sim \).

(a) Suppose that \( f \) and \( g \) are positive functions such that \( f(n) = O(g(n)) \). Show that \( f(n)^2 = O(g(n)^2) \).

(b) Suppose that \( g(n) \to \infty \) and that \( f(n) = \Theta(g(n)) \). Show that \( \ln f(n) \sim \ln g(n) \).

(c) Give an example of functions \( f(n), g(n) \) such that \( f(n) \sim g(n) \to \infty \) and \( 2^{f(n)} = o(2^{g(n)}) \).

(4) (a) Show that the number of unlabeled unicyclic graphs is at most \( n^2 \cdot 4^n \).

(b) Give a decent lower bound for the number of unlabeled unicyclic graphs on \( n \) vertices.

2. Optional problems

- Text: 8.4.2, 8.4.3, 9.1.1, 9.1.2, 9.2.5,