Absolute order and factorizations in $\text{GL}_n(\mathbb{F}_q)$

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Joint with J. Huang and V. Reiner (arXiv:1506.03332) and A. Morales (in prep.)

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For $w \in S_n$, let $c(w)$ be the number of cycles; equiv., dimension of fixed space: $2461537 = (124)(36)(5)(7) \iff$

\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\]

fixes span of
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

$\text{codimfix}(w) = n - c(w)$, codimension of fixed space

In $S_n$, $\text{codimfix}(w)$ is equal to the transposition length $\ell_T(w)$: smallest $k$ s.t. $w = t_1 \cdots t_k$, all $t_i$ transpositions

- E.g.: $(1243) = (12) \cdot (23) \cdot (24)$

If $u \cdot v = w$ then $\ell_T(w) \leq \ell_T(u) + \ell_T(v)$
Absolute order on $S_n$

- $\ell_T(w) = \text{“length”} = \text{codimfix}(w) = n - c(w) = \text{fix space codim.}$
- $u \cdot v = w$ implies $\ell_T(w) \leq \ell_T(u) + \ell_T(v)$

- A structuralist approach: build a poset
- Define order $\preceq$ on $S_n$ by $u \preceq w$ iff there is $v$ such that $u \cdot v = w$, $\ell_T(w) = \ell_T(u) + \ell_T(v)$.

This poset is called the absolute order.
Absolute order on $S_n$ 3

\[ \ell_T(w) = \text{length, fixed space codim.}; \text{if } u \cdot v = w \text{ then} \]
\[ \ell_T(w) \leq \ell_T(u) + \ell_T(v); u \preceq w \text{ iff there is } v \text{ such that } u \cdot v = w, \]
\[ \ell_T(w) = \ell_T(u) + \ell_T(v). \]

- Graded by $\ell_T$; rank sizes are Stirling numbers
- Hasse diagram is the Cayley graph for $S_n$ generated by transpositions
- Maximal chains below an element count shortest factorizations as a product of transpositions
- Intervals are self-dual. Interval below a long cycle $c = (1 \cdots n)$ is particularly nice: isomorphic to the lattice of noncrossing partitions, size $C_n$ Catalan, rank sizes Narayana, maximal chains $n^{n-2}$, nice zeta polynomial, Möbius function
The interval below a long cycle $c = (1 \cdots n)$ is isomorphic to the lattice of noncrossing partitions:
How much of this story extends to other matrix groups?

- codim. fix space $\mapsto$ codim. fix space
- transpositions $\mapsto$ reflections: matrices that fix a hyperplane
- $S_n \mapsto$ a matrix group generated by its subset $T$ of reflections
- transposition length $\ell_T \mapsto$ reflection length $\ell_T$

**Theorem 1** (Huang–L–Reiner). For any reflection group $W \subseteq \text{GL}(V)$ with reflections $T$, one has for all $g \in G$ that $\text{codim}_{\text{fix}}(g) \leq \ell_T(g)$. 
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**Theorem 1** (Huang–L–Reiner). *For any reflection group $W \subseteq \text{GL}(V)$ with reflections $T$, one has for all $g \in G$ that $\text{codimfix}(g) \leq \ell_T(g)$.*

- get $\text{codimfix} = \ell_T$ for
  - finite real reflection groups (Carter)
  - real orthogonal groups & complex unitary groups (Brady–Watt)
  - NOT all finite complex reflection groups (Foster-Greenwood)
Consider a (f.d.) vector space $V$ over any field. Then $G = \text{GL}(V)$ is generated by its set of reflections.

$$\begin{bmatrix} 1 & a & \cdots \\ 1 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} b \\ 1 \\ \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}$$

**Theorem 2** (Huang–L–Reiner). For $G = \text{GL}(V)$ with reflections $T$, for all $g \in G$, one has $\text{codimfix}(g) = \ell_T(g)$.

So we have an absolute order on $\text{GL}(V)$. Intervals are still self-dual. Other analogous nice properties? Particularly, when $V = \mathbb{F}_q^n$?
Interested in enumeration, $q$-analogues: consider $GL_n(F_q) = GL(F_q^n)$ over a finite field $F_q$.

- Analogue of “long cycle” or “Coxeter element” is a *Singer cycle* $c$: among elements of $GL_n(F_q)$ with no fixed space, they have largest multiplicative order.

- Interval $[e, c]$ below a Singer cycle should be a $q$-analogue of non-crossing partitions

- Structural questions: $[e, c]$ usually not a lattice; shellable??; symmetric chain decomposition??

- Chain counts = factorization counts are nice
$G = \text{GL}_n(\mathbb{F}_q)$; $c$ a Singer cycle (or more generally regular elliptic element) in $G$

**Theorem 3** (Huang–L–Reiner). Let $b = (b_1, \ldots, b_k)$ be a composition of $n$. The number of chains in the interval $[e, c]$ in the absolute order on $G$ passing through ranks $0, b_1, b_1 + b_2, \ldots, b_1 + \ldots + b_k = n$ is

$$q^{e(b)} \cdot (q^n - 1)^{k-1}$$

where $e(b) = \sum_i (b_i - 1)(n - b_i)$.

In particular, the number of maximal chains is $(q^n - 1)^{n-1}$, while the number of elements of rank $i$ is $q^{2i(n-i) - n}(q^n - 1)$.

(Compare $n^{n-2}$ maximal chains in $S_n$, rank sizes Narayana (?!).)
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More generally, consider all factorizations (not just chains).
Theorem 4 (Jackson). Let $a_{r_1,r_2,\ldots,r_k}$ be the number of $k$-tuples $(u_1, u_2, \ldots, u_k)$ in $S_n$ such that $u_i$ has $r_i$ cycles and $u_1 \cdots u_k = c$. Then

$$(n!)^{1-k} \sum_{1 \leq r_1, r_2, \ldots, r_k \leq n} a_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k} = \sum_{1 \leq p_1, \ldots, p_k \leq n} \frac{M_{p_1-1,\ldots,p_k-1}^{n-1}}{p_1^{x_1} \cdots p_k^{x_k}},$$

where

$$M_{r_1,\ldots,r_k}^m = \sum_{d=0}^{\min(r_i)} (-1)^d \binom{m}{d} \prod_{i=1}^{k} \binom{m-d}{r_i-d}.$$ 

(There is also a more refined version due to Bernardi–Morales using cycle types.)
Theorem 5 (L–Morales). Fix a regular elliptic element $c$ in $G = \text{GL}_n(F_q)$. Let $a_{r_1,\ldots,r_k}(q)$ be the number of tuples $(u_1, \ldots, u_k)$ of elements of $G$ such that $u_i$ has fixed space dimension $r_i$ and $u_1 \cdots u_k = c$. Then

$$|G|^{1-k} \sum_{r_1,\ldots,r_k} a_{r_1,\ldots,r_k}(q) x_1^{r_1} \cdots x_k^{r_k} = \sum_{(p_1,\ldots,p_k): 0 \leq p_i \leq n} \frac{M_{\tilde{p}}^{n-1}(q)}{\prod_{p \in \tilde{p}} \left[ \frac{n-1}{p} \right]_q} \frac{(x_1; q^{-1})_{p_1}}{(q; q)_{p_1}} \cdots \frac{(x_k; q^{-1})_{p_k}}{(q; q)_{p_k}},$$

where $\tilde{p}$ is the result of deleting all copies of $n$ from $(p_1, \ldots, p_k)$,

$$M_{r_1,\ldots,r_k}^m(q) := \sum_{d=0}^{\min_i(r_i)} (-1)^d q^{\frac{d+1}{2} - kd} \left[ m \atop d \right]_q \prod_{i=1}^k \left[ m - d \atop r_i - d \right]_q$$

for $k > 0$, and $M_{\emptyset}^m(q) := 0$.

(N.B.: if $x = q^N$ then $(x; q^{-1})_k/(q; q)_k = \left[ N \atop k \right]_q$.)
Let $G$ be any finite group, $A_1, \ldots, A_\ell \subset G$ unions of conjugacy classes of $G$, and $c$ any element of $G$.

**Lemma** (Frobenius, 1896). *The number of factorizations $c = t_1 \cdots t_\ell$ such that $t_i \in A_i$ for all $i$ is*

$$\frac{1}{|G|} \sum_{V \in \text{Irr}(G)} \deg(V) \chi_V(c^{-1}) \cdot \tilde{\chi}_V(z_1) \cdots \tilde{\chi}_V(z_\ell)$$

*where $\text{Irr}(G)$ is the set of irreducible (complex) representations of $G$, $\tilde{\chi}_V(g) = \frac{\chi_V(g)}{\deg(V)}$ is a normalized character, and $z_i = \sum_{t \in A_i} t \in \mathbb{C}[G]$ (Same idea used by Jackson in $S_n$)*
Theorem (Dénes 1959). The number of factorizations of the long cycle in $S_n$ as a product of $n - 1$ transpositions is

$$n^{n-2}$$

Proof. Count trees on vertex set $[n]$ with a linear order $(e_1, \ldots, e_{n-1})$ on edges:

$$(n - 1)! \cdot n^{n-2}$$
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Map this to the factorization $t_{e_1} \cdots t_{e_{n-1}}$ of an $n$-cycle via

$$e_i = \{a, b\} \mapsto t_{e_i} = (a, b) \in S_n$$

(example on next slide!)
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(example on next slide!)

This gives every factorization of every $n$-cycle, so is $(n - 1)!$ times our answer.
Example of Dénes’s map

Background

New results
Absolute order on general linear groups
Interval below a Singer cycle
Chain counting
Factorization counting in $S_n$
Factorization counting in $GL_n(F_q)$

\[ T = \begin{align*} &1 \quad 2 \quad 4 \quad 5 \quad 6 \quad 3 \end{align*} \]

\[(4, 6) \cdot (2, 4) \cdot (4, 5) \cdot (2, 3) \cdot (1, 4) = (1, 5, 2, 3, 6, 4)\]