Pattern avoidance in alternating permutations and tableaux

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- Definitions, background.
- Pattern avoidance in alternating permutations.
- A nice generalization of alternating permutations.
- Pattern avoidance in reading words of Young tableaux.
- A few more results on pattern avoidance in alternating permutations.
A permutation of length $n$ is a word $w = w_1w_2 \cdots w_n$ that contains each element of $\{1, \ldots, n\}$ exactly once.

A permutation $w$ is alternating if
\[ w_1 < w_2 > w_3 < w_4 > w_5 < \ldots. \]

We write $S_n$ for the set of all permutations of length $n$.

We write $A_n$ for the set of all alternating permutations of length $n$. (Note that this has nothing at all to do with the alternating group.)
A permutation $w = w_1 \cdots w_n$ is said to contain a permutation $p = p_1 \cdots p_k$ as a pattern if there exist $i_1 < \ldots < i_k$ such that $w_{i_\ell} < w_{i_m}$ if and only if $p_\ell < p_m$. Otherwise, $w$ is said to avoid $p$.

For example, the permutation 624153 contains the pattern 123.
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Given a set $S$ of permutations and a pattern $p$, we denote by $S(p)$ the set of permutations in $S$ that avoid $p$.

Thus, $S_n(123)$ is the set of permutations of length $n$ with no three-term increasing subsequence and $A_n(1234)$ is the set of alternating permutations of length $n$ with no four-term increasing subsequence.
Theorem (Essentially MacMahon 1915, Knuth 1968). For any $p \in S_3$ we have $\#S_n(p) = C_n$ where $C_n$ is the $n$-th Catalan number, i.e., $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Theorem (Deutsch and Reifegerste 2003, Mansour 2003). For any $p \in S_3$ we have that $\#A_n(p)$ is a Catalan number.

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**Theorem** (Essentially MacMahon 1915, Knuth 1968). For any $p \in S_3$ we have $\#S_n(p) = C_n$ where $C_n$ is the $n$-th Catalan number, i.e., $C_n = \frac{1}{n+1} \binom{2n}{n}$.

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This is an interesting coincidence.

**Question 1.** Can we say anything else interesting about pattern avoidance in alternating permutations? For example, what about alternating permutations avoiding a pattern of length four?
A *partition* is a decreasing sequence of positive integers: \( \langle 4, 4, 2 \rangle = \langle 4^2, 2 \rangle \).
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Every partition has an associated diagram:

\[
\langle 4^2, 2 \rangle \leftrightarrow \\
\begin{array}{cccc}
    & & & \\
    & & & \\
    & & & \\
\end{array}
\]

A Young tableau is a filling of the diagram with 1, 2, \ldots that increases along rows and columns:

\[
\begin{array}{cccc}
    1 & 2 & 5 & 6 \\
    3 & 7 & 9 & 10 \\
    4 & 8 \\
\end{array}
\]
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Proof idea: Apply RSK. For example, $46185723 \in A_8(1234)$ corresponds under RSK to

$$(P, Q) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ 6 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 7 & 8 \end{pmatrix}$$

and similarly

$A_8(1234) \ni 68243715 \overset{\text{RSK}}{\leftrightarrow} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 7 \\ 4 & 8 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 \\ 5 & 8 \\ 7 \end{pmatrix}$
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3 & 5 & 6 \\
7 & 8 \\
\end{array} \right)
\]

and similarly

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4 & 8 \\
6 \\
\end{array}, \begin{array}{ccc}
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3 & 4 \\
5 & 8 \\
7 \\
\end{array} \right)
\]

Two observations:

- The tableaux have at most three columns
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  (by 1234-avoidance and Greene’s Theorem)
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and similarly

$$A_8(1234) \ni 68243715 \leftrightarrow RSK \left( \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 7 & \end{array}, \begin{array}{ccc} 1 & 2 & 6 \\ 3 & 4 & \begin{array}{ccc} 5 & 8 \end{array} \\ 6 & 7 \end{array} \right)$$

Two observations:

- The tableaux have at most three columns (by 1234-avoidance and Greene’s Theorem)
- For $1 \leq i \leq n$, the value $2i - 1$ appears strictly to the left of the value $2i$ in $Q$ (because descents in $w$ become “tableau descents” in $Q$)
Theorem 1. $A_{2n}(1234)$ is in bijection with standard Young tableaux of shape $\langle 3^n \rangle$.

Proof, continued:
If $Q$ is the recording tableau associated to a permutation $w \in A_{2n}(1234)$ then

- $Q$ has at most three columns
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and, conversely, any pair \( (P, Q) \) of SYT of the same shape such that \( Q \) satisfies these two conditions corresponds to a permutation in \( A_{2n}(1234) \).
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and, conversely, any pair $(P, Q)$ of SYT of the same shape such that $Q$ satisfies these two conditions corresponds to a permutation in $A_{2n}(1234)$.

Thus, we have a bijection between $A_{2n}(1234)$ and pairs $(P, Q)$ of SYT with $2n$ boxes such that $\text{shape}(P) = \text{shape}(Q)$ and $Q$ satisfies the two properties above.

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 & \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 2 & 4 \\
3 & 5 & 6 \\
7 & 8 & \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 3 & 5 \\
2 & 7 \\
4 & 8 \\
6 & \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 2 & 6 \\
3 & 4 \\
5 & 8 \\
7 & \\
\end{pmatrix}
\]
Theorem 1. $A_{2n}(1234)$ is in bijection with standard Young tableaux of shape $\langle 3^n \rangle$.

Proof, continued:
Now we modify a trick due to Ouchterlony (2006) and match each pair of columns with their complementary column.

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 4 \\
3 & 5 & 6 \\
7 & 8
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\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 4 \\
3 & 5 & 6 \\
7 & 8
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 & 9
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 4 \\
3 & 5 & 6
\end{pmatrix}
\]
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7 & 8 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 & 9 \\
10 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 4 \\
3 \\
\end{pmatrix}
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1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 \\
\end{pmatrix}
, \quad \begin{pmatrix}
1 & 2 & 4 \\
3 & 5 & 6 \\
7 & 8 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 & 9 \\
10 & 11 \\
\end{pmatrix}
, \quad \begin{pmatrix}
1 \\
2 \\
\end{pmatrix}
\]
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\end{pmatrix}, \quad \begin{pmatrix}
1 & 2 & 4 \\
3 & 5 & 6 \\
7 & 8 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 & 9 \\
\end{pmatrix}, \quad \begin{pmatrix}
10 & 11 & 12 \\
\end{pmatrix}
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\[
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6 & 8 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 2 & 4 \\
3 & 5 & 6 \\
7 & 8 \\
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\[
\begin{align*}
\left( \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 7 \\
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\begin{array}{ccc}
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\end{align*}
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6 & \end{array} \right), \quad 
\begin{array}{ccc}
1 & 2 & 6 \\
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\begin{array}{ccc}
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2 & 7 & 9 \\
4 & 8 & 11 \\
6 & 10 & 12 &
\end{array}
\end{align*}
\]

And we’re done! □
Theorem 1. $A_{2n}(1234)$ is in bijection with standard Young tableaux of shape $\langle 3^n \rangle$.

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Now we modify a trick due to Ouchterlony (2006) and match each pair of columns with their complementary column.

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\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 7 \\
6 & 8 & \end{array},
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 6 \\
7 & 8 & \end{array}\bigg) \rightarrow 
\begin{array}{ccc}
1 & 2 & 3 \\
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6 & 8 & 9 \\
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$$

$$(
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 7 & \\
4 & 8 & \\
6 & \end{array},
\begin{array}{ccc}
1 & 2 & 6 \\
3 & 4 & \\
5 & 8 & \\
7 & \end{array}\bigg) \rightarrow 
\begin{array}{ccc}
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2 & 7 & 9 \\
4 & 8 & 11 \\
6 & 10 & 12 & \end{array}
$$

And we’re done!
(It follows that $\#A_{2n}(1234) = \frac{2 \cdot (3n)!}{n!(n+1)!(n+2)!}$.)
Question 2. We have
\[ \#S_n(123) = C_n = \#SYT(2^n) \quad \text{and} \quad \#A_{2n}(1234) = \#SYT(3^n), \]
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\[ \#S_n(123) = C_n = \#SYT(2^n) \quad \text{and} \quad \#A_{2n}(1234) = \#SYT(3^n), \]
so one might guess that there is some set \( \mathcal{P} \) such that
\[ \#\mathcal{P}(12345) = \#SYT(4^n) \]
(and so on). Is this intuition correct? That is, is there a reasonable set \( \mathcal{P} \) with these properties?
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\[ \#S_n(123) = C_n = \#SYT(2^n) \quad \text{and} \quad \#A_{2n}(1234) = \#SYT(3^n), \]
so one might guess that there is some set \(?\) such that
\[ \# ?(12345) = \#SYT(4^n) \]
(and so on). Is this intuition correct? That is, is there a reasonable set \(?\) with these properties?

Yes! But to describe the set in question, it helps to have another definition.
Question 2. We have

\[ \#S_n(123) = C_n = \#SYT(2^n) \quad \text{and} \quad \#A_{2n}(1234) = \#SYT(3^n), \]

so one might guess that there is some set \( \mathcal{A} \) such that

\[ \# \mathcal{A}(12345) = \#SYT(4^n) \]

(and so on). Is this intuition correct? That is, is there a reasonable set \( \mathcal{A} \) with these properties?

Yes! But to describe the set in question, it helps to have another definition.

Idea: permutations are *reading words* of certain (standard skew Young) tableaux.

\[
\text{RW} \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 4 & 5 \end{pmatrix} = 452613
\]
We have (for example) that $S_4$ is the set of reading words of tableaux of shape

$$= \langle 4, 3, 2, 1 \rangle / \langle 3, 2, 1 \rangle$$

and similarly that $A_8$ is the set of reading words of tableaux of shape

$$= \langle 5, 4, 3, 2 \rangle / \langle 3, 2, 1 \rangle.$$
generalized alternating permutations

Let $\mathcal{L}_{n,k}$ be the set of reading words of tableaux of skew shape $\langle n + k, n + k - 1, \ldots, k + 1 \rangle / \langle n - 1, n - 2, \ldots, 1 \rangle$. For example, $\mathcal{L}_{4,3}$ is the set of reading words of SYT of shape

SO

\[
\begin{array}{cccccccc}
1 & 6 & 9 & \\
3 & 8 & 12 & \\
4 & 1011 & \\
\end{array}
\]

\[\text{RW} \begin{pmatrix}
2 & 5 & 7 \\
1 & 6 & 9 \\
3 & 8 & 12 \\
4 & 1011 \\
\end{pmatrix} = 410113812169257 \in \mathcal{L}_{4,3}
\]
Let $\mathcal{L}_{n,k}$ be the set of reading words of tableaux of skew shape $\langle n + k, n + k - 1, \ldots, k + 1 \rangle / \langle n - 1, n - 2, \ldots, 1 \rangle$. For example, $\mathcal{L}_{4,3}$ is the set of reading words of SYT of shape

![Diagram of a skew shape](image)

So

$$\text{RW} \begin{pmatrix}
2 & 5 & 7 \\
1 & 6 & 9 \\
3 & 8 & 12 \\
4 & 10 & 11
\end{pmatrix} = 4 \ 10 \ 11 \ 3 \ 8 \ 12 \ 1 \ 6 \ 9 \ 2 \ 5 \ 7 \in \mathcal{L}_{4,3}$$

Observe that $\mathcal{L}_{n,1} = S_n$ and $\mathcal{L}_{n,2} = A_{2n}$. 
**Theorem 2.** $\mathcal{L}_{n,k}(12\cdots(k+2))$ is in bijection with the set of SYT of shape $\langle(k+1)^n\rangle$. 
Theorem 2. \( \mathcal{L}_{n,k}(12 \cdots (k + 2)) \) is in bijection with the set of SYT of shape \( (k + 1)^n \).

The proof is very similar to the proof of Theorem 1, but instead of using \( \binom{3}{1} = \binom{3}{2} \) we use \( \binom{k+1}{1} = \binom{k+1}{k} \). That is, after applying RSK we pair each set of \( k \) of the \( k + 1 \) available columns with the lone unused column.
Theorem 2. $\mathcal{L}_{n,k}(12 \cdots (k+2))$ is in bijection with the set of SYT of shape $(k+1)^n$.

The proof is very similar to the proof of Theorem 1, but instead of using $\binom{3}{1} = \binom{3}{2}$ we use $\binom{k+1}{1} = \binom{k+1}{k}$. That is, after applying RSK we pair each set of $k$ of the $k+1$ available columns with the lone unused column.

$\mathcal{L}_{4,3} \ni 4\ 10\ 11\ 3\ 8\ 12\ 1\ 6\ 9\ 2\ 5\ 7 \xrightarrow{\text{RSK}} \begin{pmatrix} 1 & 2 & 5 & 7 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 11 \\ 10 \end{pmatrix}$,
Now that we’re talking about pattern avoidance in reading words of tableaux of a given shape, we can ask

**Question 3.** For an arbitrary (skew) shape $\lambda/\mu$ and pattern $p$, can we say anything about the number of tableaux of shape $\lambda/\mu$ whose reading words avoid the pattern $p$?
Theorem 3. Suppose $\lambda/\mu$ is a connected skew shape. The number of SYT of shape $\lambda/\mu$ whose reading words avoid $213$ is equal to the number of partitions $\tau$ contained in $\mu$. 
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Observation. It’s somewhat remarkable that such a theorem should exist.
**Theorem 3.** Suppose $\lambda/\mu$ is a connected skew shape. The number of SYT of shape $\lambda/\mu$ whose reading words avoid $213$ is equal to the number of partitions $\tau$ contained in $\mu$.

Proof idea: Given $\lambda/\mu$ and $\tau$, 

![Diagram of skew shapes and partitions]
Theorem 3. Suppose \( \lambda/\mu \) is a connected skew shape. The number of SYT of shape \( \lambda/\mu \) whose reading words avoid 213 is equal to the number of partitions \( \tau \) contained in \( \mu \).

Proof idea: Given \( \lambda/\mu \) and \( \tau \), choose maximal \( i \) such that \( \tau_{i-1} > \mu_i \) (or \( i = 1 \)) and put 1 in the \( i \)-th row of \( \lambda/\mu \).
Pattern avoidance in reading words, continued
Then split $\lambda/\mu$ and $\tau$ into three parts.
Then split $\lambda/\mu$ and $\tau$ into three parts and continue recursively.
Pattern avoidance in reading words, continued
Put the larger values in the lower piece and the smaller values in the upper piece.

(This is what ensures 213-avoidance.)
Put the larger values in the lower piece and the smaller values in the upper piece.

(This is what ensures $213$-avoidance.) And we’re done!
Back to alternating permutations
What about alternating permutations of odd length?
Theorem 4. $A_{2n+1}(1234)$ is in bijection with the set of SYT of shape $(3^{n-1}, 2, 1)$. 
**Theorem 4.** $A_{2n+1}^{(1234)}$ is in bijection with the set of SYT of shape $\langle 3^{n-1}, 2, 1 \rangle$.

Proof idea: Modify the proof of Theorem 1; consider down-up permutations instead of up-down (i.e., alternating) permutations.

3271546 $\leftrightarrow_{RSK}$ \[
\begin{array}{ccc}
1 & 4 & 6 \\
2 & 5 \\
3 & 7 \\
\end{array},
\begin{array}{ccc}
1 & 3 & 7 \\
2 & 5 \\
4 & 6 \\
\end{array}
\]
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\[
3271546 \xrightarrow{RSK} \begin{pmatrix} 1 & 4 & 6 \\ 2 & 5 \\ 3 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 7 \\ 2 & 5 \\ 4 & 6 \end{pmatrix}
\]
Theorem 4. \( A_{2n+1}(1234) \) is in bijection with the set of SYT of shape \( (3^{n-1}, 2, 1) \).

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\[
3271546 \xleftrightarrow{RSK} \begin{pmatrix}
1 & 4 & 6 \\
2 & 5 \\
3 & 7
\end{pmatrix}, \begin{pmatrix}
1 & 3 & 7 \\
2 & 5 \\
4 & 6
\end{pmatrix} \xleftrightarrow{RSK} \begin{pmatrix}
1 & 4 & 6 \\
2 & 5 & 9 \\
3 & 7 \\
4 & 8
\end{pmatrix}
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Theorem 4. $A_{2n+1}(1234)$ is in bijection with the set of SYT of shape $(3^{n-1}, 2, 1)$.

Proof idea: Modify the proof of Theorem 1; consider down-up permutations instead of up-down (i.e., alternating) permutations.

These permutations are also reading words of tableaux, and similar results hold for (at least) the $L_{n,k}$-type analogue coming from shapes like

\[
\begin{array}{c|c|c}
1 & 4 & 6 \\
2 & 5 & \\
3 & 7 & \\
\hline
\end{array}, \quad \begin{array}{c|c}
1 & 3 & 7 \\
2 & 5 & \\
4 & 6 & \\
\hline
\end{array} \quad \begin{array}{c|c|c}
1 & 4 & 6 \\
2 & 5 & 9 \\
3 & 7 & \\
4 & 8 & \\
\hline
\end{array}
\]
There is an alternate proof of Theorem 1 (the enumeration of $A_{2n}(1234)$) using *generating trees* (a type of recursive enumeration scheme).
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**Theorem 1.** $A_{2n}(1234)$ is in bijection with SYT of shape $\langle 3^n \rangle$.

Proof idea: Consider the (infinite) tree whose vertices are the 1234-avoiding alternating permutations of even length, ordered by the relation “is order-isomorphic to a prefix of.”

```
12 1423 2314 2413 3412
   |     |     |     |
  1324 1423 2314 2413 3412
    |     |     |     |
   163524 164523 263514 264513 364512

1423 ← 263514
```
To each $w \in A_{2n}(1234)$, assign a pair $(i, j)$ of integers (a *label*), where

- $i$ is the smallest entry of $w$ that appears as the 2 in a 12-pattern in $w$ (i.e., the smallest entry with something smaller to its left), and
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For example, 68243715 has label $(3, 5)$ while 46185723 has label $(2, 3)$. 
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For example, 68243715 has label \((3, 5)\) while 46185723 has label \((2, 3)\). Then show that we can express the labels of the children of \( w \) just in terms of the label of \( w \).
Finally, find an isomorphic tree/labeling pair for rectangular standard Young tableaux with three columns, and conclude by induction.
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Also can give even more:

**Theorem 5.** The sets $A_{2n}(1234)$ and $A_{2n}(2143)$ are equinumerous for all $n$. 
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Proof idea: The two sets have isomorphic generating trees. Label $w \in A_{2n}(2143)$ with $(i, j)$ where $i = w_{2n-1} + 1$ and $j$ is the number of values from $[2n + 1]$ we can append to $w$ without creating a 2143 pattern.
Theorem 5. The sets $A_{2n}(1234)$ and $A_{2n}(2143)$ are equinumerous for all $n$.

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For example, to $68143527 \in A_8(2143)$ can safely append 1, 2, 3, 8, or 9 since 792546381, 791546382, 791546283, 691435278 and 681435279 avoid 2143 while 691435287, 791435286, 791436285 and 791536284 contain it. Thus $i = 2$ and $j = 5$. 

Final remarks on other patterns
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**Conjecture 1.** If $\# A_{2n}(p) = \# A_{2n}(q)$ for all $n$ then $\# S_n(p) = \# S_n(q)$ for all $n$. 
Thanks to Alex Postnikov and to Suho Oh, Craig Desjardins and Alejandro Morales for many helpful discussions.

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Thanks to all of you for coming!